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On Analytic Functions Subclass Defined by Differential Operator

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ABSTRACT: In this work, we present a novel differential operator-defined a specific set of negative-coefficient analytic univalent functions. For this class, we get subordination results, integral means inequalities, extreme points, and coefficient inequalities.

Key Words: Univalent, coefficient estimates, subordination, differential operator.

Contents

1. Introduction

The functions of type $\ddot{A}$ of class $\ddot{u}(ß)$ are indicated in the sections that follow.

$$\ddot{u}(ß) = ß + \sum_{\iota=2}^{\infty} \ddot{a}_\iota ß^\iota \tag{1.1}$$

which are in the open unit disc, are analytical $\ddot{E} = \{ß \in \mathbb{C} : |ß| < 1\}$.

If a function $\ddot{u}(ß)$ in the class $\ddot{A}$ satisfies the inequality, it is said to be in the class $\ddot{S}\ddot{T}(\ell)$ of starlike functions of order ℓ in $\ddot{E}$,

$$Re \left\{ \frac{ß\ddot{u}'(ß)}{\ddot{u}(ß)} > \ell, \right\} \quad (0 \leq \ell < 1) \quad (ß \in \ddot{E}). \tag{1.2}$$

Keep in mind that the class of starlike function is $\ddot{S}\ddot{T}(0) = \ddot{S}\ddot{T}$.
Indicate by $\ddot{T}$ the subclass of $\ddot{A}$ made up of the function $\ddot{u}$ with the following form.

$$\ddot{u}(ß) = ß - \sum_{\iota=2}^{\infty} \ddot{a}_\iota ß^\iota \quad (\ddot{a}_\iota \geq 0). \tag{1.3}$$

Silverman introduced and researched this subclass in great detail [5].
Let $\ddot{A}$ be one of the class $\ddot{u}$'s function. The differential operator proposed by Deniz and Ozkan [1] is defined here,

$$\begin{aligned} D^0_\lambda \ddot{u}(ß) &= \ddot{u}(ß) \\ D^1_\lambda \ddot{u}(ß) &= D_\lambda \ddot{u}(ß) = \lambda ß^3 (\ddot{u}(ß))''' + (2\lambda+1)ß^2(\ddot{u}(ß))'' + ß\ddot{u}'(ß) \\ D^3_\lambda \ddot{u}(ß) &= D_\lambda(D^1_\lambda \ddot{u}(ß)) \\ D^m_\lambda &= D_\lambda(D^{m-1}_\lambda \ddot{u}(ß)) \end{aligned} \tag{1.4}$$

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in which $m \in \ddot{N}_0 =: \ddot{N} \cup \{0\}$ and $\lambda \geq 0$. In the occurrence when $\ddot{u}$ provides by (1.1), it can be shown from the operator $D_\lambda^m \ddot{u}(ß)$ it is to see that

$$D_\lambda^m \ddot{u}(ß) = ß + \sum_{\iota=2}^{\infty} \Phi^m(\lambda, \iota)\ddot{a}_\iota ß^\iota) \tag{1.5}$$

where

$$\Phi^m(\lambda, \iota) = \iota^{2m}[\lambda(\iota - 1) + 1]^m. \tag{1.6}$$

If $\ddot{u} \in \ddot{T}$ is given by (1.2) then we have

$$D_\lambda^m \ddot{u}(ß) = ß - \sum_{\iota=2}^{\infty} \Phi^m(\lambda, \iota)\ddot{a}_\iota ß^\iota \tag{1.7}$$

where (1.5) yields $\Phi^m(\lambda, \iota)$.

Motivated by Murugusunderamoorthy and Magesh [4],we define the following new class in this using the operator $D_\lambda^m \ddot{u}(ß)$.
Let $\ddot{S}_\lambda^m$ be the subclass of $\ddot{A}$ made up of function of the form (1.1) for $\tau \geq 0, \quad 0 \leq \upsilon < 1$ and satisfy

$$Re\left(\frac{D_\lambda^m \ddot{u}(ß)}{ß}\right) \geq \tau \left|(D_\lambda^m \ddot{u}(ß))' - \frac{D_\lambda^m \ddot{u}(ß)}{ß}\right| + \upsilon \tag{1.8}$$

where $\ddot{S}_\lambda^m(\tau, \upsilon)$ is given by (1.6).
We also allowed $\ddot{T}\ddot{S}_\lambda^m(\tau, \upsilon) = \ddot{S}_\lambda^m(\tau, \upsilon) \cap \ddot{T}$.
In this paper, for the function in the class $\ddot{T}\ddot{S}_\lambda^m(\tau, \upsilon)$, In addition to coefficient, extreme point, and integral means inequalities, we obtain subordination result for the function $\ddot{u} \in \ddot{S}_\lambda^m(\tau, \upsilon)$ class. Moreover, the function's class subordination result.

2. Coefficient Estimates

Theorem 2.1 *The form (1.1) of a function $\ddot{u}(ß)$ be is in $\ddot{S}_\lambda^m(\tau, \upsilon)$ if*

$$\sum_{\iota=2}^{\infty}[1 + \tau(\iota - 1)]\Phi^m(\lambda, \iota)|\ddot{a}_\iota| \leq 1 - \upsilon \tag{2.1}$$

where $\tau \geq 0, \quad 0 \leq \upsilon < 1$, and $\Phi^m(\lambda, \iota)$ is given by(1.7).

Proof: It's enough to demonstrate that

$$\tau\left|(\mathcal{D}_\lambda^m \ddot{u}(ß))' - \frac{\mathcal{D}_\lambda^m \ddot{u}(ß)}{ß}\right| - Re\left\{\frac{\mathcal{D}_\lambda^m \ddot{u}(ß)}{ß} - 1\right\} \leq 1-\upsilon$$

we have

$$\begin{aligned}
&\tau\left|(\mathcal{D}_\lambda^m \ddot{u}(ß))' - \frac{\mathcal{D}_\lambda^m \ddot{u}(ß)}{ß}\right| - Re\left\{\frac{\mathcal{D}_\lambda^m \ddot{u}(ß)}{ß} - 1\right\} \qquad (2.2)\\
&\leq \tau\left|(\mathcal{D}_\lambda^m \ddot{u}(ß))' - \frac{\mathcal{D}_\lambda^m \ddot{u}(ß)}{ß}\right| - Re\left\{\frac{\mathcal{D}_\lambda^m \ddot{u}(ß)}{ß} - 1\right\}\\
&\leq \tau\left|\frac{\sum_{\iota=2}^{\infty}(\iota-1)\Phi^m(\lambda,\iota)\ddot{a}_\iota ß^\iota}{ß}\right| + \left|\frac{\sum_{\iota=2}^{\infty} C_{\mu,b}^{m,n}\Phi^m(\lambda,\iota)\ddot{a}_\iota ß^\iota}{ß}\right|\\
&\leq \tau\sum_{\iota=2}^{\infty}(\iota-1)\Phi^m(\lambda,\iota)|\ddot{a}_\iota| + \sum_{\iota=2}^{\infty}\Phi^m(\lambda,\iota)|\ddot{a}_\iota|\\
&= \sum_{\iota=2}^{\infty}[1+\tau(\iota-1)]\Phi^m(\lambda,\iota)|\ddot{a}_\iota|.
\end{aligned}$$

The final expression has $(1-\upsilon)$ as its upper bound if

$$\sum_{\iota=2}^{\infty}[1+\tau(\iota-1)]\Phi^m(\lambda,\iota)|\ddot{a}_\iota| \leq 1-\upsilon$$

and the theorem has been fully proved. □

We derive sufficient and necessary conditions for functions in $\ddot{T}\ddot{S}_\lambda^m(\tau,\upsilon)$ as in the subsequent theorem.

Theorem 2.2 *For $\tau \geq 0$, $0 \leq \upsilon < 1$, a function ü the structure (1.2) is assigned to the class $\ddot{T}\ddot{S}_\lambda^m(\tau,\upsilon)$ within the occurrence when*

$$\sum_{\iota=2}^{\infty}[1+\tau(\iota-1)]\Phi^m(\lambda,\iota)|\ddot{a}_\iota| \leq 1-\upsilon).$$

Proof: Assume that $\ddot{u}(ß)$ in the following way (1.2) belongs inside the class $\ddot{T}\ddot{S}_\lambda^m(\tau,\upsilon)$ then.

$$Re\left\{\frac{\mathcal{D}_\lambda^m \ddot{u}(ß)}{ß}\right\} - \tau\left|(\mathcal{D}_\lambda^m \ddot{u}(ß))' - \frac{\mathcal{D}_\lambda^m \ddot{u}(ß)}{ß}\right| \geq \upsilon.$$

Equivalently

$$Re\left[1-\sum_{\iota=2}^{\infty}\Phi^m(\lambda,\iota)|\ddot{a}_\iota|ß^{\iota-1}\right] - \tau\left[\sum_{\iota=2}^{\infty}(\iota-1)\phi^m(\lambda,\iota)\ddot{a}_\iota ß^{\iota-1}\right] \geq \upsilon.$$

□

As $|ß| \to 1$, allowing ß to be actual numbers, we obtain

$$1-\sum_{\iota=2}^{\infty}\Phi^m(\lambda,\iota)|\ddot{a}_\iota| - \tau\sum_{\iota=2}^{\infty}(\iota-1)\Phi^m(\lambda,\iota)|\ddot{a}_\iota| \geq \upsilon$$

which implies

$$\sum_{\iota=2}^{\infty}[1+\tau(\iota-1)\Phi^m(\lambda,\iota)|\ddot{a}_\iota| \leq 1-\upsilon]$$

where $\tau \geq 0$, $0 \leq \upsilon < 1$, $\Phi^m(\lambda,\iota)$ is given by (1.7) and the sufficiency originates with Theorem 2.1.

Corollary 2.1 *If $\ddot{u} \in \ddot{T}\ddot{S}^m_\lambda(\tau, \upsilon)$ then*

$$|\ddot{a}_\iota| \leq \frac{(1-\upsilon)}{[1+\tau(\iota-1)]\Phi^m(\lambda,\iota)}.$$

This holds true for the function

$$\ddot{u}(ß) = ß - \frac{(1-\upsilon)}{[1+\tau(\iota-1)]\Phi^m(\lambda,\iota)}ß^\iota$$

$\tau \geq 0, \quad 0 \leq \upsilon < 1, \quad \Phi^m(\lambda,\iota)$ *is given by* (1.7).

3. Extreme Points

Theorem 3.1 *Let $\ddot{u}_1(ß) = ß$ and $\ddot{u}_\iota(ß) = ß - \frac{(1-\upsilon)}{[1+\tau(\kappa-1)\Phi^m(\lambda,\iota)]}ß^\iota$, $\iota \geq 2$ for $\tau \geq 0$, $0 \leq \upsilon < 1$ and $\Phi^m(\lambda,\iota)$ is given by* (1.7)*. Subsequently, $\ddot{u}(\iota)$ belong to the class $\ddot{T}\ddot{S}^m_\lambda(\tau,\upsilon)$ if and only if it can be represented as $\ddot{u}(ß) = \sum_{\iota=1}^{\infty} \lambda_\iota \ddot{u}_\iota(ß)$, where $\lambda_\iota \geq 0$ and $\sum_{\iota=1}^{\infty} \lambda_\iota = 1$ are located.*

Proof: If $\ddot{u}(ß) = \sum_{\iota=1}^{\infty} \lambda_\iota \ddot{u}_\iota(ß)$ with $\lambda_\iota \geq 0$ and $\sum_{\iota=1}^{\infty} \lambda_\iota = 1$. Then

$$\begin{aligned}
\ddot{u}(ß) &= \sum_{\iota=1}^{\infty} \lambda_\iota \ddot{u}_\iota(ß) = \lambda_1 \ddot{u}_1(ß) + \sum_{\iota=1}^{\infty} \lambda_\iota \ddot{u}_\iota(ß) \\
&= \left(1 - \sum_{\iota=2}^{\infty} \lambda_\iota\right) ß + \sum_{\iota=2}^{\infty} \left[\lambda_\iota (ß - \frac{(1-\upsilon)}{[1+\tau(\iota-1)]\Phi^m(\lambda,\iota)})\right] \\
&= ß - \sum_{\iota=2}^{\infty} \frac{(1-\upsilon)}{[1+\tau(\iota-1)]\Phi^m(\lambda,\iota)} ß^\iota.
\end{aligned}$$

$$\begin{aligned}
\text{Now } & \sum_{\iota=2}^{\infty} \frac{[1+\tau(\iota-1)]\Phi^m(\lambda,\iota)}{1-\upsilon} \frac{1-\upsilon}{[1+\tau(\iota-1)]\Phi^m(\lambda,\iota)} \lambda_\iota \\
&= \sum_{\iota=2}^{\infty} \lambda_\iota = 1 - \lambda_1 \leq 1. \\
\text{Then } & \ \ddot{u}(ß) \in \ddot{T}\ddot{S}^m_\lambda(\tau,\upsilon).
\end{aligned}$$

Conversely, suppose $\ddot{u}(ß) \in \ddot{T}\ddot{S}^m_\lambda(\tau,\upsilon)$. Then corollary 2.1 gives

$$\begin{aligned}
\ddot{a}_\iota &\leq \frac{(1-\upsilon)}{[1+\tau(\iota-1)\Phi^m(\lambda,\iota)]}, \ \iota \geq 2 \\
\text{set} \quad \lambda_n &= \frac{[1+\tau(\iota-1)\Phi^m(\lambda,\iota)]}{(1-\upsilon)} \ddot{a}_\iota, \quad \iota \geq 2,
\end{aligned}$$

where $\lambda_\iota = 1 - \sum_{\iota=2}^{\infty} \lambda_\iota$ then

$$\begin{aligned} \ddot{u}(ß) = ß - \sum_{\iota=2}^{\infty} \ddot{a}_\iota ß^\iota &= ß - \sum_{\iota=2}^{\infty} \lambda_\iota \frac{(1-\upsilon)}{[1+\tau(\iota-1)\Phi^m(\lambda,\iota)]} \\ &= ß - \sum_{\iota=2}^{\infty} \lambda_\iota ß + \sum_{\iota=2}^{\infty} \lambda_\iota \ddot{u}_\iota(ß) \\ &= ß - \left[1 - \sum_{\iota=2}^{\infty} \lambda_\iota\right] + \sum_{\iota=2}^{\infty} \lambda_\iota \ddot{u}_\iota(ß) \\ &= \lambda_1 \ddot{u}_1(ß) \sum_{\iota=2}^{\infty} \lambda_\iota \ddot{u}_\iota(ß) = \sum_{\iota=2}^{\infty} \lambda_\iota \ddot{u}_\iota(ß). \end{aligned}$$

□

The theorem has been fully demonstrated.

4. Integral Means Inequalities

The notion of analytical functions being subordinated at once $\ddot{g}$ and $\ddot{h}$ having $\ddot{g}(0) = \ddot{h}(0)$, $\ddot{g}$ then $\ddot{g}$ is a subordinate of $\ddot{h}$, as shown through $\ddot{g} \prec \ddot{h}$, if a function known as analytical exists that $\omega(0) = 0$, $|\omega(ß)| < 1$ and $\ddot{g}(ß) = \ddot{h}(\omega(ß))$, for all $ß \in \ddot{E}$.

Lemma 4.1 *Assuming that $\ddot{g}(ß) \prec \ddot{h}(ß)$ and $\ddot{u}(ß)$ are both analytical functions contained in $\ddot{E}$, consequently*

$$\int_0^{2\pi} |g(r\theta^{i\theta})|^p \, d\theta \leq \int_0^{2\pi} |\ddot{u}(r\theta^{i\theta})|^p \, d\theta, \quad (0 \leq r < 1, \quad p > 0).$$

Theorem 4.1 *Suppose $\ddot{u}(ß) \in \ddot{T}\ddot{S}_\lambda^m(\tau,\upsilon)$, $p > 0$, $\tau \geq 0$, $0 \leq \upsilon < 1$ and $\ddot{u}_2(ß)$ is defined by*

$$\ddot{u}_2(ß) = ß - \frac{(1-\upsilon)}{(1+\tau)\Phi^m(\lambda,\iota)}.$$

Consequently, for $ß = re^{i\theta}$, $0 \leq r < 1$, we find

$$\int_0^{2\pi} |\ddot{u}(ß)|^p \, d\theta \leq \int_0^{2\pi} |\ddot{u}_2(ß)|^p \, d\theta. \tag{4.1}$$

Proof: For $\ddot{u}(ß) = ß - \sum_{\iota=2}^{\infty} |\ddot{a}_n| ß^n$, (4.1) is similar to demonstrating that

$$\int_0^{2\pi} \left| ß - \sum_{n=2}^{\infty} |\ddot{a}_\iota| ß^\iota \right|^p d\theta \leq \int_0^{2\pi} \left| ß - \frac{(1-\upsilon)}{(1+\tau)\Phi^m(\lambda,\iota)} ß^2 \right|^p d\theta, \ (p > 0).$$

Using the subordination theorem of Little Wood (Lemma 4.1,) It would be adequate to demonstrate that

$$1 - \sum_{\iota=2}^{\infty} |\ddot{a}_\iota| ß^{\iota-1} \prec 1 - \frac{(1-\upsilon)}{(1+\tau)\Phi^m(\lambda,\iota)} ß. \tag{4.2}$$

Setting

$$1 - \sum_{\iota=2}^{\infty} |\ddot{a}_\iota| ß^{\iota-1} \prec 1 - \frac{(1-\upsilon)}{(1+\tau)\Phi^m(\lambda,\iota)} w(ß)$$

we obtain

$$w(ß) = \frac{(1+\tau)\Phi^m(\lambda,\iota)}{(1-\upsilon)} \sum_{\iota=2}^{\infty} \ddot{a}_\iota ß^{\iota-1}.$$

Moreover, with $w(0) = 0$ is analytic in $\ddot{E}$. Additionally, it is sufficient to demonstrate that $w(ß)$ satisfies $|w(ß)| < 1,\ ß \in \ddot{E}$. Now

$$|w(ß)| = \left| \sum_{\iota=2}^{\infty} \frac{(1+\tau)\Phi^m(\lambda,\iota)}{(1-\upsilon)} \ddot{a}_\iota ß^{\iota-1} \right|$$

$$\leq |(ß)| \sum_{\iota=2}^{\infty} \frac{(1+\tau)\Phi^m(\lambda,\iota)}{(1-\upsilon)} |\ddot{a}_\iota| \leq |ß| < 1. \tag{4.3}$$

As a result, the subordination (4.2) derives from the inequality (4.3), proving the theorem. □

5. Subordination Results

Definition 5.1 A complex number sequence $\{\ddot{b}_\iota\}_{\iota=1}^{\infty}$ is perceived as subordinate (Subordination factor sequence) if, for every $\ddot{u}(ß) = \sum_{\iota=1}^{\infty} \ddot{a}_\iota ß^\kappa, \ddot{a}^1 = 1$ is convex, regular, and univalent in $\ddot{E}$. It concludes as

$$\sum_{\kappa=1}^{\infty} \ddot{b}_\iota \ddot{a}_\iota ß^\iota \prec \ddot{u}(ß), ß \in \ddot{E}.$$

Lemma 5.1 *Only in the case that Subordinating factor sequence $\{\ddot{b}_\iota\}_{\iota=1}^{\infty}$ corresponds to this*

$$Re\left\{1 + 2\sum_{\iota=1}^{\infty} \ddot{b}_\iota ß^\iota\right\} > 0,\ ß \in \ddot{E}.$$

Theorem 5.1 *Considering the function $\ddot{u} \in \ddot{S}_\lambda^m(\tau,\upsilon)$ and $\ddot{g}(ß)$ in the classification of convex functions C, then*

$$\frac{(1+\tau)\Phi_m(\lambda,2)}{2[(1-\upsilon)+(1+\alpha)\Phi^m(\lambda,2)]}(\ddot{u} * \ddot{g})(ß) \prec \ddot{g}(ß) \tag{5.1}$$

where $\tau \geq 0, \quad 0 \leq \upsilon < 1$, with $\Phi^m(\lambda,2)$ is given by (1.7)

$$Re\{\ddot{u}(ß)\} > -\frac{(1-\upsilon)+(1+\tau)\Phi^m(\lambda,2)}{(1+\alpha)\Phi^m(\lambda,2)};\ ß \in \ddot{E}. \tag{5.2}$$

The constant $\frac{(1+\tau)c\Phi^m(\lambda,2)}{2[(1-\upsilon)+(1+\tau)\Phi^m(\lambda,2)]}$ is the best estimate.

Proof: Let $\ddot{u} \in \ddot{S}_\lambda^m(\tau,\upsilon)$ and suppose that $\ddot{g}(ß) = ß + \sum_{\iota=2}^{\infty} \ddot{c}_\iota ß^\iota \in C$, then

$$\frac{(1+\tau)\Phi^m(\lambda,2)}{2[(1-\upsilon)+(1+\alpha)\Phi^m(\lambda,2)]}(\ddot{u} * \ddot{g})(ß) = \frac{(1+\tau)\Phi^m(\lambda,2)}{2[(1-\upsilon)+(1+\tau)\Phi^m(\lambda,2)]}(ß + \sum_{\iota=2}^{\infty} \ddot{c}_\iota \ddot{a}_\iota ß^\iota).$$

According to definition $\ddot{a}_1 = 1$, the subordination outcome is valid if

$$\left\{\frac{(1+\tau)\Phi^m(\lambda,\iota)}{2[(1-\upsilon)+(1+\tau)\Phi^m(\lambda,\iota)]}\right\}_{\iota=1}^{\infty}$$

is an ordered sequence of subordinating factors where, in the context of $\ddot{a}_1 = 1$. □

Lemma 5.2 *This can be translated into the inequality that follows*

$$Re\left\{1+\sum_{\iota=1}^{\infty}\frac{(\tau+1)\Phi^{m}(\lambda,2)}{(1-\upsilon)+(1+\tau)\Phi^{m}(\lambda,2)}\ddot{a}_{\iota}ß^{\iota}\right\}>0,\quad ß\in\ddot{E}. \tag{5.3}$$

We now have $|ß|=r<1$, *for which*

$$\begin{aligned}
&Re\left\{1+\sum_{\iota=1}^{\infty}\frac{(\tau+1)\Phi^{m}(\lambda,2)}{(1-\upsilon)+(\tau+1)\Phi^{m}(\lambda,2)}\ddot{a}_{\iota}ß^{\iota}\right\}\\
&=Re\left\{1+\sum_{n=1}^{\infty}\frac{(\tau+1)\Phi^{m}(\lambda,2)}{(1-\upsilon)+(\tau+1)\Phi^{m}(\lambda,2)}ß+\frac{\sum_{\iota=1}^{\infty}(\tau+1)\Phi^{m}(\lambda,2)\ddot{a}_{\iota}ß^{\iota}}{(1-\upsilon)+(\tau+1)\Phi^{m}(\lambda,2)}\right\}\\
&\geq 1-\sum_{n=1}^{\infty}\frac{(\tau+1)\Phi^{m}(\lambda,2)}{(1-\upsilon)+(\tau+1)\Phi^{m}(\lambda,2)}r-\frac{\sum_{\iota=1}^{\infty}(\tau+1)\Phi^{m}(\lambda,2)\ddot{a}_{\iota}r^{\iota}}{(1-\upsilon)+(\tau+1)\Phi^{m}(\lambda,2)}\\
&\geq 1-\sum_{n=1}^{\infty}\frac{(\tau+1)\Phi^{m}(\lambda,2)}{(1-\upsilon)+(\tau+1)\Phi^{m}(\lambda,2)}r-\frac{1-\upsilon}{(1-\upsilon)+(1+\tau)\Phi^{m}(\lambda,2)}r\\
&>0.
\end{aligned}$$

Given that $1+\tau(\iota-1)\Phi^{m}(\lambda,\iota)$ *is a function which grows as* $\iota\geq 2$ *and based on* (2.1). *This establishes the subordination result* (5.1), *which is also supported by the inequality* (5.3), *as stated by Theorem 5.1. From* (5.1), *the inequality* (5.2) *is obtained by taking*

$$\ddot{g}(ß)=\frac{ß}{1-ß}=2+\sum_{\iota=2}^{\infty}ß^{\iota}\in C.$$

Now we consider the function $\ddot{u}(ß)=ß-\frac{(1-\upsilon)}{(1+\tau)\Phi^{m}(\lambda,2)}ß^{2}$ *where* $\tau\geq 0,\ 0\leq\upsilon<1$ *and clearly* $\ddot{U}\in\ddot{S}_{\lambda}^{m}(\tau,\upsilon)$. *For this function* (5.1) *becomes*

$$\frac{(1+)\Phi^{m}(\lambda,2)}{2[(1-\upsilon)+(1+\tau)\Phi^{m}(\lambda,2)]}\ddot{U}(ß)\prec\frac{ß}{1-ß}.$$

It is easily verified that
min $Re\left\{\frac{(1+)\Phi^{m}(\lambda,2)}{2[(1-\upsilon)+(1+\tau)\Phi^{m}(\lambda,2)]}\ddot{U}(ß)\right\}\frac{-1}{2},\quad ß\in\ddot{E}.$
This show that the constant $\frac{(1+\alpha)\Phi^{m}(\lambda,2)}{2[(1-\beta)+(1+\alpha)\Phi^{m}(\lambda,2)]}$ *is best possible.*

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Competing interests

The author declares no competing interests.

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