Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.78645

Compatible Mappings and Their Variants in Perturbed Metric Spaces

Kajal Kharb and Sanjay Kumar

ABSTRACT: In this paper, we introduce the notions of compatible mappings and their variants in perturbed metric spaces. Next, we prove common fixed point theorems for these mappings.

Key Words: Perturbed metric space, common fixed points, compatible mappings.

Contents

1. Introduction

The measurement of the distance between two points is not always exact. During measurement, some errors may occur. These errors may be slightly positive, slightly negative, or sometimes zero. If error is zero, then it corresponds to the metric. To account for these, a positive error is subtracted and a negative error is added during determining the exact value of the distance function. These errors may play a significant role during measurement.

In order to overcome the difficulty, whenever error is added in metric, Mohamed Jleli and Bessem Samet [7] gave the notion of a perturbed metric space. Perturbed metric spaces represent a useful and practical improvement over the metric spaces. The significance of perturbed metric spaces lies across a wide range of mathematical and applied disciplines.

Even though for small positive errors, the structure of these spaces still retains the properties of metric spaces. In this way, perturbed metric spaces help to bridge the gap between the mathematical models and real-world situations, where exact distance are not measurable.

In 2025, Mohamed Jleli and Bessem Samet [7] introduced a more general form of distance function, known as perturbed metric space as follows:

Definition 1.1. Let $D, P : X \times X \to [0, \infty)$ be two given functions. The function D is called a perturbed metric on X with respect to P, if the function

$$D - P : X \times X \to \mathbb{R},$$

defined by the relation

$$(D - P)(x, y) = D(x, y) - P(x, y),$$

for all $x, y, z \in X$, is a exact metric on X, i.e., for all $x, y, z \in X$, it satisfies the following conditions

(i) $(D - P)(x, y) \geq 0$;

(ii) $(D - P)(x, y) = 0$ if and only if $x = y$;

(iii) $(D - P)(x, y) = (D - P)(y, x)$;

(iv) $(D - P)(x, y) \leq (D - P)(x, z) + (D - P)(z, y)$.

2020 *Mathematics Subject Classification:* 47H10, 54H25.
Submitted August 27, 2025. Published January 20, 2026

P is called a *perturbing function* and $D = d + P$ be an *perturbed metric.*
The set X endowed with D and *perturbed function* P denoted by (X, D, P) is known as *perturbed metric spaces.*

Notice that a perturbed metric on X is not necessarily a metric on X. But a metric is always perturbed metric when perturbed error is zero.

Example 1.1. Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + x^2 y^4, \ for\ all\ x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = x^2 y^4, \quad x, y \in \mathbb{R}.$$

In this case, the exact metric is the mapping $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$d(x, y) = D(x, y) - P(x, y)\ ,\ where$$

$$d(x, y) = |x - y|, \quad x, y \in \mathbb{R}.$$

Here we note that D is not necessarily a metric, because $D(1, 1) = 1 \neq 0$ as $x = y$, but D is perturbed metric on X with respect to perturbed function P.

We now introduce topological structure in perturbed metric space.

The topological structure of the perturbed metric space (X, D, P) is induced by the exact metric $d = D - P$. That is, the topology $\tau_{D,P}$ on X is defined as:

$$\tau_{D,P} := \tau_d = \{U \subseteq X \mid \forall x \in U, \exists r > 0 \text{ such that } B_d(x, r) \subseteq U\},$$

where the open ball with respect to d is given by:

$$B_d(x, r) = \{y \in X \mid d(x, y) = D(x, y) - P(x, y) < r\}.$$

Definition 1.2. Let (X, D, P) be a perturbed metric space with perturbed function P. A sequence $\{x_n\}$ in X is said to be

(i) *perturbed convergent sequence*, if $\{x_n\}$ is convergent in the metric space (X, d), where $d = D - P$ is the exact metric.

(ii) *perturbed Cauchy sequence*, if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d).

(iii) (X, D, P) is a *complete perturbed metric space* if (X, d) is a complete metric space, i.e., every perturbed Cauchy sequence converges in perturbed metric space.

A mapping T defined on (X, D, P) is a *perturbed continuous mapping*, if T is continuous with respect to the exact metric d.

We recall some elementary properties of perturbed metric spaces [7] .

Proposition 1.1. [7] Let $D, P, Q : X \times X \to [0, \infty)$ be three given mappings and $\alpha > 0$.

(i) If (X, D, P) and (X, D, Q) be two perturbed metric spaces, then $\left(X, D, \frac{P+Q}{2}\right)$ is a perturbed metric space.

(ii) If (X, D, P) is a perturbed metric space, then $(X, \alpha D, \alpha P)$ is a perturbed metric space.

Here for the convenience of readers, we provide the proof of the proposition 1.1.

Proof.

(i) Since $D - P$ and $D - Q$ are two metrics on X, then

$$\frac{1}{2}[(D-P)+(D-Q)] = D - \frac{P+Q}{2}$$

is a metric on X, which proves (i).

(ii) Since $D - P$ is a metric on X and $\alpha > 0$, then

$$\alpha(D-P) = \alpha D - \alpha P$$

is a metric on X, which proves (ii).

2. Relationships and Properties of Compatible Mappings and their Variants

Now we introduce the notions of compatible mappings and their variants in the setting of perturbed metric spaces as follows:

In 1986, Jungck [2] introduced the notion of compatible mappings in metric spaces as follows:

Definition 2.1. Let S and T be two mappings of a metric space (X, d) into itself. Then S and T are called compatible if and only if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

In 1993, Jungck et al. [3] introduced the notion of compatible mappings of type (A) in metric spaces as follows:

Definition 2.2. A pair (S, T) of self-mappings of a metric space (X, d) is said to be compatible mappings of type (A) if and only if

$$\lim_{n\to\infty} d(SSx_n, TSx_n) = 0, \quad \text{and} \quad \lim_{n\to\infty} d(STx_n, TTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

In 1994, Pathak, Pant and Singh [6] introduced the notion of compatible mappings of type (P) in metric spaces as follows:

Definition 2.3. A pair (S, T) of self-mappings of a metric space (X, d) is said to be compatible mappings of type (P) if and only if

$$\lim_{n\to\infty} d(SSx_n, TTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \quad \text{for some } z \in X.$$

In 1995, Pathak and Khan [4] introduced the notion of compatible mappings of type (B) in metric spaces as follows:

Definition 2.4. A pair (S, T) of self-mappings of a metric space (X, d) is said to be compatible mappings of type (B) if and only if

$$\lim_{n\to\infty} d(TSx_n, SSx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(TSx_n, Tz) + \lim_{n\to\infty} d(Tz, TTx_n)\right],$$

and

$$\lim_{n\to\infty} d(STx_n, TTx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(STx_n, Sz) + \lim_{n\to\infty} d(Sz, SSx_n)\right],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

In 1998, Pathak, Cho and Kang [5] introduced the notion of compatible mappings of type (C) in metric spaces as follows:

Definition 2.5. A pair (S, T) of self-mappings of a metric space (X, d) is said to be compatible mappings of type (C) if and only if

$$\lim_{n\to\infty} d(STx_n, TTx_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(STx_n, Sz) + \lim_{n\to\infty} d(Sz, SSx_n) + \lim_{n\to\infty} d(Sz, TTx_n)\right],$$

and

$$\lim_{n\to\infty} d(TSx_n, SSx_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(TSx_n, Tz) + \lim_{n\to\infty} d(Tz, SSx_n) + \lim_{n\to\infty} d(Tz, TTx_n)\right],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Now we introduce the analogues notions of compatible mappings and their variants in setting of perturbed metric spaces.

Definition 2.6. Let S and T be two mappings of a perturbed metric space (X, D, P) into itself. Then S and T are called compatible if and only if

$$\lim_{n\to\infty} D(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Example 2.1. Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^4, \ \ for\ all\ x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed function

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = x^2y^4, \quad x, y \in \mathbb{R}.$$

Let $S, T : X \to X$ be defined by $Sx = \frac{x}{2}$ and $Tx = \frac{x}{3}$, for all $x \in X$, where $X = [0, \infty)$. Taking the sequence $\{x_n\}$ as $x_n = \frac{1}{n}, n > 0$, such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X,$$

then S and T are said to be compatible

$$\lim_{n\to\infty} D(STx_n, TSx_n) = 0.$$

Remark 2.1. Weakly compatible maps need not be compatible.

Example 2.2. Let $X = [2, 20]$ and $D : \mathbb{R} \times \mathbb{R} \to [2, 20]$ be the mapping defined by

$$D(x, y) = |x - y| + x^2 y^2, \ for\ all\ x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed function

$$P : \mathbb{R} \times \mathbb{R} \to [2, 20]$$

given by

$$P(x, y) = x^2 y^2, \quad x, y \in \mathbb{R}.$$

Defining $S, T : X \to X$ as below:

$$Sx = \begin{cases} 2 & \text{if } x = 2 \ or > 5 \\ 6 & \text{if } 2 < x \le 5. \end{cases} \qquad Tx = \begin{cases} 12, & \text{if } 2 < x \le 5 \\ x - 3, & \text{if } x > 5 \\ x, & \text{if } x = 2. \end{cases}$$

The mappings S and T are non-compatible since sequence $\{x_n\}$ defined by $\{x_n\} = 5 + (\frac{1}{n}),\ n \ge 1$. Then $Tx_n \to 2$, $Sx_n \to 2$. But they are weakly compatible since they commute at coincidence point $x = 2$. But they are not compatible at that point.

Definition 2.7. A pair (S, T) of self-mappings of a perturbed metric space (X, D, P) is said to be compatible mappings of type (A) if and only if

$$\lim_{n\to\infty} D(SSx_n, TSx_n) = 0, \quad \text{and} \quad \lim_{n\to\infty} D(STx_n, TTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Proposition 2.1. Let S and T be compatible mappings of type (A). If one of S or T is continuous, then S and T are compatible.

Proof: Since (S, T) be compatible of type (A), we have

$$\lim_{n\to\infty} D(S(Tx_n), T(Tx_n)) \to 0 \quad and \quad \lim_{n\to\infty} D(T(Sx_n), S(Sx_n)) \to 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Suppose that S is continuous. Then $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} STx_n = Sz$ for some $z \in X$. Now we get $\lim_{n\to\infty} D(S(Tx_n), T(Sx_n)) = 0$, *i.e.*, S and T be compatible mappings.
Similarly, if T is continuous, the S and T be compatible mappings.

Proposition 2.2. Let S and T be continuous mappings. If S and T are compatible, then they are compatible mappings of type (A).

The direct consequence of propositions 2.1 and 2.2 is in the form of following :

Proposition 2.3. Let S and T be continuous mappings. Then S and T are compatible if and only if they are compatible mappings of type (A).

Proposition 2.4. Let S and T be compatible mappings of type (A) of a perturbed metric space (X, D, P) into itself. If $Sz = Tz$ for some $z \in X$, then

$$STz = SSz = TTz = TSz.$$

Proof. Let $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \quad \text{for some } z \in X$$

and suppose $Sz = Tz$. Then we have

$$TSx_n \to Tz \quad \text{and} \quad TTx_n \to Tz \quad \text{as } n \to \infty.$$

Since S and T are compatible of type (A), we have

$$\lim_{n\to\infty} D(TSx_n, SSx_n) = 0,$$

and

$$\lim_{n\to\infty} D(STx_n, TTx_n) = 0,$$

Hence $STz = TTz$. Now, since $Sz = Tz$, we also have

$$SSz = TSz \quad \text{and} \quad TSz = TTz.$$

Therefore,

$$STz = SSz = TTz = TSz.$$

This completes the proof. □

Proposition 2.5. Let S and T be compatible mappings of type (A) of a perturbed metric space (X, D, P) into itself. Suppose that

$$\lim_{n\to\infty} Tx_n = z \quad \text{and} \quad \lim_{n\to\infty} Sx_n = z \quad \text{for some } z \in X.$$

Then

(a) $\lim_{n\to\infty} STx_n = Tz$ if T is continuous at z.

(b) $STz = TSz$ and $Sz = Tz$ if S and T are continuous at z.

Proof: (a) Suppose that T is continuous at z. Since

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \in X$, we have $TTx_n, TSx_n \to Tz$ as $n \to \infty$.
Since S and T are compatible of type (A), we have

$$\lim_{n\to\infty} D(TSx_n, SSx_n) = 0,$$

and

$$\lim_{n\to\infty} D(STx_n, TTx_n) = 0,$$

Therefore, $\lim_{n\to\infty} STx_n = Tz$. This completes the proof of (a).

(b) Suppose that S and T are continuous at z. Since $Tx_n \to z$ as $n \to \infty$ and S is continuous at z, by (a), $TTx_n \to Sz$ as $n \to \infty$. On the other hand, T is also continuous at z, so $TTx_n \to Tz$. Thus, we have $Sz = Tz$ by the uniqueness of limits, and by Proposition 2.4, $STz = TSz$. This completes the proof.

Definition 2.8. A pair (S, T) of self-mappings of a perturbed metric space (X, D, P) is said to be weak compatible of type (A) if

$$\lim_{n\to\infty} D(TSx_n, SSx_n) \le \lim_{n\to\infty} D(STx_n, SSx_n) \quad \text{and} \quad \lim_{n\to\infty} D(STx_n, TTx_n) \le \lim_{n\to\infty} D(TSx_n, TTx_n),$$

whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Proposition 2.6. Every pair of compatible mappings of type (A) is weak compatible of type (A).
Proof. Suppose that S and T are compatible mappings of type (A),
i.e.

$$0 = \lim_{n\to\infty} D(TSx_n, SSx_n) \leq \lim_{n\to\infty} D(STx_n, SSx_n)$$

and

$$0 = \lim_{n\to\infty} D(STx_n, TTx_n) \leq \lim_{n\to\infty} D(TSx_n, TTx_n).$$

Which shows that pair (S, T) is weak compatible of type (A).

Proposition 2.7. Let S and T are continuous mappings of a perturbed metric space (X, D, P) into itself. If S and T are weak compatible of type (A), then they are compatible of type (A).

Proof : Suppose that S and T are weak compatible of type (A). Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Since S and T are continuous mappings, then we have

$$\lim_{n\to\infty} D(TSx_n, SSx_n) \leq \lim_{n\to\infty} D(STx_n, SSx_n) = D(Sz, Sz) = 0$$

and

$$\lim_{n\to\infty} D(STx_n, TTx_n) \leq \lim_{n\to\infty} D(TSx_n, TTx_n) = D(Tz, Tz) = 0.$$

Therefore, S and T are compatible mappings of type (A). This completes the proof.

Proposition 2.8. Let S and T be weak compatible mappings of type (A) from a perturbed metric space (X, D, P) into itself. If one of S and T is continuous, then S and T are compatible.

Proof. Without loss of generality, suppose that T is continuous. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Since T is continuous, we have

$$\lim_{n\to\infty} TSx_n = Tt = \lim_{n\to\infty} TTx_n.$$

Now

$$\begin{aligned}\lim_{n\to\infty} D(TSx_n, STx_n) &\leq \lim_{n\to\infty} D(TSx_n, TTx_n) + \lim_{n\to\infty} D(TTx_n, STx_n)\\ &\leq 0 + \lim_{n\to\infty} D(TTx_n, STx_n).\end{aligned}$$

Since (S, T) are weak compatible of type (A), therefore, we have

$$\lim_{n\to\infty} D(TSx_n, STx_n) \leq \lim_{n\to\infty} D(STx_n, TTx_n)$$

$$\leq \lim_{n\to\infty} D(TSx_n, TTx_n) \leq 0.$$

Therefore, S and T are compatible.

Proposition 2.9. Let S and T be weak compatible mappings of type (A) from perturbed metric space (X, D, P) into itself and let

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \; for\ some\ z \in X,$$

then we have the following :

1. $\lim_{n\to\infty} STx_n = Sz$ *if* T *is continuous at* z *.*
2. $\lim_{n\to\infty} TSx_n = Tz$ *if* S *is continuous at* z *.*
3. $TSz = STZ$ and $Sz = Tz$ if S and T are continuous at z.

Definition 2.9. A pair (S, T) of self-mappings of a perturbed metric space (X, D, P) is said to be compatible mappings of type (B) if and only if

$$\lim_{n\to\infty} D(TSx_n, SSx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} D(TSx_n, Tz) + \lim_{n\to\infty} D(Tz, TTx_n)\right],$$

and

$$\lim_{n\to\infty} D(STx_n, TTx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} D(STx_n, Sz) + \lim_{n\to\infty} D(Sz, SSx_n)\right],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Definition 2.10. A pair (S, T) of self-mappings of a perturbed metric space (X, D, P) is said to be compatible mappings of type (C) if and only if

$$\lim_{n\to\infty} D(STx_n, TTx_n) \le \frac{1}{3}\left[\lim_{n\to\infty} D(STx_n, Sz) + \lim_{n\to\infty} D(Sz, SSx_n) + \lim_{n\to\infty} D(Sz, TTx_n)\right],$$

and

$$\lim_{n\to\infty} D(TSx_n, SSx_n) \le \frac{1}{3}\left[\lim_{n\to\infty} D(TSx_n, Tz) + \lim_{n\to\infty} D(Tz, SSx_n) + \lim_{n\to\infty} D(Tz, TTx_n)\right],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Remark 2.2. *Compatible mappings of type (A) $\Longrightarrow$ compatible mappings of type (B) $\Longrightarrow$ compatible mappings of type (C), but the converse is not true in general.*

Example 2.3. : Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^2, \; for\ all\ x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = x^2y^2, \quad x, y \in \mathbb{R}.$$

Let $S, T : X \to X$ be defined by $Sx = \frac{x}{2}$ and $Tx = \frac{x}{3}$, for all $x \in X$, where $X = [0, \infty)$. Taking the sequence $\{x_n\}$ as $x_n = \frac{1}{n}, n > 0$. Then, S and T are compatible of type (A), compatible of type (B) and compatible of type (C) also. But the converse is not true in general.

Let $X = [1, 20]$, and $D : \mathbb{R} \times \mathbb{R} \to [1, 20]$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^2, \; for\ all\ x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P : \mathbb{R} \times \mathbb{R} \to [1, 20]$$

given by

$$P(x, y) = x^2y^2, \quad x, y \in \mathbb{R}.$$

Defining $S, T : X \to X$ as below:

$$Sx = \begin{cases} 1, & \text{if } x = 1, \\ 3, & \text{if } 1 < x \leq 7, \\ x - 6, & \text{if } 7 < x \leq 20. \end{cases} \quad \textit{and} \quad Tx = \begin{cases} 1, & \text{if } x = 1 \text{ or } x \in (7, 20], \\ 2, & \text{if } 1 < x \leq 7. \end{cases}$$

Taking sequence $\{x_n\}$ as $x_n = 7 + \frac{1}{n}$, $n > 0$. Then, S and T are compatible of type (C), but neither compatible nor compatible of type (A) nor compatible of type (B).

Definition 2.11. A pair (S, T) of self-mappings of a metric space (X, D, P) is said to be compatible mappings of type (P) if and only if

$$\lim_{n\to\infty} D(SSx_n, TTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \quad \text{for some } z \in X.$$

Proposition 2.10. Every pair of compatible mappings of type (A) is compatible of type (B).

Proof : Suppose that S and T are compatible of type (A). Then we have

$$0 = \lim_{n\to\infty} D(TSx_n, SSx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} D(TSx_n, Tz) + \lim_{n\to\infty} D(Tz, TTx_n)\right]$$

and

$$0 = \lim_{n\to\infty} D(STx_n, TTx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} D(STx_n, Sz) + \lim_{n\to\infty} D(Sz, SSx_n)\right],$$

as derived.

Proposition 2.11. Let S and T be continuous mappings of a perturbed metric space (X, D, P) into itself. If S and T are compatible mappings of type (B), then they are compatible of type (A).

Proof : Let $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \quad \text{for some } z \in X.$$

Since S and T are continuous, we have

$$\lim_{n\to\infty} D(TSx_n, SSx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} D(TSx_n, Tz) + \lim_{n\to\infty} D(Tz, TTx_n)\right] = 0,$$

and

$$\lim_{n\to\infty} D(STx_n, TTx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} D(STx_n, Sz) + \lim_{n\to\infty} D(Sz, SSx_n)\right] = 0,$$

Therefore, S and T compatible of type (A). This completes the proof.

Proposition 2.12. Let S and T be continuous mappings of a perturbed metric space (X, D, P) into itself. If S and T are compatible of type (B), then they are compatible.

Proof. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Since S and T are continuous, we have

$$\lim_{n\to\infty} SSx_n = St = \lim_{n\to\infty} STx_n$$

and

$$\lim_{n\to\infty} TSx_n = Tt = \lim_{n\to\infty} TTx_n.$$

By triangle inequality, we have

$$D(STx_n, TSx_n) \leq D(STx_n, TTx_n) + D(TTx_n, TSx_n).$$

Letting $n \to \infty$ and taking S and T are compatible of type (B), we have

$$\begin{aligned}\lim_{n\to\infty} D(STx_n, TSx_n) &\leq \lim_{n\to\infty} D(STx_n, TTx_n) + D(TTx_n, TSx_n)\\ &\leq \frac{1}{2}\left[\lim_{n\to\infty} D(STx_n, Sz) + \lim_{n\to\infty} D(Sz, SSx_n)\right]\\ &\quad + \lim_{n\to\infty} D(TTx_n, TSx_n)\\ &= 0.\end{aligned}$$

Therefore, S and T are compatible. This completes the proof. □

Proposition 2.13. Let S and T be continuous mappings of a perturbed metric space (X, D, P) into itself. If S and T are compatible, then they are compatible of type (B).

Proof: One can easily prove it using Propositions 2.2 and 2.10.

Proposition 2.14. Let S and T be continuous mappings of a perturbed metric space (X, D, P) into itself. Then

(1) *S and T are compatible if and only if they are compatible of type (B);*

(2) *S and T are compatible of type (A) if and only if they are compatible of type (B).*

Proposition 2.15. Let S and T be compatible mappings of a perturbed metric space (X, D, P) into itself. If $Sz = Tz$ for some $z \in X$, then $STz = SSz = TTz = TSz$.
Proof. Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = z, n = 1, 2, \ldots$ for some $z \in X$ and $Sz = Tz$. Then we have Sx_n *and* $Tx_n \to Sz$ as $n \to \infty$. Since S and T are compatible, we have

$$D(STz, TSz) = \lim_{n\to\infty} D(STx_n, TSx_n) = 0.$$

Hence we have $STz = TTz$. Therefore, since $Sz = Tz$, we have $STz = SSz = TTz = TSz$. This completes the proof. □

Proposition 2.16. Let S and T be compatible mappings of a perturbed metric space (X, D, P) into itself. Suppose that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \text{ for some } z \in X.$$

Then

(a) $\lim_{n\to\infty} TSx_n = Sz$ *if S is continuous at z;*

(b) $\lim_{n\to\infty} STx_n = Tz$ *if T is continuous at z;*

(c) $STz = TSz$ *and* $Sz = Tz$ *if S and T are continuous at z.*

Proof.

(a) Suppose that S is continuous at z. Since $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$, we have $STx_n \to Sz$ as $n \to \infty$. Since S and T are compatible, we have

$$\lim_{n\to\infty} D(TSx_n, Sz) \leq \lim_{n\to\infty} [D(TSx_n, STx_n) + D(STx_n, Sz)] = 0.$$

Therefore, $\lim_{n\to\infty} TSx_n = Sz$. This completes the proof.

(b) The proof of $\lim_{n\to\infty} STx_n = Tz$ follows by similar arguments as in (1).

(c) Suppose that S and T are continuous at z. Since $Tx_n \to z$ as $n \to \infty$ and S is continuous at z, by (1), $TSx_n \to Sz$ as $n \to \infty$. On the other hand, T is also continuous at z, so $TSx_n \to Tz$. Thus, we have $Sz = Tz$ by uniqueness of the limit, and by Proposition 2.14, $STz = TSz$. This completes the proof. □

Proposition 2.17. Let S and T be compatible mappings of type (B) of a perturbed metric space (X, D, P) into itself. If $Sz = Tz$ for some $z \in X$, then

$$STz = SSz = TTz = TSz.$$

Proof. Let $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \quad \text{for some } z \in X$$

and suppose $Sz = Tz$. Then we have

$$TSx_n \to Tz \quad \text{and} \quad TTx_n \to Tz \quad \text{as } n \to \infty.$$

Since S and T are compatible of type (B), we have

$$\lim_{n\to\infty} D(TSx_n, SSx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} D(TSx_n, Tz) + \lim_{n\to\infty} D(Tz, TTx_n)\right] = 0,$$

and

$$\lim_{n\to\infty} D(STx_n, TTx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} D(STx_n, Sz) + \lim_{n\to\infty} D(Sz, SSx_n)\right] = 0,$$

Hence $STz = TTz$. Now, since $Sz = Tz$, we also have

$$SSz = TSz \quad \text{and} \quad TSz = TTz.$$

Therefore,

$$STz = SSz = TTz = TSz.$$

This completes the proof. □

Proposition 2.18. Let S and T be compatible mappings of type (B) of a perturbed metric space (X, D, P) into itself. Suppose that

$$\lim_{n\to\infty} Tx_n = z \quad \text{and} \quad \lim_{n\to\infty} Sx_n = z \quad \text{for some } z \in X.$$

Then

(a) $\lim_{n\to\infty} TTx_n = Sz$ if S is continuous at z.

(b) $\lim_{n\to\infty} SSx_n = Tz$ if T is continuous at z.

(c) $STz = TSz$ and $Sz = Tz$ if S and T are continuous at z.

Proof: (a) Suppose that S is continuous at z. Since

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \in X$, we have $SSx_n, STx_n \to Sz$ as $n \to \infty$.
Since S and T are compatible of type (B), we have

$$\lim_{n\to\infty} D(TSx_n, SSx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} D(TSx_n, Tz) + \lim_{n\to\infty} D(Tz, TTx_n)\right] = 0,$$

and

$$\lim_{n\to\infty} D(STx_n, TTx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} D(STx_n, Sz) + \lim_{n\to\infty} D(Sz, SSx_n)\right] = 0,$$

Therefore, $\lim_{n\to\infty} TTx_n = Sz$. This completes the proof of (a).

(b) The proof of $\lim_{n\to\infty} SSx_n = Tz$ follows by similar arguments as in (a).

(c) Suppose that S and T are continuous at z. Since $Tx_n \to z$ as $n \to \infty$ and S is continuous at z, by (a), $TTx_n \to Sz$ as $n \to \infty$. On the other hand, T is also continuous at z, so $TTx_n \to Tz$. Thus, we have $Sz = Tz$ by the uniqueness of limits, and by Proposition 2.16, $STz = TSz$. This completes the proof.

Remark 2.3. In Proposition 2.17, assume that S and T be compatible mappings of type (C) or of type (P) instead of of type (B). The conclusion of Proposition 2.17 still holds.

Remark 2.4. In Proposition 2.18, assume that S and T be compatible mappings of type (C) or of type (P) instead of of type (B). The conclusion of Proposition 2.18 still holds.

3. Main Results

Now we prove theorems for these mappings.

Theorem 3.1. Let (X, D, P) be a complete perturbed metric space and f and g be compatible self-mappings of X satisfying the following conditions:

(3.1) $g(X) \subseteq f(X)$;

(3.2) f or g is continuous;

(3.3) $D(gx, gy) \le \alpha\, D(fx, fy)$ for every $x, y \in X$ and $0 \le \alpha < 1$.

Then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (3.1) one can choose a point x_1 in X such that $fx_1 = gx_0$. In general choose

$$y_n = gx_{n-1} = fx_n, \quad \text{for all } n = 0, 1, 2, \dots, \text{ where } x_0 \in X. \tag{3.4}$$

Therefore, from (3.4), we have

$$\begin{aligned} D(y_n, y_{n+1}) &= D(fx_n, fx_{n+1}) = D(gx_{n-1}, gx_n) \le \alpha D(fx_{n-1}, fx_n) \\ &= \alpha D(y_{n-1}, y_n) \\ &\;\vdots \\ &\le \alpha^n D(y_0, y_1). \end{aligned} \tag{3.5}$$

i.e.,

$$D(y_n, y_{n+1}) \le \alpha^n D(y_0, y_1).$$

Let $d = D - P$ be the exact metric. Then, from (3.5), we obtain that

$$d(y_n, y_{n+1}) + P(y_n, y_{n+1}) \le \alpha^n D(y_0, y_1) \quad \forall n \ge 0.$$

Thus, we have

$$\begin{aligned} d(y_n, y_{n+p}) &\le \alpha^n D(y_0, y_1) + \alpha^{n+1} D(y_0, y_1) + \cdots + \alpha^{n+p-1} D(y_0, y_1) \\ &= \alpha^n D(y_0, y_1)(1 + \alpha + \cdots + \alpha^{p-1}) \\ &= \alpha^n D(y_0, y_1)\left(\frac{1-\alpha^p}{1-\alpha}\right) \end{aligned}$$

$$\leq \frac{\alpha^n}{1-\alpha} D(y_0, y_1).$$

Since $\alpha \in [0,1)$, we obtain that $< y_n >$ is a cauchy sequence in the metric space (X,d), so $< y_n >$ is a perturbed cauchy sequence in (X,D,P). By the completeness of the perturbed metric space (X,D,P), there exists $t \in X$ such that

$$f(x_n) \to t. \tag{3.6}$$

But (3.4) implies that

$$g(x_n) \to t. \tag{3.7}$$

Let us suppose that the mapping g is continuous. Therefore

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gt.$$

Since f and g are compatible mappings,

$$\lim_{n\to\infty} D(g(f(x_n)), f(g(x_n))) = 0. \tag{3.8}$$

whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \quad \text{for some } t \in X.$$

we get

$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = gt.$$

We now prove that $t = gt$. Suppose $t \neq gt$, then $D(t, gt) > 0$.
From (3.3), on letting $x = x_n, y = gx_n$

$$D(g(x_n), g(g(x_n))) \leq \alpha\ D(f(x_n), fg(x_n)). \tag{3.9}$$

Proceeding limit as $n \to \infty$, we get

$$D(t, gt) \leq \alpha\ D(t, gt) < D(t, gt),\ \textit{a contradicition}.$$

Therefore $t = gt$. Since $g(X) \subseteq f(X)$, we can find t_1 in X such that $t = gt = ft_1$.
Now from (3.3), take $x = gx_n,\ y = t_1$, we have

$$D(g(g(x_n)), g(t_1)) \leq \alpha\ D(f(g(x_n)), f(t_1)). \tag{3.10}$$

Taking limit as $n \to \infty$, we get

$$D(gt, gt_1) \leq \alpha\ D(gt, ft_1) = \alpha D(gt, gt) = 0,$$

which implies that $gt = gt_1$, *i.e.*,

$$t = gt = gt_1 = ft_1.$$

Also, by using definition of compatibility,

$$D(gt, ft) = \lim_{n\to\infty} D(g(f(x_n)), f(g(x_n))) = 0,$$

which again implies that

$$ft = gt = t.$$

Thus t is a common fixed point of f and g.
Uniqueness: We assume that $t_2 (\neq t)$ be another common fixed point of f and g.
Then $D(t, t_2) > 0$ and

$$D(t, t_2) = D(gt, gt_2) \leq \alpha\ D(ft, ft_2) = \alpha D(t, t_2) < D(t, t_2),$$

a contradiction, therefore $t = t_2$. Hence uniqueness follows.
Theorem 3.2. Theorem 3.1 remains true if compatible mappings is replaced by any one (retaining the rest of the hypothesis) of the following:

(a) compatible mappings of type (A),

(b) compatible mappings of type (B),

(c) compatible mappings of type (C),

(d) compatible mappings of type (P).

Proof: Let x_0 be an arbitrary point in X. By (3.1) one can choose a point x_1 in X such that $fx_1 = gx_0$. In general choose

$$y_n = gx_{n-1} = fx_n, \quad \text{for all } n = 0, 1, 2, \ldots, \text{ where } x_0 \in X. \tag{3.11}$$

From the proof of Theorem 3.1 we conclude that $< y_n >$ is a perturbed cauchy sequence in (X, D, P).By the completeness of the perturbed metric space (X, D, P), there exists $t \in X$ such that

$$f(x_n) \to t. \tag{3.12}$$

But (3.11) implies that

$$g(x_n) \to t. \tag{3.13}$$

Let us suppose that the mapping g is continuous. Therefore

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gt.$$

(a) In case (f, g) is compatible mappings of type (A), then

$$\lim_{n\to\infty} D(ffx_n, gfx_n) = 0, \quad \text{and} \quad \lim_{n\to\infty} D(fgx_n, ggx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.
we get

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} fgx_n = gt.$$

We now prove that $t = gt$. Suppose $t \neq gt$, then $D(t, gt) > 0$.
From (3.3), on letting $x = x_n, y = gx_n$

$$D(g(x_n), g(g(x_n))) \leq \alpha\ D(f(x_n), fg(x_n)). \tag{3.15}$$

Proceeding limit as $n \to \infty$, we get

$$D(t, gt) \leq \alpha\ D(t, gt) < D(t, gt),\ \textit{a contradicition}.$$

Therefore $t = gt$. Since $g(X) \subseteq f(X)$, we can find t_1 in X such that $t = gt = ft_1$.
Now from (3.3), take $x = gx_n,\ y = t_1$, we have

$$D(g(g(x_n)), g(t_1)) \leq \alpha\ D(f(g(x_n)), f(t_1)). \tag{3.16}$$

Taking limit as $n \to \infty$, we get

$$D(gt, gt_1) \leq \alpha\ D(gt, ft_1) = \alpha D(gt, gt) = 0,$$

which implies that $gt = gt_1$, *i.e.*,

$$t = gt = gt_1 = ft_1.$$

Also, by using definition of compatibility of type (A),

$$D(gt, ft) = \lim_{n\to\infty} D(g(f(x_n)), f(f(x_n))) = 0,$$

which again implies that

$$ft = gt = t.$$

Thus t is a common fixed point of f and g.
(b) In case (f, g) is compatible mappings of type (B), then

$$\lim_{n\to\infty} D\big(gfx_n,\ ffx_n\big) \leq \frac{1}{2}\Big[\lim_{n\to\infty} D\big(gfx_n,\ gt\big) + \lim_{n\to\infty} D\big(gt,\ ggx_n\big)\Big], \tag{3.17}$$

and

$$\lim_{n\to\infty} D\big(fgx_n,\ ggx_n\big) \leq \frac{1}{2}\Big[\lim_{n\to\infty} D\big(fgx_n,\ ft\big) + \lim_{n\to\infty} D\big(ft,\ ffx_n\big)\Big], \tag{3.18}$$

whenever $\{x_n\} \subset X$ satisfies $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$.
Using (3.17), we get

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} gfx_n = gt.$$

We now prove that $t = gt$. Suppose $t \neq gt$, then $D(t, gt) > 0$.
From (3.3), on letting $x = x_n, y = fx_n$

$$D(g(x_n), g(f(x_n))) \leq \alpha\ D(f(x_n), ff(x_n)). \tag{3.19}$$

Proceeding limit as $n \to \infty$, we get

$$D(t, gt) \leq \alpha\ D(t, gt) < D(t, gt),\ \textit{a contradicition}.$$

Therefore $t = gt$. Since $g(X) \subseteq f(X)$, we can find t_1 in X such that $t = gt = ft_1$.
Now from (3.3), take $x = fx_n,\ y = t_1$, we have

$$D(g(f(x_n)), g(t_1)) \leq \alpha\ D(f(f(x_n)), f(t_1)). \tag{3.20}$$

Taking limit as $n \to \infty$, we get

$$D(gt, gt_1) \leq \alpha\ D(gt, ft_1) = \alpha D(gt, gt) = 0,$$

which implies that $gt = gt_1$, *i.e.*,

$$t = gt = gt_1 = ft_1.$$

Also, by using definition of compatibility of type (B), from (3.18)

$$D(ft, gt) = \lim_{n\to\infty} D\big(fgx_n,\ ggx_n\big) \leq \frac{1}{2}\Big[\lim_{n\to\infty} D\big(fgx_n,\ ft\big) + \lim_{n\to\infty} D\big(ft,\ ffx_n\big)\Big] = 0,$$

which again implies that

$$ft = gt = t.$$

Thus t is a common fixed point of f and g.

The proof of part (c) and part (d) follows by similar arguments as in part (a) and (b).

4. Conflict of interest

The authors declare that they have no conflicts of interest.

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Kajal Kharb
Department of Mathematics,
Deenbandhu Chhotu Ram University of Science and Technology,Murthal,Sonipat
India.
E-mail address: kajalkharb18799@gmail.com

and

Sanjay Kumar
Department of Mathematics,
Deenbandhu Chhotu Ram University of Science and Technology,Murthal,Sonipat
India.
E-mail address: drsanjaykumar.math@dcrustm.org