Bol. Soc. Paran. Mat. (3s.) v. **2026 (44) 2** : 1–11.
©SPM – E-ISSN-2175-1188 ISSN-0037-8712
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.78699

Attractivity Results for Fuzzy Caputo-Katugampola Fractional Differential Equations

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ABSTRACT: This work investigates sufficient conditions for the existence of locally attractive mild solutions of the Caputo-Katugampola fuzzy fractional differential equation. The main findings are derived using the fixed-point method, supported by key tools such as Wright-type functions, semigroup theory, and fractional calculus. Finally, the theoretical results are illustrated through an illustrative example.

Key Words: Fuzzy fractional differential equation, Caputo-Katugampola fractional derivative, mild solution, fixed point theorem.

Contents

1. Introduction

The study of Fractional differential equations has become a very active area of research because they are useful in many fields of science and engineering. Equations that use fractional (non-integer) order derivatives are good at describing real-world processes such as diffusion, transport, earthquakes, fluid flow, traffic models, mechanics, chemistry, sound waves, and even psychology. These equations are harder to analyze than ordinary differential equations because fractional derivatives are non-local which means they depend on the whole past of the function, not just its value at a single point. They also involve special kernels that behave differently near the starting point. For more details on fractional calculus and its many applications, there are several good reference books and articles available [1,2,3,4,5,6].

Fractional differential equations have seen substantial development through the work of Zhou [1] and Kilbas et al., Miller and Ross, Podlubny, Lakshmikantham et al. [7,8,9,10], significantly advancing both theory and applications across science and engineering.

Allahviranloo et al. [11] investigated fuzzy fractional differential equations using the Caputo fractional gH-derivative, placing particular emphasis on the existence and uniqueness of solutions. Arshad [12] also studied the fundamental properties of fuzzy fractional differential equations, focusing on existence and uniqueness results through fuzzy integral equivalent equations. In a related contribution, Salahshour and his team [13] addressed the complexities of fractional differential equations within a fuzzy framework, proposing more effective and efficient methods for obtaining solutions. Hariharan and Udhayakumar [14, 15] have studied approximate controllability in fuzzy fractional systems. One paper focuses on Sobolev-type systems with Hilfer fractional derivatives and Clarke subdifferential using fixed point theorems. Another addresses fuzzy fractional evolution equations of order between 1 and 2, employing semigroup theory. Their work advances control theory for fuzzy fractional systems.

Katugampola introduced the Katugampola fractional integral and derivative operators [16,17], which include an additional parameter $\varrho > 0$. When ϱ approaches 0_+, these operators coincide with the Hadamard fractional integral, and for $\varrho = 1$, they reduce to the Riemann-Liouville fractional integral.

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2010 *Mathematics Subject Classification:* 34A07, 47H08, 47H10.
Submitted August 30, 2025. Published December 19, 2025

This parameterization unifies the two classical operators, allowing results proved for the Katugampola derivative to simultaneously apply to both the Riemann-Liouville and Hadamard derivatives, thus simplifying the theoretical analysis.

Katugampola [18] investigated the existence and uniqueness of solutions for fractional differential equations involving the Caputo-Katugampola derivative using Schauder fixed point theorem. Almeida et al. [19] further studied initial value problems for Caputo-Katugampola fractional differential equations, establishing existence and uniqueness results along with a numerical solution method. Zeng et al. [20] introduced a discrete form of the Caputo-Katugampola derivative and developed a numerical approach for solving linear fractional differential equations with the Caputo-Katugampola derivative. Baleanu et al. [21] explored chaotic dynamics and stability analysis in fractional differential equations based on the Caputo-Katugampola derivative. More recently, Hariharan and Udhayakumar [22] examined the existence of mild solutions for fuzzy fractional differential equations employing the Hilfer-Katugampola fractional derivative. These studies have emerged in recent years and provide a foundational framework for the development of fuzzy fractional differential equation involving the Katugampola fractional derivative.

Abbas and Benchohra [23] investigated the existence and attractivity of solutions to fractional order integral equations within the framework of Fréchet spaces. Abbas et al. [24] focused on establishing existence and attractivity results for solutions to Hilfer fractional differential equations. Using fixed point theorem and analytical tools, they provided conditions ensuring the existence and asymptotic attractivity of solutions.

Van Hoa et al. [25] introduced a novel concept of fuzzy fractional derivatives and investigated the existence and uniqueness of solutions for an initial value problem involving Caputo-Katugampola fuzzy fractional differential equations. Motivated by this work, the present study examines the existence of locally attractive mild solutions for fuzzy fractional differential equations with the Caputo-Katugampola fractional derivative, subject to the following initial condition:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\varrho}x(t) = Ax(t) + f(t, x(t)), \qquad t \in [0,\infty) = \mathbb{U}, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where ${}^{C}D_{0^+}^{\alpha,\varrho}$ is a Caputo-Katugampola fractional derivative of order $0 < \alpha < 1$. The state variable $x(\cdot)$ takes values within the space $\mathbb{F}$ and a fuzzy number is defined as a fuzzy set $x : \mathbb{R} \to [0, 1]$. A is the infinitesimal generator of a C_0-semigroup consisting of uniformly bounded linear operators $T(t)_{t\geq 0}$ on the space $\mathbb{B}$ and $f : \mathbb{U} \times \mathbb{F} \to \mathbb{F}$ is a fuzzy valued function, where $\mathbb{F}$ denotes the space of all fuzzy numbers on $\mathbb{R}$.

The manuscript is organized as follows. Section 2 reviews the fundamental concepts of fuzzy fractional calculus relevant to this study. Section 3 establishes the existence of the proposed mild solution. Section 4 provides an illustrative example to facilitate understanding. Finally, Section 5 presents the conclusions.

2. Preliminaries

Definition 2.1 *[26] A fuzzy number is a fuzzy set on $\mathbb{R}$ that satisfies properties such as convexity, normality, and upper semicontinuity. A common way to define a distance between fuzzy numbers $\widetilde{x}$ and $\widetilde{y}$ is using a norm-based metric, such as:*

$$d(\widetilde{x}, \widetilde{y}) = \|\widetilde{x} - \widetilde{y}\|,$$

where $\|\cdot\|$ is a norm that measures differences between fuzzy numbers.

Definition 2.2 *[14] In a fuzzy number space, the Hausdorff distance can be defined similarly to classical sets:*

$$d_H(\widetilde{x}, \widetilde{y}) = \max\left\{ \sup_{\zeta\in\widetilde{x}} \inf_{\varsigma\in\widetilde{y}} \|\zeta - \varsigma\|, \sup_{\varsigma\in\widetilde{y}} \inf_{\zeta\in\widetilde{x}} \|\varsigma - \zeta\| \right\}.$$

Where ζ, ς represent elements in the support of the fuzzy numbers, and $\|\varsigma - \zeta\|$ is the norm-based distance between them.

Let us consider the function $g : N \to \mathbb{F}$ (where $N = [0, \beta]$ and $\beta > 0$) with the supremum norm

$$\|g\|_\infty = \sup_{t \in \beta} |g(t)|.$$

A space is defined by

$$AC^1(N) = \Big\{x : N \to \mathbb{F} : \frac{dx}{dt} \in AC(N)\Big\},$$

where $AC^1(N)$ is absolutely continuous function from N into $\mathbb{F}$. A mapping $g : N \to \mathbb{F}$ with the norm and $\mathcal{L}^1$ denote the Lebesgue integrable functions defined as

$$\|g\|_1 = \int_0^\beta |g(t)|dt.$$

We define weighted space of continuous functions $\mathbb{C}_\alpha(N)$ and $\mathbb{C}^1_\alpha(N)$ as follows:

$$\mathbb{C}_\alpha(N) = \Big\{x : (0, \beta] \to \mathbb{F} : x(t) \in \mathbb{C}\Big\}$$

with the norm

$$\|x\|_{\mathbb{C}_\alpha} = \sup_{t \in N} |x(t)|$$

and weighted space we define as

$$\mathbb{C}^1_\alpha(N) = \Big\{x \in \mathbb{C} : \frac{dx}{dt} \in \mathbb{C}_\alpha\Big\}$$

with the norm x

$$\|x\|_{\mathbb{C}^1_\alpha} = \|x\|_\infty + \|\mathfrak{h}'\|_{\mathbb{C}_\alpha}$$
$$\mathcal{FC}_\alpha = \mathcal{FC}_\alpha(\mathbb{U}),$$

where $\mathcal{FC}$ is a space of all fuzzy numbers of all continuous and bounded functions from $\mathbb{U}$ into $\mathbb{F}$ and the weighted space of $\mathcal{FC}_\alpha$ defined by

$$\mathcal{FC}_\alpha = \Big\{x : (0, +\infty) \to \mathbb{F} : x(t) \in \mathcal{FC}\Big\}$$

with the norm

$$\|x\|_{\mathcal{FC}_\alpha} = \sup_{t \in \beta} |x(t)|.$$

Important notions and preliminaries concerning fractional calculus are introduced below:

Definition 2.3 *[16] The Katugampola fractional integral of order α and $x : (0, +\infty) \to \mathbb{R}$ for $-\infty < 0 < t < \infty$ is defined by*

$$I^{\alpha,\varrho}_{0^+} x(t) = \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\varrho-1}}{(t^\varrho - s^\varrho)^{1-\alpha}} x(s)ds, \ \ t > b; \ \ \varrho > 0, \alpha > 0.$$

The Katugampola fractional integral is defined with respect to an additional parameter $\varrho > 0$. These operators have special properties based on the value of ϱ.

Remark 2.4 *[16] Specifically, as $\varrho \to 0^+$, the Katugampola fractional integral converges to the Hadamard fractional integral,*

$$\lim_{\varrho \to 0} I^{\alpha,\varrho}_{0^+} x(t) = \int_0^t \frac{\big(\log \frac{t}{s}\big)^{\alpha-1}}{\Gamma(\alpha)} x(s) \frac{ds}{s}.$$

When the parameter $\varrho = 1$, they coincide with the Riemann-Liouville fractional integral,

$$I^{\alpha,1}_{0^+} x(t) = \int_0^t \frac{x(s)}{\Gamma(\alpha)(t-s)^{1-\alpha}} ds.$$

Definition 2.5 *[25] The Caputo-Katugampola fractional derivative of order $\alpha \in (0,1)$ and parameter $\varrho > 0$ of a function $x : [0,\infty) \to \mathbb{R}$ is defined by*

$$ {}^{C}D_{0_+}^{\alpha,\varrho} x(t) = \frac{\varrho^{\alpha}}{\Gamma(1-\alpha)} \int_0^t (t^{\varrho} - s^{\varrho})^{-\alpha} x^{'}(s) ds, \quad t > 0, $$

where $\Gamma(\cdot)$ is the Gamma function and x' denotes the classical derivative of x.

Note:-

- If $\varrho = 1$, then the Caputo-Katugampola fractional derivative reduces to the well-known Caputo fractional derivative.
- If $\varrho = 0^{+}$, it becomes the Caputo-Hadamard fractional derivative.

Definition 2.6 *[27] A function $f : \mathbb{U} \times \mathbb{F} \to \mathbb{F}$ is said to satisfy the Caratheodory conditions, if the following holds:*

- *The map $t \to f(t,x)$ is measurable for $x \in \mathcal{FC}_{\alpha}$.*
- *The map $x \to f(t,x)$ is continuous for each $t \in \mathbb{U}$.*

Lemma 2.7 *[28] Assume that the linear operator A acts as the infinitesimal generator of a C_0-semigroup if and only if*

- *The set A has the property of being closed and $D(A) = \mathbb{B}$.*
- *The resolvent set $p(A)$ of A includes positive real numbers, $\forall\ \alpha > 0$,*

$$ \|T(\beta, A)\| \leq \frac{1}{\beta}, $$

where $T(\beta, A) = (\beta^{q} I - A)^{-1} s = \int_0^{\infty} e^{-\beta^{q} t} J(t) s dt$.

Lemma 2.8 *[25] The system (1.1) can be expressed in the form of the following integral equation:*

$$ x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^{\varrho} - s^{\varrho}}{\varrho} \right)^{\alpha - 1} s^{\varrho - 1} [Ax(s) + f(s, x(s))] ds. \tag{2.1} $$

Corollary 2.1 *[14] The wright function $\mathcal{T}_{\alpha}(\mu)$ is defined by,*

$$ \mathcal{T}_{\alpha}(\mu) = \sum_{t=1}^{\infty} \frac{(-\mu)^{t-1}}{(t-1)!\ \alpha(1-\alpha t)}, \quad \alpha \in (0,1), \quad \mu \in [0, +\infty). $$

Which satisfies the equality given below

$$ \int_0^{\infty} \mu^{s} \mathcal{T}_{\alpha}(\mu) d\mu = \frac{\Gamma(1+s)}{(1+\alpha s)}, \quad s \geq 0. $$

Where $\mathcal{T}_{\alpha}(\mu)$ is a probability density function and is given as follows

$$ \mathcal{T}_{\alpha}(\mu) = \frac{1}{\alpha} \mu^{-1-\frac{1}{\alpha}} \kappa_{\alpha}(\mu^{-\frac{1}{\alpha}}) \leq 0, \quad \text{for} \ \ \alpha \in (0,1), \mu \in [0, +\infty), $$

where

$$ \kappa_{\alpha}(\mu) = \frac{1}{\pi} \sum_{m=1}^{\infty} (-1)^{m-1} \mu^{-m\alpha - 1} \frac{\Gamma(1+m\alpha)}{m!} \sin(m\pi\alpha). $$

Definition 2.9 *[29] A mild solution of the system (1.1) is defined as a function $x \in \mathcal{FC}_\alpha$ as follows:*

$$x(t) = \mathcal{W}_\alpha\left(\frac{t^\varrho}{\varrho}\right)x_0 + \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\mathcal{Q}_\alpha\left(\frac{t^\varrho - s^\varrho}{\varrho}\right) f(s, x(s))ds, \tag{2.2}$$

where

$$\mathcal{W}_\alpha\left(\frac{t^\varrho}{\varrho}\right) = \int_0^\infty \mathcal{T}_\alpha(\mu) J\left(\left(\frac{t^\varrho}{\varrho}\right)^\alpha \mu\right) d\mu$$

and

$$\mathcal{Q}_\alpha\left(\frac{t^\varrho}{\varrho}\right) = \alpha\int_0^\infty \mu\mathcal{T}_\alpha(\mu) J\left(\left(\frac{t^\varrho}{\varrho}\right)^\alpha \mu\right) d\mu.$$

Definition 2.10 *[29] The operators $\mathcal{W}_\alpha(\frac{t^\varrho}{\varrho})$ and $\mathcal{Q}_\alpha(\frac{t^\varrho}{\varrho})$ satisfy the following characteristics:*

- *The operatos $\{\mathcal{W}_\alpha(\frac{t^\varrho}{\varrho})\}_{t>0}$ and $\{\mathcal{Q}_\alpha(\frac{t^\varrho}{\varrho})\}_{t>0}$ are linear, bounded and compact. Hence we obtain:*

$$\left|\mathcal{W}_\alpha\left(\frac{t^\varrho}{\varrho}\right)x\right| \leq \mathcal{K}|x| \text{ and } \left|\mathcal{Q}_\alpha\left(\frac{t^\varrho}{\varrho}\right)x\right| \leq \frac{\mathcal{K}}{\Gamma(\alpha)}|x|, \text{ with } \mathcal{K} > 0 \text{ and } x \in \mathbb{B}.$$

- *The operators $\{\mathcal{W}_\alpha(\frac{t^\varrho}{\varrho})\}_{t>0}$ and $\{\mathcal{Q}_\alpha(\frac{t^\varrho}{\varrho})\}_{t>0}$ are strongly continuous $\forall\ t_1, t_2 \in \mathbb{U}$, we have*

$$\left|\mathcal{W}_\alpha\left(\frac{t_2^\varrho}{\varrho}\right)x - \mathcal{W}_\alpha\left(\frac{t_1^\varrho}{\varrho}\right)x\right| \to 0, \qquad \left|\mathcal{Q}_\alpha\left(\frac{t_2^\varrho}{\varrho}\right)x - \mathcal{Q}_\alpha\left(\frac{t_1^\varrho}{\varrho}\right)x\right| \to 0, \quad \text{as } t_2^\varrho \to t_1^\varrho.$$

Let $\Xi : \Psi \to \Psi$ and $\Psi \subset \mathcal{FC}_\alpha$ (where, Ψ is non-empty). Let the solution of the equation be

$$(\Xi x)(t) = x(t). \tag{2.3}$$

We initiate the following concepts of attractivity of the solutions for the equation (2.3).

Definition 2.11 *[24] The equation (2.3) are Locally Asysmptotically Stable (LAS) or Locally Attractive (LA) in the space $\mathcal{FC}_\alpha$, there exist a ball $\mathcal{B}(x_0, i)$ in $\mathcal{FC}_\alpha$ such that solution of the equation is $u = u(t)$ and $v = v(t)$ of the equation (2.3) belongs to $\mathcal{B}(x_0, i) \cup \Psi$ then the equation is*

$$\lim_{t\to\infty}(u(t) - v(t)) = 0, \tag{2.4}$$

equation (2.4) is uniform with respect to the ball $\mathcal{B}(x_0, i) \cup \Psi$. Then the solution of the equation (2.3) is said to be uniformly LAS or uniformly LA.

Lemma 2.12 *[24] If $\mathscr{P} \subset \mathcal{FC}_\alpha$ then $\mathscr{P}$ is relatively compact in Space of all fuzzy numbers of continuous and bounded satisfied the bellow conditions:*

- *The mapping $\Xi : \Psi \to \Psi$ belong to $\mathscr{P}$ are almost equicontinuous in $\mathbb{U}$ that is a equicontinuous on every compact set in $\mathbb{U}$.*
- *The mapping Ξ is equiconvergent, that is, given $\omega > 0$, then there exist $\beta(\omega) > 0$ such that*

$$|x(t) - \lim_{t\to\infty} x(t)| < \omega,$$

  for every $x \in \mathscr{P}$, $t \geq \mathscr{P}(\omega)$.

- *$\mathscr{P}$ is uniformly bounded in $\mathcal{FC}_\alpha$.*

Theorem 2.13 *[22] Let $\mathbb{B}$ be a Banach space and $\mathbb{D}$ is a nonempty, closed, bounded, and convex subset of a Banach space $\mathbb{B}$, such that $\mathbb{S} : \mathbb{D} \to \mathbb{D}$ is a compact operator. Then $\mathbb{S}$ has at least one fixed point in $\mathbb{D}$.*

3. Existence of Mild Solution

The following results are established under the following hypotheses:

(K1) Let A be the infinitesimal generator of C_0-semigroup $\{T(t),\ t > 0\}$ in $\mathbb{F}$ such that $\|T(t)\| \leq \mathcal{K}$, where $\mathcal{K} \geq 1$ be the constant.

(K2) The function $f : \mathbb{U} \times \mathbb{F} \to \mathbb{F}$ satisfies Caratheodory conditions.

(K3) Then there exists a mapping $\Phi_f : \mathbb{U} \to \mathbb{U}$, $x \in \mathbb{F}$ such that:

$$\big|f(t, x(t))\big| \leq \Phi_f(t),\ \forall t \in \mathbb{U},$$

and

$$\lim_{t\to\infty} (\mathcal{I}_0^\alpha \Phi_f)(t) = 0.$$

We define

$$\mu = \frac{\mathcal{K} t^{\alpha\varrho} \mathbf{B}(1, \alpha)}{\Gamma(\alpha)\varrho^\alpha}, \quad \text{and} \quad \Phi_f^* = \sup_{t\in\mathbb{U}} \Phi_f(t).$$

Theorem 3.1 *Assume that the hypotheses $(K1) - (K3)$ are satisfied. Then, the equation (1.1) has at least one solution on $\mathbb{U}$, and the equation (2.2) is uniformly locally asymptotically stable.*

Proof: *Now, we consider the operator $\Xi : \mathcal{FC}_\alpha \to \mathcal{FC}_\alpha$, for any element of $t \in \mathbb{U}$.*

$$\Xi x(t) = \mathcal{W}_\alpha\left(\frac{t^\varrho}{\varrho}\right)x_0 + \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\mathcal{Q}_\alpha\left(\frac{t^\varrho - s^\varrho}{\varrho}\right) f(s, x(s))ds.$$

Next, we define $\mathcal{B}_T = \mathcal{B}(0, T) = \{x \in \mathcal{FC}_\alpha : \|x\|_{\mathcal{FC}_\alpha} \leq T\}$, it is clear that $\mathcal{B}_T$ is closed, bounded and convex subset of $\mathcal{FC}_\alpha$ with $\forall\ T > 0$, such that:

$$\big(\mathcal{K}|x_0| + \mu\Phi_f^*\big) \leq T.$$

Next to show that Ξ has a fixed point on $\mathcal{B}_T$.

Step-1 *The operator Ξ maps the set $\mathcal{B}_T$ into itself. Let $x \in \mathcal{B}_T$. Then, $\forall t \in U$ we obtain*

$$\begin{aligned}
\|\Xi x(t)\|_{\mathcal{FC}_\alpha} &= \sup_{t\in\mathbb{U}}\left|\mathcal{W}_\alpha\left(\frac{t^\varrho}{\varrho}\right)x_0 + \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\mathcal{Q}_\alpha\left(\frac{t^\varrho - s^\varrho}{\varrho}\right) f(s, x(s))ds\right| \\
&\leq \sup_{t\in\mathbb{U}}\left[\left|\mathcal{W}_\alpha\left(\frac{t^\varrho}{\varrho}\right)x_0\right| + \left|\int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\mathcal{Q}_\alpha\left(\frac{t^\varrho - s^\varrho}{\varrho}\right) f(s, x(s))ds\right|\right] \\
&\leq \mathcal{K}\bigg(|x_0| + \int_0^t \frac{(t^\varrho - s^\varrho)^{\alpha-1}}{\Gamma(\alpha)\varrho^{\alpha-1}} s^{\varrho-1}|f(s, x(s))|ds\bigg). \qquad (3.1)
\end{aligned}$$

By using $(K3)$. $\forall t \in \mathbb{U}$, we have

$$\big|f(t, x(t))\big| \leq \Phi_f(t).$$

Hence,

$$\|\Xi x(t)\|_{\mathcal{FC}_\alpha} \leq \big(\mathcal{K}|x_0| + \mu\Phi_f^*\big).$$

This implies that:

$$\|\Xi x(t)\|_{\mathcal{FC}_\alpha} \leq T. \qquad (3.2)$$

This demonstrates that Ξ maps the ball $\mathcal{B}_T$ onto itself, implying that Ξ is bounded.

Step-2 *The Ξ is continuous and uniformly bounded. If we consider the sequence of $\{x_n\}_{n\in N}$ such that $x_n \to x$ in $\mathcal{B}_T$, for every $t \in \mathbb{U}$ we have*

$$\|\Xi x_n(t) - \Xi x(t)\|_{\mathcal{FC}_\alpha} \leq \frac{\mathcal{K}}{\Gamma(\alpha)} \int_0^t \frac{(t^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1} |f_n(s, x(s)) - f(s, x(s))| ds. \tag{3.3}$$

If two cases, $t \in [0, \beta]$, and $t \in (\beta, \infty)$, where $\beta > 0$.

Case-(i)

By the Lebesgue dominated convergence theorem, if $t \in [0, \beta]$, $x_n \to x$ as $n \to \infty$ and the function f is continuous above equation implies that

$$\|\Xi x_n(t) - \Xi x(t)\|_{\mathcal{FC}_\alpha} \to 0, \quad n \to \infty.$$

Case-(ii)

If $t \in (\beta, \infty)$, from equation (3.3) and the hypotheses satisfies this equation which implies that

$$\|\Xi x_n(t) - \Xi x(t)\|_{\mathcal{FC}_\alpha} \leq \frac{2\mathcal{K}}{\Gamma(\alpha)} \int_0^t \frac{(t^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1} \Phi_f^* ds.$$

From the above equation converges to zero as $t \to \infty$, where $x_n \to x$ as $n \to \infty$ such that,

$$\|\Xi x_n(t) - \Xi x(t)\|_{\mathcal{FC}_\alpha} \to 0, \quad n \to \infty.$$

By the equation (3.2) satified bounded conditions. Therefore, $\Xi(\mathcal{B}_T) \subset \mathcal{B}_T$ is bounded. Hence, $\Xi(\mathcal{B}_T)$ is uniformly bounded.

Step-3 *$\Xi(\mathcal{B}_T)$ is equiconvergent and equicontinuous, where $\beta > 0$, $[0, \beta] \in \mathbb{U}$. Let $x \in \mathcal{B}_T$ and $t_1, t_2 \in [0, \beta], t_1 < t_2$ this yields*

$$\begin{aligned}
\|\Xi x(t_2) - \Xi x(t_1)\|_{\mathcal{FC}_\alpha} \leq & \frac{\mathcal{K}}{\Gamma(\alpha)} \bigg| \int_0^{t_2} \frac{(t_2^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1} f(s, x(s)) ds \\
& - \int_0^{t_1} \frac{(t_1^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1} f(s, x(s)) ds \bigg| \\
\leq & \frac{\mathcal{K}}{\Gamma(\alpha)} \int_0^{t_1} \frac{[(t_2^\varrho - s^\varrho)^{\alpha-1} - (t_1^\varrho - s^\varrho)^{\alpha-1}]}{\varrho^{\alpha-1}} s^{\varrho-1} |f(s, x(s))| ds \\
& + \frac{\mathcal{K}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{(t_2^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1} |f(s, x(s))| ds \\
\leq & \frac{\mathcal{K}\Phi_f^*}{\Gamma(\alpha)} \bigg[\frac{t_2^{\alpha\varrho}}{\varrho^{\alpha-1}} \int_0^{\frac{t_1}{t_2}} (1 - u^\varrho)^{\alpha-1} u^{\varrho-1} ds - \frac{t_1^{\alpha\varrho}}{\varrho^{\alpha-1}} \int_0^1 (1 - u^\varrho)^{\alpha-1} u^{\varrho-1} ds \\
& + \frac{t_2^{\alpha\varrho}}{\varrho^{\alpha-1}} \int_{\frac{t_1}{t_2}}^1 (1 - u^\varrho)^{\alpha-1} u^{\varrho-1} ds \bigg] \\
\leq & \frac{\mathcal{K}\Phi_f^*}{\Gamma(\alpha)} \int_0^1 (1 - u^\varrho)^{\alpha-1} u^{\varrho-1} ds \left[\frac{t_2^{\alpha\varrho}}{\varrho^{\alpha-1}} - \frac{t_1^{\alpha\varrho}}{\varrho^{\alpha-1}} \right] \\
\leq & \frac{\mathcal{K}\Phi_f^* \boldsymbol{B}(1, \alpha)}{\Gamma(\alpha)\varrho^\alpha} [t_2^{\varrho\alpha} - t_1^{\varrho\alpha}].
\end{aligned}$$

Therefore, $\|\Xi x(t_2) - \Xi x(t_1)\|_{\mathcal{FC}_\alpha} \to 0$ as $t_2^\varrho \to t_1^\varrho$. Hence, we conclude that $\Xi(\mathcal{B}_T)$ is equicontinuous.

Next to prove $\Xi(\mathcal{B}_T)$ is equiconvergent, from the equation (3.1) is given by

$$\begin{aligned}|(\Xi x)(t)| &\leq \mathcal{K}|x_0| + \frac{\mathcal{K}}{\Gamma(\alpha)}\int_0^t \frac{(t^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1}\Phi_f(s)ds \\ &\leq \mathcal{K}|x_0| + \mathcal{K}({}^\varrho I_0^\alpha \Phi_f)(s).\end{aligned}$$

Since, $\mathcal{K}({}^\varrho I_0^\alpha \Phi_f)(s) \to 0$, therefore

$$|(\Xi x)(t)| \leq \mathcal{K}|x_0| + \mathcal{K}({}^\varrho I_0^\alpha \Phi_f)(s) \to 0.$$

Hence, $|(\Xi x)(t) - (\Xi x)(+\infty)| \to 0$ as $t \to +\infty$.

From Steps 1, 2, and 3, and Lemma 2.12, we conclude that $\Xi : \mathcal{B}_T \to \mathcal{B}_T$ is both continuous and compact. By applying the Schauder fixed point theorem, Ξ has a fixed point x, which is a solution of the equation (1.1) on $\mathbb{U}$.

Step-4 *We prove local asymptotic stability by assuming that x_0 is a solution of the system (1.1) under the conditions specified in this theorem. By taking $x \to \mathcal{B}(x_0, 2\mu\Phi_f^*)$, we obtain*

$$\begin{aligned}\|(\Xi x)(t) - (x_0)(t)\|_{\mathcal{FC}_\alpha} &\leq \frac{\mathcal{K}}{\Gamma(\alpha)}\int_0^t \frac{(t^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1}|f_n(s,x(s)) - f(s,x(s))|ds \\ &\leq \frac{2\mathcal{K}}{\Gamma(\alpha)}\int_0^t \frac{(t^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1}\Phi_f(s)ds \\ &\leq 2\mu\Phi_f^*.\end{aligned}$$

We get,

$$\|\Xi(x) - x_0\|_{\mathcal{FC}_\alpha} \leq 2\mu\Phi_f^*$$

we determine that Ξ is continuous mapping such that

$$\Xi(\mathcal{B}(x_0, 2\mu\Phi_f^*)) \subset \mathcal{B}(x_0, 2\mu\Phi_f^*).$$

If x is a solution of the equation (1.1) then

$$\begin{aligned}|x(t) - x_0(t)| &= |(\Xi x)(t) - (\Xi x_0)(t)| \\ &\leq \frac{\mathcal{K}}{\Gamma(\alpha)}\int_0^t \frac{(t^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1}|f(s,x(s)) - f(s,x_0(s))|ds \\ &\leq 2\mathcal{K}({}^\varrho I_0^\alpha \Phi_f)(s).\end{aligned}$$

By using (K3), we obtain

$$\lim_{t\to\infty}(\mathcal{I}_0^\alpha \Phi_f)(t) = 0.$$

Therefore, we determined that

$$\lim_{t\to\infty}|x(t) - x_0(t)| = 0.$$

Finally, the system (1.1) is uniformly locally attractivity.

□

4. Example

Consider the equation of the form

$$\begin{cases} \left({}^{\varrho}D_0^{\frac{1}{3}}x\right)(t) = \frac{\partial^2}{\partial t^2}x(t) + f(t, x(t)), & t \in \mathbb{U}, \\ x(0) = x_0, \end{cases} \tag{4.1}$$

where ${}^{\varrho}D_0^{\frac{1}{3}}$ is the Caputo-Katugampola fuzzy fractional derivative of order $\alpha = \frac{1}{3}$, the additional parameter $\varrho > 0$, $f : \mathbb{U} \times \mathbb{F} \to \mathbb{F}$ is a fuzzy mapping and the linear operator $\mu = \frac{\partial^2}{\partial t^2} : D(\mu) \subset \mathbb{B} \to \mathbb{B}$ is continuous functions.

$$\begin{cases} f(t, x) = \frac{Mt^{\frac{-1}{3}} \sin t}{64(1+\sqrt{t})}, & t \in \mathbb{U}, \quad x \in \mathbb{F}, \\ f(0, x) = 0, & \qquad\qquad x \in \mathbb{F}, \end{cases}$$

and

$$M = \frac{9\sqrt{\pi}}{16}.$$

Therefore, the function f is continuous. The hypothesis (K3) is satisfied with

$$\begin{cases} \Phi_f(t) = \frac{Mt^{\frac{-1}{3}} \sin t}{64(1+\sqrt{t})}, & t \in (0, +\infty), \\ \Phi_f(0) = 0. \end{cases}$$

In addition, we have

$$\mathcal{I}_0^{\frac{1}{3}} \Phi_f(t) = \frac{1}{\Gamma\left(\frac{1}{3}\right)} \int_0^t (t-s)^{\frac{-2}{3}} \Phi_f(s) ds \to 0 \quad \text{as} \quad t \to \infty.$$

All conditions of Theorem 3.1 are satisfied. Hence, the problem (4.1) has at least one solution defined on $\mathbb{U}$, and moreover, the solution of this problem is locally asymptotically stable.

5. Conclusion

The study established sufficient conditions for the existence of locally attractive mild solutions of Caputo-Katugampola fuzzy fractional differential equations. The results were based on fixed-point methods and the use of fractional calculus together with semigroup theory. The analysis had been restricted to the fractional order $\alpha \in (0, 1)$, and the results addressed only local attractivity rather than global attractivity. Future research could explore the applicability of the proposed approach to Caputo-Katugampola fuzzy fractional differential equations of order $\alpha \in (1, 2)$. Moreover, numerical methods for approximating mild solutions and stability analysis in broader functional spaces present interesting avenues for further study.

Acknowledgements

The authors would like to thank the editor and reviewer for their valuable feedback and suggestions on enhancing the quality of the article.

Authors Contribution

Writers R. Hariharan and R. Udhayakumar made equal contributions to the research, conceptualization, and creation of the manuscript.

Conflict of interest

I consequently declare that there is no conflict of interest related to the publication of this essay, nor is there a conflict of interest with any other author or organization.

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