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Caputo Generalized Proportional Fractional Differential Equation: Analytical Approach and Stability Results

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ABSTRACT: This paper investigates the existence, uniqueness, and stability of solutions for a novel class of Caputo generalized proportional fractional differential equations involving two distinct fractional orders. We present key properties of the generalized proportional fractional derivative and establish our main results using Schaefer's fixed point theorem and the Banach contraction principle. Furthermore, we analyze Ulam-Hyers and generalized Ulam-Hyers stability for the proposed problem. To illustrate the applicability of our theoretical findings, we conclude with a numerical example demonstrating these results.

Keywords: Caputo generalized proportional fractional derivative, Schaefers's fixed point theorem, generalized Ulam-Hyers stability, fixed point theorem.

Contents

1. Introduction

Fractional differential equations (FDEs) have attracted significant interest in recent years due to their ability to model complex phenomena across various scientific disciplines. Their effectiveness has been demonstrated in physics, mechanics, biology, chemistry, control theory, and other fields , see [6, 13,5,2,1,3,9,12,14,20,11,15,4,8]. Among the different definitions of fractional integrals and derivatives, the Riemann-Liouville and Caputo operators are the most widely used. These derivatives have proven particularly valuable in modeling systems with long-term memory effects, addressing key challenges in science and engineering [25,26,27,28], for more details for Caputo fractional derivative, we direct readers to the papers [7,29,30]. In [16], Jarad et al. proposed a modification of the conformable derivatives [17,18], introducing a new category of fractional derivatives called the generalized proportional fractional (GPF) derivative. Later, Anderson et al. [19] addressed the limitation that the fractional conformable derivative does not approach the original function as the order ρ tends to zero, by defining a proportional derivative of order ρ as follows:

$$D_{\tau}^{\rho}\mathfrak{h}(\tau) = \xi_1(\rho,\tau)\mathfrak{h}(\tau) + \xi_2(\rho,\tau)\mathfrak{h}'(\tau),$$

where $\xi_1, \xi_2 : [0,1] \times \mathbb{R} \longrightarrow [0,\infty)$ are continuous functions such that, for all $\tau \in \mathbb{R}$,

$$\lim_{\rho\to 0^+} \xi_1(\rho,\tau) = 1, \lim_{\rho\to 0^+} \xi_2(\rho,\tau) = 0, \lim_{\rho\to 1^-} \xi_1(\rho,\tau) = 0, \lim_{\rho\to 1^-} \xi_2(\rho,\tau) = 1,$$

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and $\xi_1(\rho,\tau)\neq 0, \xi_2(\rho,\tau)\neq 0, \rho\in[0,1]$, with these modifications, the proposed proportional derivative approaches the original function as ρ tends to 0.
In recent years, there has been an increasing interest among researchers in exploring this fractional derivative. Building on these contributions, the present work aims to integrate these concepts in order to establish existence results.

$$\begin{cases}\mathfrak{D}^{\varpi,\chi}_{\gamma^+}w(\tau)=\mathfrak{h}(\tau,w(\tau)), \tau\in\Lambda:=[\gamma,\delta],\\ w(\gamma)=0, w(\delta)=\sum_{j=1}^{m}\nu_j\mathfrak{I}^{\alpha_j,\chi}_{\gamma^+}w\left(\varrho_j\right)+\sum_{i=1}^{n}\iota_i\mathfrak{I}^{\beta_i,\chi}_{\gamma^+}w\left(\kappa_i\right).\end{cases} \tag{1}$$

Where ${}^{\mathfrak{C}}\mathfrak{D}^{\varpi,\chi}_{\gamma^+}$ is the Caputo generalized proportional fractional derivative of order ϖ, $0<\varpi\leq 1$, respectively. $\mathfrak{I}^{\alpha_j,\chi}_{\gamma^+}$ and $\mathfrak{I}^{\beta_i,\chi}_{\gamma^+}$ are the generalized proportional fractional integral of order α_j, $\beta_i>0$, $\chi\in(0,1]$, $\gamma\geq 0$, $\nu_j,\iota_i\in\mathbb{R}$, $j=1,...,m$, $i=1,...,n$, $\gamma<\varrho_1<...<\varrho_n<\delta$, $\gamma<\kappa_1<...<\kappa_n<\delta$ and $\mathfrak{h}\in\mathcal{C}(\Lambda\times\mathbb{R},\mathbb{R})$.

The paper is structured as follows: Section 2 presents the necessary preliminaries, including notations, definitions, and auxiliary lemmas from fractional calculus relevant to our analysis. In Section 3, we establish existence results for problem (1) using Schaefer's fixed-point theorem, while the uniqueness result is derived through Banach's contraction principle. Section 4 is devoted to studying Ulam–Hyers stability and its generalized form for the considered problem. Finally, a numerical scheme is introduced to demonstrate the applicability of the obtained theoretical results.

2. Preliminaries

Definition 2.1 [19] For $\chi\in(0,1]$. Let $\xi_1,\xi_2:[0,1]\times\mathbb{R}\longrightarrow[0,\infty)$ be continuous functions such that, for all $\tau\in\mathbb{R}$,

$$\lim_{\chi\to 0^+}\xi_1(\chi,\tau)=1,\ \lim_{\chi\to 0^+}\xi_2(\chi,\tau)=0,\ \lim_{\chi\to 1^-}\xi_1(\chi,\tau)=0,\ \lim_{\chi\to 1^-}\xi_2(\chi,\tau)=1,$$

and $\xi_1(\chi,\tau)\neq 0, \xi_2(\chi,\tau)\neq 0, \chi\in[0,1]$, then the proportional derivative of order χ of $\mathfrak{h}$ is defined by

$$D^{\chi}\mathfrak{h}(\tau)=\xi_1(\chi,\tau)\mathfrak{h}(\tau)+\xi_2(\chi,\tau)\mathfrak{h}'(\tau), \tag{2}$$

By setting $\xi_1(\chi,\tau)=\chi-1$ and $\xi_2(\chi,\tau)=\chi$, (2) becomes

$$D^{\chi}\mathfrak{h}(\tau)=(1-\chi)\mathfrak{h}(\tau)+\chi\mathfrak{h}'(\tau) \tag{3}$$

Lemma 2.1 *[16] For $\chi\in(0,1]$, $\sigma,\varpi\in\mathbb{C}$ with $\mathcal{R}e(\sigma)>0$ and $\mathcal{R}e(\varpi)>0$. If $\mathfrak{h}\in C([\gamma,\delta],\mathbb{R})$ then we have*

$$\mathfrak{I}^{\sigma,\chi}_{\gamma^+}\mathfrak{I}^{\varpi,\chi}_{\gamma^+}\mathfrak{h}(\tau)=\mathfrak{I}^{\sigma+\varpi,\chi}_{\gamma^+}\mathfrak{h}(\tau), \tau>\gamma. \tag{2.1}$$

Lemma 2.2 *[16] Let $\chi\in(0,1]$, $n\in\mathbb{N}^+$, $\mathfrak{h}\in L^1([\gamma,\delta],\mathbb{R})$ and $\mathfrak{I}^{\sigma,\chi}_{\gamma^+}\mathfrak{h}\in AC^n([\gamma,\delta],\mathbb{R})$. Then*

$$\mathfrak{I}^{\sigma,\chi}_{\gamma^+}({}^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi}_{\gamma^+}\mathfrak{h})(\tau)=\mathfrak{h}(\tau)-e^{\frac{\chi-1}{\chi}(\tau-\gamma)}\sum_{k=1}^{n}\left(\mathfrak{I}^{k-\sigma,\chi}_{\gamma^+}\mathfrak{h}\right)(\gamma)\frac{(\tau-\gamma)^{\sigma-k}}{\chi^{\sigma-k}\Gamma(\sigma-k+1)}. \tag{2.2}$$

Lemma 2.3 *[16] Let $\sigma,\varpi\in\mathbb{C}$ with $\mathcal{R}e(\sigma)>0$ and $\mathcal{R}e(\upsilon)>0$. Then for each $\chi\in(0,1]$ and $n=[\mathcal{R}e(\sigma)]+1$, we have*

$$(i)\left(\mathfrak{I}^{\sigma,\chi}_{\gamma^+}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}(\tau-\gamma)^{\upsilon-1}\right)(\tau)=\frac{\Gamma(\upsilon)}{\chi^{\sigma}\Gamma(\upsilon+\sigma)}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}(\tau-\gamma)^{\upsilon+\sigma-1},\quad \mathcal{R}e(\sigma)>0.$$

$$(ii)\left({}^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi}_{\gamma^+}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}(\tau-\gamma)^{\upsilon-1}\right)(\tau)=\frac{\chi^{\sigma}\Gamma(\upsilon)}{\Gamma(\upsilon-\sigma)}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}(\tau-\gamma)^{\upsilon-\sigma-1},\quad \mathcal{R}e(\sigma)>n.$$

$$(iii)\left({}^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi}_{\gamma^+}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}(\tau-\gamma)^{k}\right)(\tau)=0,\quad \mathcal{R}e(\sigma)>n,\ k=0,1\ldots,n-1.$$

Theorem 2.1 *(Schaefers's fixed point theorem) [21,22]*
Let $\mathcal{X}$ be a Banach space and $\mathcal{K}:\mathcal{X}\longrightarrow\mathcal{X}$, be a completely continuous operator. If the set $\Upsilon_\varepsilon=\{w\in\mathcal{X}\mid w=\varepsilon\mathcal{K}w; 0\le\varepsilon\le 1\}$ is bounded, then $\mathcal{K}$ has at least a fixed point in $\mathcal{X}$.

Theorem 2.2 *(Banach's fixed point theorem) [23]*
Let $\mathcal{X}$ be a Banach space, C a closed subset of $\mathcal{X}$. Then any contraction mapping $\mathcal{K}$ from C into itself has a unique fixed point.

3. Main Results

Lemma 3.1 *Let $\gamma\ge 0, 0<\varpi\le 1$, and $\mathfrak{f}\in C(\Lambda,\mathbb{R})$. Then the function w is a solution of the following boundary value problem:*

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{\varpi,\chi}_{\gamma^+}w(\tau)=\mathfrak{f}(\tau), \quad \tau\in\Lambda:=[\gamma,\delta],\\ w(\gamma)=0\,,\;\; w(\delta)=\sum_{j=1}^{m}\nu_j\mathfrak{I}^{\alpha_j,\chi}_{\gamma^+}w(\varrho_j)+\sum_{i=1}^{n}\iota_i\mathfrak{I}^{\beta_i,\chi}_{\gamma^+}w(\kappa_i),\; \gamma<\varrho_j,\kappa_i<\delta, \end{cases} \tag{3.1}$$

if and only if

$$w(\tau)=\mathfrak{I}^{\varpi,\chi}_{\gamma^+}\mathfrak{f}(\tau)+\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \tag{3.2}$$

$$\times\left[\mathfrak{I}^{\varpi,\chi}_{\gamma^+}\mathfrak{f}(\delta)-\sum_{i=1}^{n}\iota_i\mathfrak{I}^{\varpi+\beta_i,\chi'}_{\gamma^+}\mathfrak{f}(\kappa_i)-\sum_{j=1}^{m}\nu_j\mathfrak{I}^{\varpi+\alpha_j,\chi'}_{\gamma^+}\mathfrak{f}(\varrho_j)\right], \tag{3.3}$$

$$=\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_\gamma^\tau e^{\frac{\chi-1}{\chi}(\tau-s)}(\tau-s)^{\varpi-1}\mathfrak{f}(s)ds+\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \tag{3.4}$$

$$\times\left[\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_\gamma^\delta e^{\frac{\chi-1}{\chi}(\delta-s)}(\delta-s)^{\varpi-1}\mathfrak{f}(s)ds\right. \tag{9}$$

$$-\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_\gamma^{\kappa_i}e^{\frac{\chi-1}{\chi}(\kappa_i-s)}(\kappa_i-s)^{\varpi+\beta_i-1}\mathfrak{f}(s)ds \tag{3.5}$$

$$\left.-\sum_{j=1}^{n}\nu_j\frac{1}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j)}\int_\gamma^{\varrho_j}e^{\frac{\chi-1}{\chi}(\varrho_j-s)}(\varrho_j-s)^{\varpi+\alpha_j-1}\mathfrak{f}(s)ds\right] \tag{3.6}$$

where

$$\Theta=\sum_{i=1}^{n}\iota_i\frac{(\kappa_i-\gamma)^{\varpi+\beta_i}\,e^{\frac{\chi-1}{\chi}(\kappa_i-\gamma)}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}+\sum_{j=1}^{n}\nu_j\frac{(\varrho_j-\gamma)^{\varpi+\alpha_j}\,e^{\frac{\chi-1}{\chi}(\varrho_j-\gamma)}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)} \tag{3.7}$$

$$-\frac{(\delta-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\delta-\gamma)}}{\chi^{\varpi}\Gamma(\varpi+1)}\neq 0. \tag{3.8}$$

Proof: Let w be the solution of problem (3.1). By applying the GPF integral of order σ, ϖ and Lemma 2.2 with Lemma 2.3, the first equation of problem (3.1) can be expressed as

$$w(\tau)=\mathfrak{I}^{\varpi,\chi}_{\gamma^+}\mathfrak{f}(\tau)+d_0\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\chi^{\varpi}\Gamma(\varpi+1)}+d_1e^{\frac{\chi-1}{\chi}(\tau-\gamma)}, \tag{3.9}$$

where d_0 and d_1 are constants. Next, by using the boundary condition $w(\gamma)=0$ in (3.9) we obtain $d_1=0$ then

$$w(\tau)=\mathfrak{I}^{\varpi,\chi}_{\gamma^+}\mathfrak{f}(\tau)+d_0\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\chi^{\varpi}\Gamma(\varpi+1)}, \tag{3.10}$$

next, by using the boundary condition $w(\delta) = \sum_{j=1}^{m} \nu_j \mathfrak{I}_{\gamma^+}^{\alpha_j,\chi} w(\varrho_j) + \sum_{i=1}^{n} \iota_i \mathfrak{I}_{\gamma^+}^{\beta_i,\chi} w(\kappa_i)$ in (3.10) we obtain

$$d_0 = \frac{1}{\Theta}\left[\mathfrak{I}_{\gamma^+}^{\varpi,\chi}\mathfrak{f}(\delta) - \sum_{i=1}^{n}\iota_i \mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi}\mathfrak{f}(\kappa_i) - \sum_{j=1}^{m}\nu_j \mathfrak{I}_{\gamma^+}^{\varpi+\alpha_j,\chi}\mathfrak{f}(\varrho_j)\right], \tag{3.11}$$

where Θ is given by (3.7). Substituting the value of d_0 in (3.10) we obtain

$$w(\tau) = \mathfrak{I}_{\gamma^+}^{\varpi,\chi}\mathfrak{f}(\tau) + \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \tag{3.12}$$

$$\times \left[\mathfrak{I}_{\gamma^+}^{\varpi,\chi}\mathfrak{f}(\delta) - \sum_{i=1}^{n}\iota_i \mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi}\mathfrak{f}(\kappa_i) - \sum_{j=1}^{m}\nu_j \mathfrak{I}_{\gamma^+}^{\varpi+\alpha_j,\chi}\mathfrak{f}(\varrho_j)\right]. \tag{3.13}$$

The converse follows by direct computation that the solution $w(\tau)$ given by (3.2) satisfies problem (3.1) under the given boundary conditions.

4. Existence and Uniqueness Results for Problem (1)

In this section, we establish the existence and uniqueness of the solution to problem (1).

Based on Lemma 3.1 we define the operator $\mathcal{K} : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$(\mathcal{K}w)(\tau) = \mathfrak{I}_{\gamma^+}^{\varpi,\chi}\mathfrak{h}(\tau, w(\tau)) + \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \tag{4.1}$$

$$\times \left[\mathfrak{I}_{\gamma^+}^{\varpi,\chi}\mathfrak{h}(\delta, w(\delta)) - \sum_{i=1}^{n}\iota_i \mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi^\chi}\mathfrak{h}(\kappa_i, w(\kappa_i)) - \sum_{j=1}^{m}\nu_j \mathfrak{I}_{\gamma^+}^{\varpi+\alpha_j,\chi^\chi}\mathfrak{h}(\varrho_j, w(\varrho_j))\right] \tag{4.2}$$

$$= \frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)}(\tau-s)^{\varpi-1}\mathfrak{h}(s,w(s))ds \tag{4.3}$$

$$+ \frac{(\tau-\gamma)^{\varpi}}{\Theta\chi^{\frac{\chi-1}{\chi}}(\tau-\gamma)}\left[\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\delta-s)}(\delta-s)^{\varpi-1}\mathfrak{h}(s,w(s))ds\right. \tag{15}$$

$$-\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\kappa_i-s)}(\kappa_i-s)^{\varpi+\beta_i-1}\mathfrak{h}(s,w(s))ds \tag{4.4}$$

$$\left.-\sum_{j=1}^{n}\nu_j\frac{1}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j)}\int_{\gamma}^{\varrho_j} e^{\frac{\chi-1}{\chi}(\varrho_j-s)}(\varrho_j-s)^{\varpi+\alpha_j-1}\mathfrak{h}(s,w(s))ds\right], \tag{4.5}$$

where $\mathcal{C} = C([\gamma,\delta],\mathbb{R})$ denotes the Banach space of all continuous functions from $[\gamma,\delta]$ into $\mathbb{R}$ with the norm $\|w\| := sup\{|w(\tau)|; \tau \in [\gamma,\delta]\}$.

To address the existence and uniqueness results for problem (1), we introduce the following notations to simplify the computations.

$$\mathfrak{A} = \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}. \tag{4.6}$$

$$\mathfrak{B} = \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} + \sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}\right. \tag{4.7}$$

$$\left.+\sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)}\right]. \tag{4.8}$$

The main results are established under the following assumptions.

(H_1): $|\mathfrak{h}(\tau, v) - \mathfrak{h}(\tau, w)| \leq \mathcal{L}|v - w|$; $\mathcal{L} > 0$, for each $\tau \in [\gamma, \delta]$ and $v, w \in \mathbb{R}$.

(H_2): there exist non-negatives continuous functions ψ_1 and ψ_2, such that $|\mathfrak{h}(\tau, w)| \leq \psi_1(\tau) + \psi_2(\tau)|w|$, $(\tau, w) \in [\gamma, \delta] \times \mathbb{R}$, with $\|\psi_1\| = sup_{\tau\in[\gamma,\delta]}|\psi_1(\tau)|$, $\|\psi_2\| = sup_{\tau\in[\gamma,\delta]}|\psi_2(\tau)|$.

(H_3): $\|\psi_2\|\big(\mathfrak{A} + \mathfrak{B}\big) < 1$, where $\mathfrak{A}$ and $\mathfrak{B}$ are given by (4.6) and (4.7).

Existence analysis through Schaefer's fixed-point theorem

Theorem 4.1 *Assume that (H_2) and (H_3) are satisfied. Then, there exists at least one solution for the problem (1) on $[\gamma, \delta]$.*

The proof of Theorem 4.1, is carried out by verifying that the operator $\mathcal{K}$ meets the conditions of Theorem 2.1 (Schaefer's fixed point theorem).

Proof: Consider the operator $\mathcal{K}$ defined in (4.1), we will show that $\mathcal{K}$ is a completely continuous operator.
Step 1: $\mathcal{K}$ is continuous.
By using the continuity of function $\mathfrak{h}$, it follows that $\mathcal{K}$ is continuous.
Step 2: $\mathcal{K}$ is bounded.
Let $\mathcal{N}$ a bounded set, such that $\mathcal{N} \subset \mathcal{B}_\rho$, we will show that $\mathcal{K}(\mathcal{N})$ is bounded for all $w \in \mathcal{N}$. For each $\tau \in \Lambda$ and $w \in \mathcal{N}$, we have

$$
\begin{aligned}
|(\mathcal{K}w)(\tau)| &\leq \mathfrak{I}^{\varpi,\chi}_{\gamma^+}|\mathfrak{h}(\tau, w(\tau))| + \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \left[\mathfrak{I}^{\varpi,\chi}_{\gamma^+}|\mathfrak{h}(\delta, w(\delta))| + \sum_{i=1}^{n} \iota_i \mathfrak{I}^{\varpi+\beta_i,\chi}_{\gamma^+}\left|\mathfrak{h}\left(\kappa_i, w\left(\kappa_i\right)\right)\right| + \sum_{j=1}^{m} \nu_j \mathfrak{I}^{\varpi+\alpha_j,\chi}_{\gamma^+}\left|\mathfrak{h}\left(\varrho_j, w\left(\varrho_j\right)\right)\right|\right] \\
&\leq \frac{1}{\chi^{\varpi}\Gamma(\varpi)} \int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)}(\tau-s)^{\varpi-1}|\mathfrak{h}(s, w(s))|ds \\
&+ \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \left[\frac{1}{\chi^{\varpi}\Gamma(\varpi)} \int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\delta-s)}(\delta-s)^{\varpi-1}|\mathfrak{h}(s, w(s))|ds\right. \\
&+ \sum_{i=1}^{n} \iota_i \frac{1}{\chi^{\varpi+\beta_i}\Gamma\left(\varpi+\beta_i\right)} \int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\kappa_i-s)}\left(\kappa_i-s\right)^{\varpi+\beta_i-1}|\mathfrak{h}(s, w(s))|ds \\
&\left.+ \sum_{j=1}^{m} \nu_j \frac{1}{\chi^{\varpi+\alpha_j}\Gamma\left(\varpi+\alpha_j\right)} \int_{\gamma}^{\varrho_j} e^{\frac{\chi-1}{\chi}(\varrho_j-s)}\left(\varrho_j-s\right)^{\varpi+\alpha_j-1}|\mathfrak{h}(s, w(s))|ds\right],
\end{aligned}
$$

using (H_2) and the property $e^{\frac{\chi-1}{\chi}(t-s)} \leq 1$ for $\gamma \leq s < t < \tau \leq \delta$ it leads to

$$
\begin{aligned}
|w(\tau)| \leq & \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\left(\|\psi_1\|+|w|\,\|\psi_2\|\right)+\frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
& \times\left[\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\left(\|\psi_1\|+|w|\,\|\psi_2\|\right)\right. \\
& +\sum_{i=1}^{n}|\iota_i|\,\frac{(\kappa_i-\gamma)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma\left(\varpi+\beta_i+1\right)}\left(\|\psi_1\|+|w|\,\|\psi_2\|\right) \\
& \left.+\sum_{j=1}^{n}|\nu_j|\,\frac{(\varrho_j-\gamma)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j}\Gamma\left(\varpi+\alpha_j+1\right)}\left(\|\psi_1\|+|w|\,\|\psi_2\|\right)\right] \\
& \leq\left(\|\psi_1\|+|w|\,\|\psi_2\|\right)\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}+\frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
& \times\left[\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}+\sum_{i=1}^{n}|\iota_i|\,\frac{(\kappa_i-\gamma)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma\left(\varpi+\beta_i+1\right)}\right. \\
& \left.+\sum_{j=1}^{n}|\nu_j|\,\frac{(\varrho_j-\gamma)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j\Gamma(\varpi+\alpha_j+1)}}\right]\left(\|\psi_1\|+|w|\,\|\psi_2\|\right) \\
& \leq\left(\|\psi_1\|+|w|\,\|\psi_2\|\right)(\mathfrak{A}+\mathfrak{B}), \\
& \leq(\mathfrak{A}+\mathfrak{B})\,\|\psi_1\|+|w|\,\|\psi_2\|\,(\mathfrak{A}+\mathfrak{B}),
\end{aligned}
$$

then $\|\mathcal{K}w\| \leq \big(\|\psi_1\|+\rho\|\psi_1\|\big)\big(\mathfrak{A}+\mathfrak{B}\big)$, where $\mathfrak{A}$ and $\mathfrak{B}$ are given by (4.6) and (4.7).
Step 3: $\mathcal{K}$ is equicontinuous.
Let $\tau_1, \tau_2 \in [\gamma, \delta]$ such that $\tau_1 < \tau_2$, and for each $w \in \mathcal{N}$ we obtain

$$
\begin{aligned}
&\left|(\mathcal{K}w)(\tau_2)-(\mathcal{K}w)(\tau_1)\right| \\
&\leq \frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau_1}\left[(\tau_2-s)^{\varpi-1}-(\tau_1-s)^{\varpi-1}\right]|\mathfrak{h}(s,w(s))|ds \\
&+\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\varpi-1}|\mathfrak{h}(s,w(s))|ds \\
&+\frac{\left|(\tau_2-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_2-\gamma)}-(\tau_1-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_1-\gamma)}\right|}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\left[\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta}(\delta-s)^{\varpi-1}|\mathfrak{h}(s,w(s))|ds\right. \\
&+\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_{\gamma}^{\kappa_i}(\kappa_i-s)^{\varpi+\beta_i-1}|\mathfrak{h}(s,w(s))|ds \\
&\left.+\sum_{j=1}^{m}\nu_j\frac{1}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j)}\int_{\gamma}^{\varrho_j}(\varrho_j-s)^{\varpi+\alpha_j-1}|\mathfrak{h}(s,w(s))|ds\right], \\
&\leq \frac{(\|\psi_1\|+\rho\|\psi_2\|)}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau_1}\left[(\tau_2-s)^{\varpi-1}-(\tau_1-s)^{\varpi-1}\right]ds \\
&+\frac{(\|\psi_1\|+\rho\|\psi_2\|)}{\chi^{\varpi}\Gamma(\varpi)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\varpi-1}ds \\
&+\frac{\left|(\tau_2-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau_2)-\Phi(\gamma))}-(\tau_1-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_1-\gamma)}\right|}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\left[\frac{(\|\psi_1\|+\rho\|\psi_2\|)}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta}(\delta-s)^{\varpi-1}ds\right. \\
&+\sum_{i=1}^{n}\iota_i\frac{(\|\psi_1\|+\rho\|\psi_2\|)}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_{\gamma}^{\kappa_i}(\kappa_i-s)^{\varpi+\beta_i-1}ds \\
&\left.+\sum_{j=1}^{m}\nu_j\frac{(\|\psi_1\|+\rho\|\psi_2\|)}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j)}\int_{\gamma}^{\varrho_j}(\varrho_j-s)^{\varpi+\alpha_j-1}ds\right], \\
&\leq \frac{(\|\psi_1\|+\rho\|\psi_2\|)}{\chi^{\varpi}\Gamma(\varpi+1)}\left(2(\tau_2-\tau_1)^{\varpi}+|(\tau_2-\gamma)^{\varpi}-(\tau_1-\gamma)^{\varpi}|\right) \\
&+\frac{\left|(\tau_2-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_2-\gamma)}-(\tau_1-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_1-\gamma)}\right|}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\left[\frac{(\|\psi_1\|+\rho\|\psi_2\|)}{\chi^{\varpi}\Gamma(\varpi+1)}(\delta-\gamma)^{\varpi}\right. \\
&+\sum_{i=1}^{n}\iota_i\frac{(\|\psi_1\|+\rho\|\psi_2\|)}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}(\kappa_i-\gamma)^{\varpi+\beta_i} \\
&\left.+\sum_{j=1}^{m}\nu_j\frac{(\|\psi_1\|+\rho\|\psi_2\|)}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)}(\varrho_j-\gamma)^{\varpi+\alpha_j}\right],
\end{aligned}
$$

the right hand side tends to zero as $\tau_2 \longrightarrow \tau_1$, indepently of $w \in \mathcal{N}$ which leads to $|(\mathcal{K}w)(\tau_2) - (\mathcal{K}w)(\tau_1)| \longrightarrow 0$ as $\tau_2 \longrightarrow \tau_1$ this implies that $\mathcal{K}(\mathcal{N})$ is equicontinuous. From step 1, step 2 and step 3 it follows-by applying the Arzela-Ascoli theorem-that the operator $\mathcal{K}$ is relatively compact, as consequence the operatorr $\mathcal{K}$ is completely continuous.

Step 4: The set $\Upsilon_\varepsilon = \{w \in \mathcal{C}(\Lambda, \mathbb{R}) \mid w = \varepsilon \mathcal{K} w \, ; \, 0 \le \varepsilon \le 1\}$ is bounded.
We are going to show that the set Υ_ε is bounded. For all $w \in \Upsilon_\varepsilon$, by using (H_2), we have

$$\begin{aligned}
|w(\tau)| &\le \mathfrak{I}_{\gamma^+}^{\varpi,\chi} |\mathfrak{h}(\tau, w(\tau))| + \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta| \chi^{\varpi} \Gamma(\varpi+1)} \\
&\times \left[\mathfrak{I}_{\gamma^+}^{\varpi,\chi} |\mathfrak{h}(\delta, w(\delta))| + \sum_{i=1}^{n} \iota_i \mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi} \left|\mathfrak{h}\left(\kappa_i, w\left(\kappa_i\right)\right)\right| + \sum_{j=1}^{m} \nu_j \mathfrak{I}_{\gamma^+}^{\varpi+\alpha_j,\chi} \left|\mathfrak{h}\left(\varrho_j, w\left(\varrho_j\right)\right)\right| \right] \\
&\le \frac{1}{\chi^{\varpi} \Gamma(\varpi)} \int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)} (\tau-s)^{\varpi-1} |\mathfrak{h}(s, w(s))| ds \\
&+ \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta| \chi^{\varpi} \Gamma(\varpi+1)} \left[\frac{1}{\chi^{\varpi} \Gamma(\varpi)} \int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\delta-s)} (\delta-s)^{\varpi-1} |\mathfrak{h}(s, w(s))| ds \right. \\
&+ \sum_{i=1}^{n} \iota_i \frac{1}{\chi^{\varpi+\beta_i} \Gamma\left(\varpi+\beta_i\right)} \int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}\left(\kappa_i-s\right)} \left(\kappa_i-s\right)^{\varpi+\beta_i-1} |\mathfrak{h}(s, w(s))| ds \\
&\left. + \sum_{j=1}^{n} \nu_j \frac{1}{\chi^{\varpi+\alpha_j} \Gamma\left(\varpi+\alpha_j\right)} \int_{\gamma}^{\varrho_j} e^{\frac{\chi-1}{\chi}\left(\varrho_j-s\right)} \left(\varrho_j-s\right)^{\varpi+\alpha_j-1} |\mathfrak{h}(s, w(s))| ds \right],
\end{aligned}$$

using (H_2) and the property $e^{\frac{\chi-1}{\chi}(t-s)} \le 1$ for $\gamma \le s < t < \tau \le \delta$ it leads to

$$\begin{aligned}
|w(\tau)| &\le \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi} \Gamma(\varpi+1)} \left(\|\psi_1\| + |w| \, \|\psi_2\|\right) + \frac{(\delta-\gamma)^{\varpi}}{|\Theta| \chi^{\varpi} \Gamma(\varpi+1)} \\
&\times \left[\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi} \Gamma(\varpi+1)} \left(\|\psi_1\| + |w| \, \|\psi_2\|\right) \right. \\
&+ \sum_{i=1}^{n} |\iota_i| \frac{\left(\kappa_i-\gamma\right)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i} \Gamma\left(\varpi+\beta_i+1\right)} \left(\|\psi_1\| + |w| \, \|\psi_2\|\right) \\
&\left. + \sum_{j=1}^{n} |\nu_j| \frac{\left(\varrho_j-\gamma\right)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j} \Gamma\left(\varpi+\alpha_j+1\right)} \left(\|\psi_1\| + |w| \, \|\psi_2\|\right) \right] \\
&\le \left(\|\psi_1\| + |w| \, \|\psi_2\|\right) \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi} \Gamma(\varpi+1)} + \frac{(\delta-\gamma)^{\varpi}}{|\Theta| \chi^{\varpi} \Gamma(\varpi+1)} \\
&\times \left[\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi} \Gamma(\varpi+1)} + \sum_{i=1}^{n} |\iota_i| \frac{\left(\kappa_i-\gamma\right)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i} \Gamma\left(\varpi+\beta_i+1\right)} \right. \\
&\left. + \sum_{j=1}^{n} |\nu_j| \frac{\left(\varrho_j-\gamma\right)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j} \Gamma(\varpi+\alpha_j+1)} \right] \left(\|\psi_1\| + |w| \, \|\psi_2\|\right) \\
&\le \left(\|\psi_1\| + |w| \, \|\psi_2\|\right) (\mathfrak{A} + \mathfrak{B}), \\
&\le (\mathfrak{A} + \mathfrak{B}) \|\psi_1\| + |w| \, \|\psi_2\| (\mathfrak{A} + \mathfrak{B}), ,
\end{aligned}$$

Where $\mathfrak{A}$ and $\mathfrak{B}$ are given by (4.6) and (4.7). Thus, we have

$$\|w\| \le \frac{(\mathfrak{A} + \mathfrak{B}) \|\psi_1\|}{1 - \|\psi_2\| (\mathfrak{A} + \mathfrak{B})}.$$

This proves that the set Υ_ε is bounded in $\mathcal{C}(\Lambda,\mathbb{R})$, by using Theorem 2.1, $\mathcal{K}$ has at least one fixed point which is the solution of the problem (1).

Uniqueness result employing Banach's contraction mapping theorem

The second existence and uniqueness result is obtained through the application of the Banach fixed point theorem

Theorem 4.2 *Assume that* (H_1) *is verified. If* $\mathcal{L}\big(\mathfrak{A}+\mathfrak{B}\big)<1$, *where* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are respectively given by* (4.6) *and* (4.7), *then the problem* (1) *has a unique solution on* $[\gamma,\delta]$.

Proof: Let $\mathcal{K}$ be the operator defined in Equation (4.1) Accordingly, problem (1) can be reformulated as a fixed point problem , namely finding $w=\mathcal{K}w$ such that. By using Banach contraction principle we will show that $\mathcal{K}$ has a unique fixed point. We set $sup_{\tau\in[\gamma,\delta]}=|\mathfrak{h}(\tau,0)|=\mathcal{M}<\infty$, and choose $\rho>0$ such that

$$\rho\geq\frac{\mathcal{M}\big(\mathfrak{A}+\mathfrak{B}\big)}{1-\mathcal{L}\big(\mathfrak{A}+\mathfrak{B}\big)},\tag{4.9}$$

$\mathcal{B}_\rho=\{w\in\mathcal{C}([\gamma,\delta],\mathbb{R});\|w\|\leq\rho)\}$, where $\mathfrak{A},\mathfrak{B}$ are respectively given by (4.6) and (4.7).
Step 1: We show that $\mathcal{K}\mathcal{B}_\rho\subset\mathcal{B}_\rho$.
For any $w\in\mathcal{B}_\rho$ we have

$$\begin{aligned}|\mathfrak{h}(\tau,w(\tau))|&\leq|\mathfrak{h}(\tau,w(\tau))-\mathfrak{h}(\tau,0)|+|\mathfrak{h}(\tau,0)|\\&\leq\mathcal{L}|w(\tau)|+\mathcal{M}\\&\leq\mathcal{L}|w|+\mathcal{M},\end{aligned}$$

then we have

$$\begin{aligned}|(\mathcal{K}w)(\tau)|&\leq\mathfrak{I}^{\varpi,\chi}_{\gamma^+}|\mathfrak{h}(\tau,w(\tau))|+\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\\&\quad\times\left[\mathfrak{I}^{\varpi,\chi}_{\gamma^+}|\mathfrak{h}(\delta,w(\delta))|+\sum_{i=1}^{n}\iota_i\mathfrak{I}^{\varpi+\beta_i,\chi}_{\gamma^+}\left|\mathfrak{h}\left(\kappa_i,w\left(\kappa_i\right)\right)\right|+\sum_{j=1}^{m}\nu_j\mathfrak{I}^{\varpi+\alpha_j,\chi}_{\gamma^+}\left|\mathfrak{h}\left(\varrho_j,w\left(\varrho_j\right)\right)\right|\right]\\&\leq\frac{1}{\chi^{\varpi}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\tau-s)}(\tau-s)^{\varpi-1}|\mathfrak{h}(s,w(s))|ds\\&+\frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\delta-s)}(\delta-s)^{\varpi-1}|\mathfrak{h}(s,w(s))|ds\right.\\&+\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma\left(\varpi+\beta_i\right)}\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}\left(\kappa_i-s\right)}\left(\kappa_i-s\right)^{\varpi+\beta_i-1}|\mathfrak{h}(s,w(s))|ds\\&\left.+\sum_{j=1}^{n}\nu_j\frac{1}{\chi^{\varpi+\alpha_j}\Gamma\left(\varpi+\alpha_j\right)}\int_{\gamma}^{\varrho_j}e^{\frac{\chi-1}{\chi}\left(\varrho_j-s\right)}\left(\varrho_j-s\right)^{\varpi+\alpha_j-1}|\mathfrak{h}(s,w(s))|ds\right],\end{aligned}$$

using (H_1) and the property $e^{\frac{\chi-1}{\chi}(t-s)}\leq 1$ for $\gamma\leq s<t<\tau\leq\delta$ it leads to

$$
\begin{aligned}
|(\mathcal{K}w)(\tau)| \leq{}& \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}(\mathcal{L}|w|+\mathcal{M}) + \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \left[\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}(\mathcal{L}|w|+\mathcal{M})\right. \\
&+ \sum_{i=1}^{n} |\iota_i| \frac{(\kappa_i-\gamma)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}(\mathcal{L}|w|+\mathcal{M}) \\
&\left.+ \sum_{j=1}^{n} |\nu_j| \frac{(\varrho_j-\gamma)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)}(\mathcal{L}|w|+\mathcal{M})\right] \\
\leq{}& (\mathcal{L}|w|+\mathcal{M})\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} + \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \left[\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} + \sum_{i=1}^{n} |\iota_i| \frac{(\kappa_i-\gamma)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}\right. \\
&\left.+ \sum_{j=1}^{n} |\nu_j| \frac{(\varrho_j-\gamma)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)}\right](\mathcal{L}|w|+\mathcal{M}) \\
\leq{}& (\mathcal{L}|w|+\mathcal{M})(\mathfrak{A}+\mathfrak{B}) \\
\leq{}& (\mathcal{L}\rho+\mathcal{M})(\mathfrak{A}+\mathfrak{B}) \\
\leq{}& \rho,
\end{aligned}
$$

which implies that $\mathcal{K}\mathcal{B}_\rho \subset \mathcal{B}_\rho$. Where $\mathfrak{A}, \mathfrak{B}$ are respectively given by (4.6) and (4.7).

Step 2: We show that the operator $\mathcal{K}$ is a contraction.

For any $v, w \in \mathcal{C}$, and for $\tau \in [\gamma, \delta]$, we have

$$
\begin{aligned}
|(\mathcal{K}v)(\tau) - (\mathcal{K}w)(\tau)| &\leq \mathfrak{I}_{\gamma^+}^{\varpi,\chi}|\mathfrak{h}(\tau, v(\tau)) - \mathfrak{h}(\tau, w(\tau))| + \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \Bigg[\mathfrak{I}_{\gamma^+}^{\varpi,\chi}|\mathfrak{h}(\delta, v(\delta)) - \mathfrak{h}(\delta, w(\delta))| + \sum_{i=1}^{n} \iota_i \mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi}\left|\mathfrak{h}\left(\kappa_i, v\left(\kappa_i\right)\right) - \mathfrak{h}\left(\kappa_i, w\left(\kappa_i\right)\right)\right| \\
&+ \sum_{j=1}^{m} \nu_j \mathfrak{I}_{\gamma^+}^{\varpi+\alpha_j,\chi}\left|\mathfrak{h}\left(\varrho_j, v\left(\varrho_j\right)\right) - \mathfrak{h}\left(\varrho_j, w\left(\varrho_j\right)\right)\right|\Bigg] \\
&\leq \frac{1}{\chi^{\varpi}\Gamma(\varpi)} \int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)}(\tau-s)^{\varpi-1}|\mathfrak{h}(s, v(s)) - \mathfrak{h}(s, w(s))|ds \\
&+ \frac{(\tau-\gamma)^{\varpi} e^{\frac{\chi-1}{\chi}}(\tau-\gamma)}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\frac{1}{\chi^{\varpi}\Gamma(\varpi)} \int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\delta-s)}(\delta-s)^{\varpi-1}|\mathfrak{h}(s, v(s)) - \mathfrak{h}(s, w(s))|ds \\
&+ \sum_{i=1}^{n} \iota_i \frac{1}{\chi^{\varpi+\beta_i}\Gamma\left(\varpi+\beta_i\right)} \int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}\left(\kappa_i-s\right)}\left(\kappa_i-s\right)^{\varpi+\beta_i-1}|\mathfrak{h}(s, v(s)) - \mathfrak{h}(s, w(s))|ds \\
&+ \sum_{j=1}^{n} \nu_j \frac{1}{\chi^{\varpi+\alpha_j}\Gamma\left(\varpi+\alpha_j\right)} \int_{\gamma}^{\varrho_j} e^{\frac{\chi-1}{\chi}\left(\varrho_j-s\right)}\left(\varrho_j-s\right)^{\varpi+\alpha_j-1}|\mathfrak{h}(s, v(s)) - \mathfrak{h}(s, w(s))|ds\Bigg] \\
&\leq (\mathcal{L}|v-w|)\frac{(\delta-\gamma)^{\varpi+\sigma}}{\chi^{\varpi}\Gamma(\varpi+1)} + \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \left[\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} + \sum_{i=1}^{n}|\iota_i| \frac{\left(\kappa_i-\gamma\right)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma\left(\varpi+\beta_i+1\right)}\right. \\
&\left.+ \sum_{j=1}^{n}|\nu_j| \frac{\left(\varrho_j-\gamma\right)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j}\Gamma\left(\varpi+\alpha_j+1\right)}\right](\mathcal{L}|v-w|) \\
&\leq \mathcal{L}(\mathfrak{A}+\mathfrak{B})|v-w|,
\end{aligned}
$$

which implies, $\|\mathcal{K}v - \mathcal{K}w\| \leq \mathcal{L}\big(\mathfrak{A}+\mathfrak{B}\big)\|v-w\|$. As $\mathcal{L}\big(\mathfrak{A}+\mathfrak{B}\big) < 1$, then $\mathcal{K}$ is a contraction. Consequently, by the Banach fixed-point theorem, the operator $\mathcal{K}$ admits a unique fixed point, which corresponds to the unique solution of the problem (1).

5. Ulam–Hyers Stability Analysis

This section is devoted to the analysis of the Ulam-Hyers (U-H) stability and the generalized Ulam-Hyers (G-U-H) stability of problem (1).

Let $\varepsilon > 0$, we consider the following inequality

$$\left|{}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi}\tilde{w}(\tau) - \mathfrak{h}(\tau, \tilde{w}(\tau))\right| \leq \varepsilon, \quad \tau \in \Lambda := [\gamma, \delta], \tag{5.1}$$

Definition 5.1 [24] The problem (1) is U-H stable if there exists $\lambda > 0$, such that for each $\varepsilon > 0$ and for each solution $\tilde{w} \in \mathcal{C}$ of inequality (5.1), there exists $w, \in \mathcal{C}$ solution of the problem (1) complying with

$$\|\tilde{w} - w\| \leq \lambda\varepsilon. \tag{5.2}$$

Definition 5.2 [24] The problem (1) is G-U-H stable if there exists $\varphi \in \mathcal{C}$ with $\varphi(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $\tilde{w} \in \mathcal{C}$ of inequality (5.1), there exists $w \in \mathcal{C}$ solution of the problem (1) complying with.

$$\|\tilde{w} - w\| \leq \varphi(\varepsilon). \tag{5.3}$$

Remark 5.1 A function $\tilde{w} \in \mathcal{C}$ is a solution of inequalities (5.1) if and only if there exists a function $\mathfrak{g} \in \mathcal{C}$ such that

i- $|\mathfrak{g}(\tau)| \leq \varepsilon$,

ii- for $\tau \in [\gamma, \delta]$:

$$ {}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi}\tilde{w}(\tau) = \mathfrak{h}(\tau, \tilde{w}(\tau)) + \mathfrak{g}(\tau). \tag{5.4} $$

To simplify the computations, we use the following notations:

$$ \begin{aligned} \Omega_1 = \mathcal{L}\Bigg\{ &\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} + \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)} \\ &+ \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} + \sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)}\Bigg]\Bigg\}, \end{aligned} \tag{5.5} $$

$$ \Omega_2 = \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}, \tag{5.6} $$

Theorem 5.1 *Assume that (H_1) hold, if $\Omega_1 < 1$ then the problem (1) is Ulam–Hyers stable on $[\gamma, \delta]$ and consequently is generalized Ulam–Hyers stable, where Ω_1 is given by (5.5).*

proof Let $\varepsilon > 0$, and $\tilde{w} \in \mathcal{C}$ satisfies inequality (5.1), and $w \in \mathcal{C}$ be the unique solution of the problem (1) with the conditions $\tilde{w}(\gamma) = w(\gamma)$, $\tilde{w}(\delta) = w(\delta)$, then by Lemma 2.2, we obtain

$$ w(\tau) = \mathfrak{I}_{\gamma^+}^{\varpi,\chi}\mathfrak{h}(\tau, w(\tau)) + \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \tag{5.7} $$

$$ \times\left[\mathfrak{I}_{\gamma^+}^{\varpi,\chi}\mathfrak{h}(\delta, w(\delta)) - \sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi^{\chi}}\mathfrak{h}\left(\kappa_i, w\left(\kappa_i\right)\right) - \sum_{j=1}^{m}\nu_j\mathfrak{I}_{\gamma^+}^{\varpi+\alpha_j,\chi^{\chi}}\mathfrak{h}\left(\varrho_j, w\left(\varrho_j\right)\right)\right] \tag{5.8} $$

$$ = \frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)}(\tau-s)^{\varpi-1}\mathfrak{h}(s, w(s))ds \tag{5.9} $$

$$ + \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}}(\tau-\gamma)}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \tag{5.10} $$

$$ \times\left[\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\delta-s)}(\delta-s)^{\varpi-1}\mathfrak{h}(s, w(s))ds\right. \tag{25} $$

$$ - \sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma\left(\varpi+\beta_i\right)}\int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\kappa_i-s)}\left(\kappa_i-s\right)^{\varpi+\beta_i-1}\mathfrak{h}(s, w(s))ds \tag{5.11} $$

$$ \left. - \sum_{j=1}^{n}\nu_j\frac{1}{\chi^{\varpi+\alpha_j}\Gamma\left(\varpi+\alpha_j\right)}\int_{\gamma}^{\varrho_j} e^{\frac{\chi-1}{\chi}(\varrho_j-s)}\left(\varrho_j-s\right)^{\varpi+\alpha_j-1}\mathfrak{h}(s, w(s))ds\right], \tag{5.12} $$

Since, $\tilde{w} \in \mathcal{C}$ satisfies inequality (5.1) by using Remark 5.1 we have

$$ \begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi}\tilde{w}(\tau) = \mathfrak{h}(\tau, \tilde{w}(\tau)) + \mathfrak{g}(\tau), \ \tau \in \Lambda := [\gamma, \delta], \\ \tilde{w}(\gamma) = w(\gamma) \quad , \quad \tilde{w}(\delta) = w(\delta), \end{cases} \tag{5.13} $$

then by Lemma 2.2, we obtain

$$\begin{aligned}\tilde{w}(\tau) &= \mathfrak{I}_{\gamma^+}^{\varpi,\chi}\mathfrak{h}(\tau,\tilde{w}(\tau)) + \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\ &\times\left[\mathfrak{I}_{\gamma^+}^{\varpi,\chi}\mathfrak{h}(\delta,\tilde{w}(\delta)) - \sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi}\mathfrak{h}(\kappa_i,\tilde{w}(\kappa_i)) - \sum_{j=1}^{m}\nu_j\mathfrak{I}_{\gamma^+}^{\varpi+\alpha_j,\chi}\mathfrak{h}(\varrho_j,\tilde{w}(\varrho_j))\right] \\ &= \frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\tau-s)}(\tau-s)^{\varpi-1}\mathfrak{h}(s,\tilde{w}(s))ds \\ &+ \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}\left[\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\delta-s)}(\delta-s)^{\varpi-1}\mathfrak{h}(s,\tilde{w}(s))ds\right. \\ &- \sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\kappa_i-s)}(\kappa_i-s)^{\varpi+\beta_i-1}\mathfrak{h}(s,\tilde{w}(s))ds \\ &- \left.\sum_{j=1}^{n}\nu_j\frac{1}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j)}\int_{\gamma}^{\varrho_j}e^{\frac{\chi-1}{\chi}(\varrho_j-s)}(\varrho_j-s)^{\varpi+\alpha_j-1}\mathfrak{h}(s,\tilde{w}(s))ds\right] \\ &+ \frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\tau-s)}(\tau-s)^{\varpi-1}\mathfrak{g}(s)ds, \end{aligned} \tag{5.14}$$

for each $\tau \in [\gamma,\delta]$, we have

$$\begin{aligned}&\left|\tilde{w}(\tau)-w(\tau)\right| \\ &\leq \mathfrak{I}_{\gamma^+}^{\varpi,\chi}\left|\mathfrak{h}(\tau,\tilde{w}(\tau))-\mathfrak{h}(\tau,w(\tau))\right| + \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\ &\times\left[\mathfrak{I}_{\gamma^+}^{\varpi,\chi}\left|\mathfrak{h}(\delta,\tilde{w}(\delta))-\mathfrak{h}(\delta,w(\delta))\right| + \sum_{i=1}^{n}|\iota_i|\mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi}\left|\mathfrak{h}(\kappa_i,\tilde{w}(\kappa_i))-\mathfrak{h}(\kappa_i,w(\kappa_i))\right|\right. \\ &+ \left.\sum_{j=1}^{m}\nu_j\mathfrak{I}_{\gamma^+}^{\varpi+\alpha_j,\chi}|\mathfrak{h}(\varrho_j,\tilde{w}(\varrho_j)-\mathfrak{h}(\varrho_j,\tilde{w}(\varrho_j)|\right] + \mathfrak{I}_{\gamma^+}^{\varpi,\chi}|\mathfrak{g}(\tau)|,\end{aligned}$$

using (H_1), the property $e^{\frac{\chi-1}{\chi}(t-s)} \leq 1$ for $\gamma \leq s < t < \tau \leq \delta$ and Remark 5.1 leads to

$$
\begin{aligned}
\|\tilde{w}-w\| &\leq \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\mathcal{L}\|\tilde{w}-w\| \\
&+ \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}\mathcal{L}\|\tilde{w}-w\|\right. \\
&\left.+ \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\mathcal{L}\|\tilde{w}-w\| + \sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)}\mathcal{L}\|\tilde{w}-w\|\right] \\
&+ \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\varepsilon, \\
&\leq \|\tilde{w}-w\|\mathcal{L}\left\{\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} + \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\right. \\
&\times\left[\sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)} + \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\right. \\
&\left.\left.+ \sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)}\right]\right\} + \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\varepsilon, \\
&\leq \|\tilde{w}-w\|\Omega_1 + \Omega_2\varepsilon, \\
&\leq \Omega_1\|\tilde{w}-w\| + \Omega_2\varepsilon, \\
&\leq \frac{\Omega_2}{1-\Omega_1}\varepsilon,
\end{aligned}
$$

which implies,

$$\|\tilde{w}-w\| \leq \frac{\Omega_2}{1-\Omega_1}\varepsilon. \tag{5.15}$$

By setting $\lambda = \frac{\Omega_2}{1-\Omega_1}$, where Ω_1 and Ω_2 are given by (5.5) and (5.6), we obtain

$$\|\tilde{w}-w\| \leq \lambda\varepsilon. \tag{5.16}$$

This proves that the problem (1), is U-H stable.
consequently, by setting $\varphi(\varepsilon) = \lambda\varepsilon$ with $\varphi(0) = 0$ we get

$$\|\tilde{w}-w\| \leq \varphi(\varepsilon). \tag{5.17}$$

This shows that the problem (1) is G-U-H stable.

6. Numerical Scheme and Example

In this section, we demonstrate our main results through an illustrative example by examining the following problem:

$$
\begin{cases}
{}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi}w(\tau) = \mathfrak{h}(\tau, w(\tau)), \tau \in \Lambda := [\gamma, \delta], \\
w(\gamma) = 0, w(\delta) = \sum_{j=1}^{m}\nu_j \mathfrak{I}_{\gamma^+}^{\alpha_j,\chi}w(\varrho_j) + \sum_{i=1}^{n}\iota_i \mathfrak{I}_{\gamma^+}^{\beta_i,\chi}w(\kappa_i).
\end{cases} \tag{6.1}
$$

Where ${}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi}$ is the Caputo generalized proportional fractional derivative of order $\varpi, 0 < \varpi \leq 1$ defined by:

To address the existence and uniqueness results for problem (6.1), we introduce the following notations to simplify the computations.

$$\mathfrak{A} = \frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}$$

$$\mathfrak{B} = \frac{(\delta-\gamma)^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\frac{(\delta-\gamma)^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} + \sum_{i=1}^{n}|\iota_i|\frac{(\kappa_i-\gamma)^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)} + \sum_{j=1}^{n}|\nu_j|\frac{(\varrho_j-\gamma)^{\varpi+\alpha_j}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)}\right]$$

where

$$\Theta = \sum_{i=1}^{n}\iota_i\frac{(\kappa_i-\gamma)^{\varpi+\beta_i}e^{\frac{\chi-1}{\chi}(\kappa_i-\gamma)}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)} + \sum_{j=1}^{n}\nu_j\frac{(\varrho_j-\gamma)^{\varpi+\alpha_j}e^{\frac{\chi-1}{\chi}(\varrho_j-\gamma)}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)} - \frac{(\delta-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\delta-\gamma)}}{\chi^{\varpi}\Gamma(\varpi+1)} \neq 0$$

We reveal the principal results under the following hypotheses.
$(H_1): |\mathfrak{h}(\tau,v)-\mathfrak{h}(\tau,w)| \leq \mathcal{L}|v-w|; \mathcal{L}>0,$ for each $\tau\in[\gamma,\delta]$ and $v,w\in\mathbb{R}$
$(H_2):$ there exist non-negatives continuous functions ψ_1 and ψ_2, such that

$$|\mathfrak{h}(\tau,w)| \leq \psi_1(\tau)+\psi_2(\tau)|w|, (\tau,w)\in[\gamma,\delta]\times\mathbb{R}, \text{ with } \|\psi_1\| = \sup_{\tau\in[\gamma,\delta]}|\psi_1(\tau)|$$

$$\|\psi_2\| = \sup_{\tau\in[\gamma,\delta]}|\psi_2(\tau)|$$

$(H_3): \|\psi_2\|(\mathfrak{A}+\mathfrak{B})<1.$
We have the following integral solution:

$$\begin{aligned} w(\tau) = {} & \frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\tau-s)}(\tau-s)^{\varpi-1}\mathfrak{h}(s)ds + \frac{(\tau-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\ & \times\left[\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\delta-s)}(\delta-s)^{\varpi-1}\mathfrak{h}(s)ds \right. \\ & - \sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\kappa_i-s)}(\kappa_i-s)^{\varpi+\beta_i-1}\mathfrak{h}(s)ds \\ & \left. - \sum_{j=1}^{n}\nu_j\frac{1}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j)}\int_{\gamma}^{\varrho_j} e^{\frac{\chi-1}{\chi}(\varrho_j-s)}(\varrho_j-s)^{\varpi+\alpha_j-1}\mathfrak{h}(s)ds\right] \end{aligned} \tag{6.2}$$

Numerical Scheme

The constant Θ can be computed directly by simple substitution of the parameters of the problem.

$$\Theta = \sum_{i=1}^{n}\frac{\iota_i(\kappa_i-\gamma)^{\varpi+\beta_i}e^{\frac{\chi-1}{\chi}(\kappa_i-\gamma)}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)} + \sum_{j=1}^{n}\frac{\nu_j(\varrho_j-\gamma)^{\varpi+\alpha_j}e^{\frac{\chi-1}{\chi}(\varrho_j-\gamma)}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j+1)} - \frac{(\delta-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\delta-\gamma)}}{\chi^{\varpi}\Gamma(\varpi+1)}.$$

Let

$$I_2 = \int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\delta-s)}(\delta-s)^{\varpi-1}\mathfrak{h}(s)\,ds,$$

$$I_{3,i} = \int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\kappa_i-s)}(\kappa_i-s)^{\varpi+\beta_i-1}\mathfrak{h}(s)\,ds,$$

$$I_{4,j} = \int_{\gamma}^{\varrho_j} e^{\frac{\chi-1}{\chi}(\varrho_j-s)}(\varrho_j-s)^{\varpi+\alpha_j-1}\mathfrak{h}(s)\,ds.$$

To compute the integrals $I_2, I_{3,i}, I_{4,j}$ we use the Simpson method defined by the following formula:

$$\int_a^b f(x)dx \approx \frac{h}{6}\left(f(a)+4f\left(a+\frac{h}{2}\right)+f(b)\right)+\frac{h}{3}\sum_{i=1}^{n-1} f(a+ih)+2f\left(a+ih+\frac{h}{2}\right), \tag{6.3}$$

where

$$h=\frac{b-a}{n}. \tag{6.4}$$

Let $(\tau_k)_{0\leq k\leq N}$ be a uniform grid on $[\gamma,\delta]$ with step size h:

$$\tau_k=\gamma+kh, \quad H=\frac{\delta-\gamma}{N}.$$

Denote w_k as the numerical approximation of $w(\tau_k)$.
We have:

$$\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau_{k+1}} e^{\frac{\chi-1}{\chi}(\tau_{k+1}-s)}(\tau_{k+1}-s)^{\varpi-1}\mathfrak{h}(s)\,ds \approx \frac{1}{\chi^{\varpi}\Gamma(\varpi)}\sum_{m=0}^{k}\mathfrak{h}_m W_{k+1-m},$$

where $W_m=\int_{mh}^{(m+1)h} e^{\frac{\chi-1}{\chi}u}u^{\varpi-1}\,du$. Which will be computed using the same formula of the Simpson method.

Let

$$A=\frac{I_2}{\chi^{\varpi}\Gamma(\varpi)}-\sum_{i=1}^{n}\frac{\iota_i I_{3,i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}-\sum_{j=1}^{n}\frac{\nu_j I_{4,j}}{\chi^{\varpi+\alpha_j}\Gamma(\varpi+\alpha_j)}.$$

We get the following implicit Numerical Scheme, for $k=0,1,\dots,N-1$:

$$w_{k+1}=\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\sum_{m=0}^{k}\mathfrak{h}_m W_{k+1-m}+\frac{(\tau_{k+1}-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_{k+1}-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}\cdot A. \tag{6.5}$$

If $\mathfrak{h}(s)$ depends on $w(s)$, approximate $\mathfrak{h}_{k+1}\approx\mathfrak{h}_k$ (explicit step).

Final Scheme

$$\boxed{w_{k+1}=\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\sum_{m=0}^{k}\mathfrak{h}_m W_{k+1-m}+\frac{(\tau_{k+1}-\gamma)^{\varpi}e^{\frac{\chi-1}{\chi}(\tau_{k+1}-\gamma)}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}\cdot A}$$

Concrete Example for Problem (6.1)

We choose the following data:

- $\Lambda=[\gamma,\delta]=[0,1]$, $\varpi=0.5$, $\chi=1$.
- Number of terms: $n=3$, $m=4$.
- Points and orders:

$$\begin{aligned}&\kappa_1=0.25,\ \beta_1=0.30, \quad \kappa_2=0.50,\ \beta_2=0.30, \quad \kappa_3=0.75,\ \beta_3=0.30,\\ &\varrho_1=0.20,\ \alpha_1=0.25, \quad \varrho_2=0.40,\ \alpha_2=0.25,\\ &\varrho_3=0.60,\ \alpha_3=0.25, \quad \varrho_4=0.80,\ \alpha_4=0.25.\end{aligned}$$

- Coefficients: $\iota_i=0.10$ for $i=1,2,3$, and $\nu_j=0.10$ for $j=1,2,3,4$.
- Nonlinearity:

$$\mathfrak{h}(\tau,w)=\sin(\tau)+(0.1+0.1\,\tau)\,w.$$

Verification of (H_1)

For any $\tau \in [0,1]$ and $v, w \in \mathbb{R}$:

$$|\mathfrak{h}(\tau, v) - \mathfrak{h}(\tau, w)| = (0.1 + 0.1\tau)\,|v - w| \leq 0.2\,|v - w|.$$

Thus we may take the Lipschitz constant

$$\mathcal{L} = 0.2,$$

verifying (H_1).

Verification of (H_2)

Set

$$\psi_1(\tau) = 1 + \cos(\tau), \qquad \psi_2(\tau) = 0.1 + 0.1\tau.$$

Then ψ_1, ψ_2 are continuous on $[0,1]$, with

$$\|\psi_1\| = \sup_{[0,1]}(1 + \cos\tau) = 2, \quad \|\psi_2\| = \sup_{[0,1]}(0.1 + 0.1\tau) = 0.2.$$

Moreover,

$$|\mathfrak{h}(\tau, w)| \leq |\sin(\tau)| + (0.1 + 0.1\tau)\,|w| \leq \psi_1(\tau) + \psi_2(\tau)\,|w|,$$

so (H_2) holds.

Computation of $\mathfrak{A}$

By definition,

$$\mathfrak{A} = \frac{(\delta - \gamma)^{\varpi}}{\chi^{\varpi}\,\Gamma(\varpi + 1)} = \frac{1^{0.5}}{1^{0.5}\,\Gamma(1.5)} = \frac{2}{\sqrt{\pi}} \approx 1.12838.$$

Computation of Θ

Using

$$\Theta = \sum_{i=1}^{3} \iota_i \frac{(\kappa_i - \gamma)^{\varpi + \beta_i}\, e^{\frac{\chi - 1}{\chi}(\kappa_i - \gamma)}}{\chi^{\varpi + \beta_i}\,\Gamma(\varpi + \beta_i + 1)} + \sum_{j=1}^{4} \nu_j \frac{(\varrho_j - \gamma)^{\varpi + \alpha_j}\, e^{\frac{\chi - 1}{\chi}(\varrho_j - \gamma)}}{\chi^{\varpi + \alpha_j}\,\Gamma(\varpi + \alpha_j + 1)} - \frac{(\delta - \gamma)^{\varpi}\, e^{\frac{\chi - 1}{\chi}(\delta - \gamma)}}{\chi^{\varpi}\,\Gamma(\varpi + 1)},$$

one finds numerically:

$$\Theta \approx 3 \times 0.10 \frac{0.25^{0.8}}{\Gamma(1.8)} + 4 \times 0.10 \frac{\{0.2, 0.4, 0.6, 0.8\}^{0.75}}{\Gamma(1.75)} - \frac{1^{0.5}}{\Gamma(1.5)} = -0.7684 \neq 0.$$

Computation of $\mathfrak{B}$

By definition,

$$\mathfrak{B} = \frac{(\delta - \gamma)^{\varpi}}{|\Theta|\,\chi^{\varpi}\,\Gamma(\varpi + 1)} \left[\mathfrak{A} + \sum_{i=1}^{3} |\iota_i| \frac{(\kappa_i - \gamma)^{\varpi + \beta_i}}{\chi^{\varpi + \beta_i}\,\Gamma(\varpi + \beta_i + 1)} + \sum_{j=1}^{4} |\nu_j| \frac{(\varrho_j - \gamma)^{\varpi + \alpha_j}}{\chi^{\varpi + \alpha_j}\,\Gamma(\varpi + \alpha_j + 1)} \right],$$

which evaluates to

$$\mathfrak{B} \approx 2.188.$$

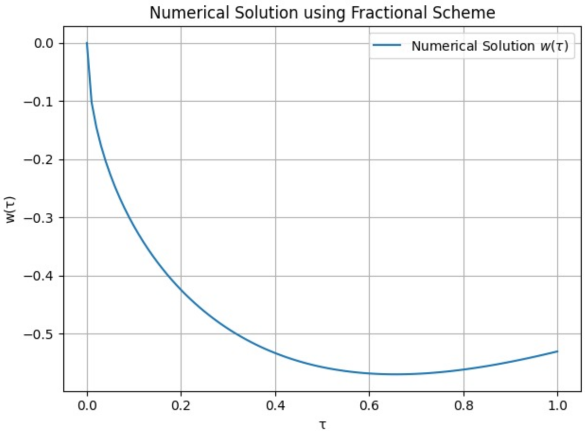
Verification of (H_3)

Finally,

$$\|\psi_2\|\,(\mathfrak{A} + \mathfrak{B}) = 0.2 \times (1.12838 + 2.188) \approx 0.663 < 1,$$

so (H_3) holds.

All hypotheses (H_1), (H_2), (H_3) are thus verified, and $\Theta \neq 0$, completing the example.



7. Conclusion

In this work, we have established existence, uniqueness, and stability results for a novel class of Caputo generalized proportional fractional differential equations involving two distinct fractional orders. The key contribution of our study lies in the consideration of the Caputo generalized proportional fractional derivative, which extends and generalizes previous works based on the classical Caputo fractional derivative.

Using fundamental fixed-point theorems namely, Schaefer's fixed-point theorem and the Banach contraction principle we rigorously proved the existence and uniqueness of solutions. Furthermore, we analyzed the stability of the proposed problem through Ulam-Hyers and generalized Ulam-Hyers stability criteria. To illustrate our theoretical findings, we provided a numerical example demonstrating the applicability of the obtained results.

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