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Some Relations and Applications of Fuzzy Automation Sub Semi-Groups

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ABSTRACT: [1,2] In this research paper, to showing that A, besides B, remain two sets. Formerly, a relative ρ from A to B may be defined as a subset of $A \times B$ [1,2,3]. For each $a \in A$, we then define $a\rho$ in the obvious way, to find the $a\rho = \{b \in B \mid (a, b) \in \rho\}$. If S and T are two fuzzy semi-groups, then a subset $\mu \subseteq S \times T$ is known as a relational morphism from S to T, if the conditions are satisfied by the relations as follows: **(RM1)** $(\forall a \in S)\ a\mu \neq \emptyset$; **(RM2)** $(\forall a, b \in S)\ (a\mu)(b\mu) \subseteq (ab)\mu$. It is known as injective if, in addition: **(RM3)** $(\forall a, b \in S)\ a\mu \cap b\mu \neq \emptyset \Rightarrow a\mu = b\mu$ [4]. To showing every relational morphism is a fuzzy sub semi-group of direct products $S \times T$. We say that S divides T if there exists a fuzzy sub semi-group U of T and a morphism ψ from U onto S. Thus, S is a quotient of a fuzzy sub semi-group of T. To shows that S divides T if as well as only if U is a relation morphism injection originating S to T [4,5,6].

Key Words: Fuzzy sub-groups, sub-product semi-groups, computational product groups, monogenic fuzzy semi-groups and fuzzy sub-semi-groups.

Contents

1. Introduction

[1,2,3] One in all these directions relies on mathematical logic and fuzzy sets another uses the notion of baggage that extend sets in a way that allows its constituents to occur more frequently. Recently, each of the aforementioned methods was linked to a fuzzy grouping automaton plan [7]. Motivated by the paper, we prefer to define pumping lemmas for fuzzy languages generated both settled which nondeterministic fuzzy multi-set finite automata, propose the plan of settled fuzzy multi-set infinite automaton, and uncover some properties of the related languages [7,8]. The following is the schedule for the granted paper. Basic concepts of multi-sets, actions over multi-sets, multi-set finite automata, including multi-set languages are presented in Section II-A. Fuzzy multi-set finite automata are covered in Section II-B [1,2,9,10].

2. Definitions and Lemmas

Definition 2.1 *Suppose fuzzy semi-group theory is one type of morphism, is known as Rees morphism, that does correspond very closely to an ideal. Initially, I is a proper ideal of fuzzy semi-group S, then $\rho_I = (I \times I) \cup I_S$.*

Lemma 2.1 *A then B are two sets formerly a relation ρ from A to B may be defined as a subset of $A \times B$ [1,2,3]. For each $a \in A$, we then define $a\rho$ in the obvious way, to find $a\rho = \{b \in B \mid (a, b) \in \rho\}$. If S and T are two fuzzy semi-groups, then a subset $\mu \subseteq S \times T$ is known as a relational morphism from S to T, if the conditions are satisfied by the relations as follows: **(RM1)** $\forall a \in S$, $a\mu \neq \emptyset$; **(RM2)** $\forall a, b \in S$, $(a\mu)(b\mu) \subseteq (ab)\mu$. It is known as injective if, in addition: **(RM3)** $\forall a, b \in S$, $a\mu \cap b\mu \neq \emptyset \Rightarrow a\mu = b\mu$ [4]. To showing every relational morphism is a fuzzy sub semi-group of the direct product $S \times T$. We say that S divides T if there exists a fuzzy sub semi-group U of T, and a morphism ψ from U onto S. Thus,*

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S is a quotient of a fuzzy sub semi-group of T. To shows if and only if here is an injective structural morphism from S to T, then S divides T.

Proof: Suppose S and T be the fuzzy semi-groups. A relational morphism $\mu \subseteq S \times T$ satisfies that relational morphism conditions are **(RM1)** $\forall a \in S,\ a\mu \neq \emptyset$; **(RM2)** $\forall a, b \in S,\ (a\mu)(b\mu) \subseteq (ab)\mu$.

Here $a\mu = \{t \in T : (a, t) \in \mu\}$, and $(a\mu)(b\mu)$ is the set $\{t_1 t_2 : t_1 \in a\mu,\ t_2 \in b\mu\}$.

The injectivity condition of **(RM3)** $\forall a, b \in S,\ a\mu \cap b\mu \neq \emptyset \Rightarrow a\mu = b\mu$ is division of fuzzy semi-groups. We say S divides T if: there exists fuzzy sub-semi-group $U \subseteq T$; there exists a morphism $\psi : U \to S$ that is onto.

Since ($\Rightarrow$) if S divides T, then there exists an injective relational morphism $\mu \subseteq S \times T$. S divides T, there exists fuzzy sub-semi-group $U \subseteq T$ and surjective morphism $\psi : U \to S$.

The relation exists $\mu \subseteq S \times T$ as $(a, t) \in \mu \Leftrightarrow \psi(t) = a$, so $a\mu \in \mu \Leftrightarrow \psi(t \in U) = a$.

(RM1) $\forall a \in S,\ a\mu \neq \emptyset$; **(RM2)**: for $a, b \in S$, take $t_1 \in a\mu,\ t_2 \in b\mu \to \psi(t_1) = a,\ \psi(t_2) = b$, then $\psi(t_1 t_2) = \psi(t_1)\,\psi(t_2) = ab \Rightarrow t_1 t_2 \in (ab)\mu$. So, $(a\mu)(b\mu) \subseteq (ab)\mu$.

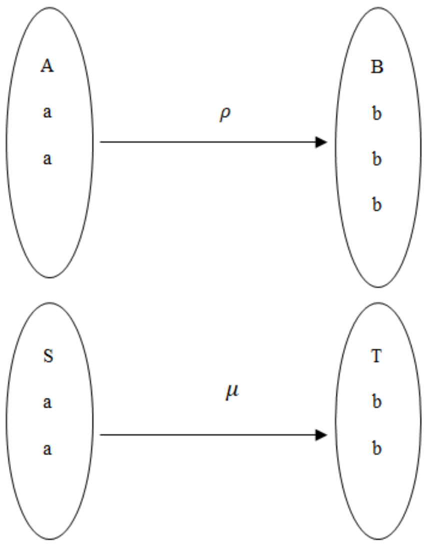


Figure 1: The above figure represents (RM1) $\forall a \in S,\ a\mu \neq \emptyset$; (RM2) $\forall a, b \in S,\ (a\mu)(b\mu) \subseteq (ab)\mu$. It is known as injective if in addition (RM3) $\forall a, b \in S,\ a\mu \cap b\mu \neq \emptyset \Rightarrow a\mu = b\mu$.

(RM3): If $(a\mu) \cap (b\mu) \neq \varnothing$, then there exists $t \in U$ such that $\psi(t) = a$ and $\psi(t) = b \to a = b \to (a\mu) = (b\mu)$. Thus, μ is an injective relational morphism from S to T.

Since ($\Leftrightarrow$) if an injective relational morphism is present $\mu \subseteq S \times T$, then S divides T .

(i) Assume $U = \bigcup_{a \in S} a\mu \subseteq T$. Since μ satisfies (RM1) and (RM2), U is closed under the fuzzy semi-group operation $\to$ a fuzzy sub-semi-group.

(ii) From the definition of $\psi : U \to S$ by $\psi(t) = a$ iff $t \in a\mu$.

(iii) Well-defined? Injectivity (RM3) guarantees that each $t \in U$ is in exactly one $a\mu$, so ψ is well-defined.

(iv) Surjective? Yes, because $a\mu \neq \varnothing$ for all a.

(v) Morphism? For $t_1 \in a\mu,\ t_2 \in b\mu,\ t_1 t_2 \in (ab)\mu \Rightarrow \psi(t_1 t_2) = ab = \psi(t_1)\,\psi(t_2)$.

Thus, ψ is an onto morphism from the fuzzy sub-semi-group $U \subseteq T$ onto $S \rightarrow$ so S divides T. $\square$

Definition 2.2 *Suppose commutative fuzzy semi-group S, define the relation θ_n^S $(n \geq 1)$, then $\frac{S}{\rho}$ and$(\frac{S}{\rho})(\frac{\sigma}{\rho}) \cong \frac{S}{\rho}$. Since the intersection of a non-empty family of congruence's on a fuzzy semi-group S is a congruence on S.*

Lemma 2.2 *For commutative fuzzy semi-group S, define the relation θ_n^s $(n \geq 1)$ by $a\theta_n^s b$ iff $(\forall x \in S^n)$ $xa = xb$.*

(i) Show that θ_n^s is a congruence on S, and that $\theta_1^s \subseteq \theta_2^s \subseteq \theta_3^s \subseteq \cdots$

(ii) Show that $\theta_n^s = 1_S$ for all n if S is a monoid.

(iii) For $n = 1, 2, \ldots$, denote $\frac{s}{\theta_n^s}$ by S_n. Show that for all $(n \geq 2)$,then $S_n \cong S_{n-1}/\theta_1^{S_{n-1}}$.

(iv) Assumimg that $S = \langle a \rangle = M(m, r)$ is a finite monogenic semi-group, where $m > 1$. Show that $\frac{S}{\theta_1^S} \cong M(m-1, r)$ and deduce that $\frac{S}{\theta_1^S} \cong M(m-n, r)$ for all $n < m$. Show also that $\frac{S}{\theta_n^S}$ is isomorphic to the fuzzy cyclic group of order r for all $n \geq m$.

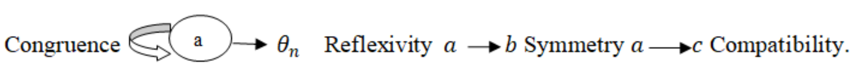
Proof: Given a commutative fuzzy semi-group S, for $n \geq 1$, defined as $a\,\theta_n^s\,b \iff \forall x \in S^n,\ xa = xb$. Here, S^n is the set of all n-tuples over S, and xa means $x_1, x_2, \cdots, x_n a$, i.e., product with a at the end commutatively ensures order doesn't matter.

(i) Show that θ_n^s is a congruence on S, and that $\theta_1^s \subseteq \theta_2^s \subseteq \theta_3^s \subseteq \cdots$

To prove that θ_n^s is a congruence:

- **Reflexivity** is if $xa = xa \Rightarrow a\,\theta_n^s\,a$.
- **Symmetry** is if $xa = xb$ then $xb = xa$.
- **Transitivity** is if $xa = xb$ and $xb = xc$, then $xa = xc$.

  To show it's a congruence (compatible with operation): Assuming that $a\,\theta_n^s\,b$ and let $c \in S$, show $ac\,\theta_n^s\,bc$: For all $x \in S^n$, we have: $x(ac) = xca = xcb = x(bc)$, using commutativity. So, θ_n^s is a congruence.



Chain of inclusions: If $a\,\theta_n^s\,b$, then $xa = xb$ for all $x \in S^n$,

so especially for all $x \in S^{n-1}$, we can define $x' = (x_1, x_2, \ldots, x_{n-1}, 1) \in S^n$ and apply the definition. Thus, $\theta_1^s \subseteq \theta_2^s \subseteq \theta_3^s \subseteq \cdots$

(ii) Show that $\theta_n^s = 1_S$ for all n, if S is a monoid.

If S is a monoid, it has an identity element $e \in S$. Consider $a\,\theta_n^s\,b$, then in particular relation of monoid is $x = (e, e, \ldots, e) \in S^n$. then $xa = ea = a$, $xb = eb = b \Rightarrow a = b$. So θ_n^s is the identity relation $\theta_n^s = 1_S = \{(a, a) \mid a \in S\}$.

(iii) For $n = 1, 2, \ldots$, denote $\frac{S}{\theta_n^s}$ by S_n. Show that for all $(n \geq 2)$ then $S_n \cong S_{n-1}/\theta_1^{S_{n-1}}$.

By the definition of $S_n \cong S/\theta_1^s$ and $S_{n-1} = S/\theta_{n-1}^s$. Consider $\pi_{n-1} : S \rightarrow S_{n-1}$ be the canonical projection, now define $\theta_1^{S_{n-1}}$ as: $\pi_{n-1}(a)\,\theta_1^{S_{n-1}}\,\pi_{n-1}(b) \Leftrightarrow \forall x \in S_{n-1},\ x\pi_{n-1}(a) = x\pi_{n-1}(b)$. But since multiplication in S_{n-1} corresponds to multiplication in S modulo θ_{n-1}^s, the condition reduces to $\forall x \in S^n,\ xa = xb \Rightarrow a\,\theta_n^s\,b$ thus $S_n = S/\theta_n^s \cong S_{n-1}/\theta_1^{S_{n-1}}$.

(iv) Assuming that $S = \langle a \rangle = M(m, r)$ is a finite monogenic semi-group, where $m > 1$. Show that $\frac{S}{\theta_1^S} \cong M(m-1, r)$ and deduce that $\frac{S}{\theta_1^S} \cong M(m-n, r)$ for all $n < m$. Show also that $\frac{S}{\theta_n^S}$ is isomorphic to the fuzzy cyclic group of order r for all $n \geq m$.

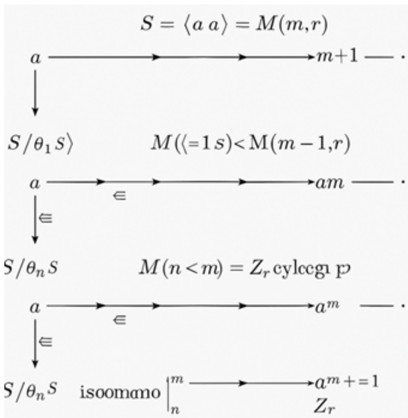


Figure 2: The above figure represents $S = \langle a \rangle = M(m, r)$ is a finite monogenic semi-group, where $m > 1$. Show that $\frac{S}{\theta_1^S} \cong M(m-1, r)$ and deduce that $\frac{S}{\theta_1^S} \cong M(m-n, r)$ for all $n < m$. Show also that $\frac{S}{\theta_n^S}$ is isomorphic to the fuzzy cyclic group of order r for all $n \geq m$.

The structure of $M(m, r)$: $S = \{a, a^2, \ldots, a^{m+r-1}\}$. with $a^m = a^{m+r}$, so the powers stabilize after m index $x = m$, period r.Next we show that $\frac{S}{\theta_1^S} \cong M(m-1, r)$ since in θ_1^S, $a\,\theta_1^S\,b$ if $xa = xb$ for all $x \in S$. Consider $a^i\,\theta_1^S\,a^j \iff xa^i = xa^j$ for all $x \in S$. But due to monogenic structure, cancellation and multiplication only depend on exponents. So, all elements from a^{m-1} onward act the same when multiplied $\Rightarrow$ collapse and the number of equivalence classes $= m - 1 + r$ elements in $M(m-1, r)$. Thus, $\frac{S}{\theta_1^S} \cong M(m-1, r)$.

We take deduction of $\frac{S}{\theta_1^S} \cong M(m-n, r)$ for all $n < m$ to apply (iii) recursively. Applications of fuzzy ideals: $S_1 = \frac{S}{\theta_1^S} \cong M(m-1, r)$, $S_2 = \frac{S}{\theta_2^S} \cong M(m-2, r)$, $\ldots$, $S_n \cong M(m-n, r)$. To show that $\frac{S}{\theta_n^S} \cong$ fuzzy cyclic group of order r for $n \geq m$ all elements a^k for $k \geq m$ behave identically under multiplication by any $x \in S^n$, because the powers have stabilized due to periodicity. Hence, beyond m, all non-transient elements fall into one cycle of length r. So, $\frac{S}{\theta_n^S}$ collapses to a structure isomorphic to a cyclic group of order r.

□

Lemma 2.3 *Consider $\rho_{m,r}$ $(m, r \geq 1)$ be the congruence $\{(a^m, a^{m+r})\}^\#$ on the free monogenic fuzzy semi-group a^+. (Thus, a^+/ρ is the monogenic semi-group $M(m, r)$).*

(i) Show that $(a^p, a^q) \in \rho$ iff $p, q \geq m$ and $p \equiv q \pmod{r}$.

(ii) Show that, for all $m, n, r, s \geq 1$, $\rho_{m,r} \subseteq \rho_{n,s}$ iff $m \geq n$ and s divides r.

(iii) Deduce that, for all $m, n, r, s \geq 1$, $\rho_{m,r} \cap \rho_{n,s} = \rho_{\max(m,n),\,\mathrm{lcm}(r,s)}$,

$$\rho_{m,r} \vee \rho_{n,s} = \rho_{\min(m,n),\,\mathrm{hcf}(r,s)}.$$

Proof: Consider $a^+ = \{a^1, a^2, \ldots\}$ be the monogenic semi-group generated by a. From the definition of congruence $\rho_{m,r} := \{(a^m, a^{m+r})\}^\#$, where $\#$ is the operation of least congruence on a^+ containing the pair (a^m, a^{m+r}). Then $\frac{a^+}{\rho(m,r)} \cong M(m, r)$, the monogenic fuzzy semi-group of index m and period r.

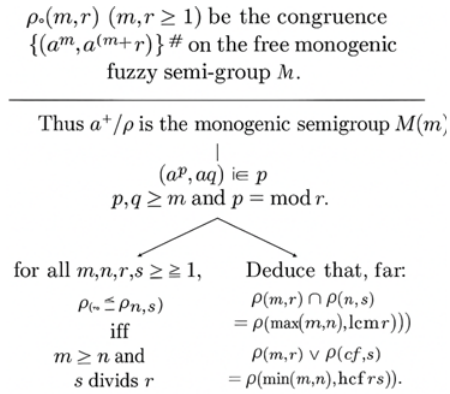


Figure 3: The above figure represents $\rho_{m,r}$ $(m, r \geq 1)$ be the congruence $\{(a^m, a^{m+r})\}^{\#}$ on the free monogenic fuzzy semi-group a^+.

(i) To show that $(a^p, a^q) \in \rho$ iff $p, q \geq m$ and $p \equiv q \pmod{r}$.

The relation $\rho_{m,r}$ is generated by the identification $a^m \sim a^{m+r}$. This allows us to repeatedly "cycle forward" by steps of r starting at m, i.e., $a^m \sim a^{m+r} \sim a^{m+2r} \cdots$ Hence, in the quotient semigroup: $a^p \sim a^q \iff p \equiv q \pmod{r}$ and $p, q \geq m$. So, $(a^p, a^q) \in \rho$ iff $p, q \geq m$ and $p \equiv q \pmod{r}$.

(ii) To show that $m, n, r, s \geq, \rho_{m,r} \subseteq \rho_{n,s}$ iff $m \geq n$ and s divides r.

Suppose $\rho_{m,r} \subseteq \rho_{n,s}$ then $(a^p, a^q) \in \rho_{(m,r)} \Rightarrow (a^p, a^q) \in \rho_{(n,s)}$. Consider $a^m \sim a^{m+r} \in \rho_{(m,r)}$. For this to be in $\rho_{(n,s)}$, both $m, m+r \geq n$, so $m \geq n$, also $m + r \equiv m \pmod{r}$, so the "step size" r must be covered by congruence modulo s, which implies $(s \mid r) \leftarrow$ suppose $m \geq n$ and $(s \mid r)$ of any pair $(a^p, a^q) \in \rho_{(m,r)}$ satisfies: $p, q \geq m \geq n, p \equiv q \pmod{r}$. Since the monoid relation satisfies $(s \mid r)$, we have, $p \equiv q \pmod{s}$, then $(a^p, a^q) \in \rho_{(n,s)}$, thus to satisfies the relation of monoid.

(iii) Deduce of the $\rho_{m,r} \cap \rho_{n,s} = \rho_{\max(m,n),\, \mathrm{lcm}(r,s)}$, $\rho_{m,r} \vee \rho_{n,s} = \rho_{\min(m,n),\, \mathrm{hcf}(r,s)}$.

We have the relation $p, q \geq m$ and $p, q \geq n \Rightarrow p, q \geq \max(m, n)$ and $p \equiv q \pmod{r}$ and $p \equiv q \pmod{s} \Rightarrow p \equiv q \pmod{\mathrm{lcm}(r, s)}$. Therefore $\rho_{m,r} \cap \rho_{n,s} = \rho_{\max(m,n),\, \mathrm{lcm}(r,s)}$. Similarly $\rho_{m,r} \vee \rho_{n,s} = \rho_{\min(m,n),\, \mathrm{hcf}(r,s)}$. to join the least congruence containing both of $\rho_{m,r}$ and $\rho_{n,s}$ and smallest index for which both relations are applicable of $\min(m, n)$ largest of modulus such that both congruence's are preserved is, $\mathrm{hcf}(r, s)$.

□

3. Conclusion

This research paper to find the $a\rho = \{b \in B \mid (a, b) \in \rho\}$. If S and T are two fuzzy semi-groups, then a subset μ of $S \times T$ is known as *relational morphism* from S to T if the conditions are satisfied by the relations are follows: (RM1) $(\forall a \in S)\ a\mu \neq \emptyset$; (RM2) $(\forall a, b \in S)\ (a\mu)(b\mu) \subseteq (ab)\mu$ and to showing more relations applications of fuzzy automation semi-groups i.e., $\theta_n^S = 1_S$ for all n if S is a monoid, $\frac{S}{\theta_1^S} \cong M(m-1, r)$, and deduce that $\frac{S}{\theta_1^S} \cong M(m-n, r)$ for all $n < m$. Show also that $\frac{S}{\theta_n^S}$ is isomorphic to the fuzzy cyclic group of order r for all $n \geq m$ etc.

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