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Number of Partitions (with Distinct Parts) Having Largest (Least) Parts from a given Set

A. David Christopher and Mateus Alegri

ABSTRACT: The function $L_k(n, A, B)$ (respectively, $l_k(n, A, B)$) denotes the number of partitions of n with parts drawn from A, such that the largest (respectively, least) parts belong to B and occur exactly k times. Similarly, $Ld(n, A, B)$ (respectively, $ld(n, A, B)$) represents the number of partitions of n into distinct parts from A, with the largest (respectively, least) parts chosen from B. Generating functions corresponding to these partition functions are derived, leading to several new equinumerous identities. Moreover, a generalized formulation of Euler's theorem on partitions yields additional families of equinumerous results. As a further consequence, various estimates and congruences are established for enumerative functions arising from these primary definitions.

Keywords: Integer partitions, congruences, asymptotic formulas, partition identities.

Contents

1. Introduction

In this paper, we use the following standard notation:

$$(a)_n = (a, q)_n = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{if } n > 0; \\ 1, & \text{if } n = 0, \end{cases}$$

$$(a)_\infty = (a, q)_\infty = \lim_{n\to\infty} (a, q)_n, \text{for a complex number } q, \text{ where } |q| < 1.$$

Throughout this paper, we assume $|q| < 1$.

Among the many q-analytical identities studied, perhaps the following two, attributed to Euler, are the most well-known:

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} \frac{q^n}{(q)_n} \tag{1.1}$$

$$(q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} \tag{1.2}$$

The first identity appears in Chapter 16 of De Partitione Numerorum [9]. This identity can also be derived using Cauchy's formula, commonly known as the q-binomial theorem:

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$$\frac{(aq)_\infty}{(a)_\infty} = \sum_{n=0}^{\infty} \frac{(a)_n q^n}{(q)_n}.$$

In this paper, we derive identities such as equation 1.1 using both combinatorial and analytical techniques, as presented in Theorems 1, 2, 3, and 4. These theorems will also guide us in establishing concise analytical proofs for Theorems 6 and 7. To that end, we introduce the key definitions below.

Definition 1 *Let A be a set of positive integers and let n and k be two positive integers. Let $B \subseteq A$. We define $L_k(n, A, B)$ (resp. $l_k(n, A, B)$) to be the number of partitions of n with parts from the set A and exactly k largest (resp. least) parts from the set B.*

We extend Definition 1 to partitions with distinct parts.

Definition 2 *Let n be a positive integer. Let A and B are two sets of positive integers such that $B \subseteq A$. We define $Ld(n, A, B)$ (resp. $ld(n, A, B)$) to be the number of partitions of n with distinct parts from the set A and largest (resp. least) part from the set B.*

Recent studies have contributed significantly to the enumerations mentioned in these definitions. Binner and Rattan (2022) presented a comparative analysis of two sets of partitions based on their smallest parts [8]. In another work, Beck studied partitions with a fixed difference between the largest and smallest parts, providing asymptotic formulas and combinatorial interpretations [3]. Additionally, Binner and Rattan (2020) explored conjectures concerning the smallest part and missing parts of integer partitions, proving several results that extend classical theorems [7]. Mahanta and Saikia (2023) extended formulas related to partitions where the smallest part occurs multiple times, focusing on cases involving fixed differences between the largest and smallest parts [13]. Further, Berkovich and Uncu (2017) established inequalities involving the largest part and restrictions on part frequencies, which yield important implications for partitions with bounded differences between the largest and least parts [5]. Archibald, Blecher, Brennan, Knopfmacher, and Mansour (2014) examined partitions in which the difference between the largest and smallest parts is a fixed integer [1]. They derived explicit formulas for the number of such partitions, providing valuable insight into structural properties of these partitions. In another significant contribution, Archibald et al. (2016) analyzed partitions according to the multiplicities and sizes of the largest parts [2]. Their work explores partitions where the largest part appears with a specific multiplicity or when the sum of the largest parts is considered under different constraints.

Further exploration of the functions in Definitions 1 and 2 yields new but non-trivial equinumerous results. Additionally, a generalized approach to Euler's proof of his equinumerous result leads to new equinumerous results, estimates, and congruences.

2. Derivations of Generating Functions and Equinumerous Theorems

2.1. Generating Functions

Following generating functions forms an essential part of this paper.

Theorem 1 *Let A be a set of positive integers and $B \subseteq A$. We have*

(a)

$$\sum_{n=1}^{\infty} L_k(n, A, B) q^n = \sum_{b \in B} q^{bk} \left(\prod_{\substack{a \in A; \\ a < b}} \frac{1}{(1 - q^a)} \right). \tag{2.1}$$

(b)

$$\sum_{n=1}^{\infty} l_k(n, A, B) q^n = \sum_{b \in B} q^{bk} \left(\prod_{\substack{a \in A; \\ a > b}} \frac{1}{(1 - q^a)} \right). \tag{2.2}$$

Let $b \in B$. Let $L_k(n, A, b)$ be the number of partitions of n with parts from the set A having b as the largest part which appears exactly k times. Let $p_A(n)$ be the number of partitions of n with parts from the set A. Now from the following well known generating function:

$$\sum_{n=0}^{\infty} p_A(n)q^n = \prod_{a\in A} \frac{1}{1-q^a}$$

it follows that

$$\sum_{n=1}^{\infty} L_k(n, A, b)q^n = q^{bk} \prod_{\substack{a\in A;\\ a<b}} \frac{1}{(1-q^a)}.$$

Since $L_k(n, A, B) = \sum_{b\in B} L_k(n, A, b)$, part (a) follows.

Let $l_k(n, A, b)$ be the number of partitions of n with parts from the set A having b as the least part with b appearing exactly k times. Now it follows that

$$\sum_{n=1}^{\infty} l_k(n, A, b)q^n = q^{bk} \prod_{\substack{a\in A;\\ a>b}} \frac{1}{(1-q^a)}.$$

Since $l_k(n, A, B) = \sum_{b\in B} l_k(n, A, b)$, part (b) follows.

□

The appearance of the generating functions above may be applied to obtain new results. First we derive few well-known q-series identities. We omit the proof of the q-series identities, as they follow directly from the generating functions associated with the enumerations in Definitions 1 and 2.

2.2. q-Series Identities

In this subsection, we present several fundamental q-series identities that are central to our analysis of partition functions and generating functions. These identities, many of which trace their origins to Euler, encapsulate deep combinatorial relationships that allow us to express infinite product expansions in terms of infinite sums, and vice versa.

The first theorem below, Theorem 2, collects some of the most classical q-series identities. These identities reveal powerful ways to decompose infinite products involving $(q)_\infty$ into simpler sum forms, which are crucial in our derivations of generating functions later in this work.

Following that, Theorem 3 presents results concerning generating functions for partitions into distinct parts, where the parts are drawn from specified sets. These results highlight the flexibility of the q-series framework in enumerating partition functions with specific restrictions.

Theorem 2 *(a)(Euler)*

$$\frac{1}{(q)_\infty} = 1 + \sum_{m=1}^{\infty} \frac{q^m}{(q)_m}. \tag{2.3}$$

(b)

$$(q)_\infty = 1 - \sum_{m=1}^{\infty} q^m (q^{m+1})_\infty \tag{2.4}$$

(c)

$$\frac{1}{(q)_\infty} = 1 + \sum_{m=1}^{\infty} \frac{q^m}{(q^m)_\infty}. \tag{2.5}$$

(d)

$$(q)_\infty = 1 - \sum_{m=1}^{\infty} q^m (q^{m-1})_\infty. \tag{2.6}$$

Theorem 3 *We have*

$$\sum_{n=1}^{\infty} Lq(n,A,B)q^n = \sum_{b\in B} q^b \prod_{\substack{a\in A;\\ a<b}} (1+q^a). \tag{2.7}$$

Let $A = \{a_1, a_2 \ldots\}$ with $a_1 < a_2 < \cdots$ and let $B = \{b_1, b_2 \ldots\}$ with $b_1 < b_2 < \cdots$. We recall that if $q_K(n)$ denotes the number of distinct partitions of n with parts from the set K. Then

$$\sum_{n=0}^{\infty} q_K(n)x^n = \prod_{k\in K}(1+x^k).$$

with $q_K(0) = 1$.

Next we observe that the generating function for the number of distinct partitions with parts from the set A and having largest part b_j is

$$\prod_{\substack{a_i\in A;\\ a_i<b_j}} x^{b_j}(1+x^{a_i}).$$

Since b_j runs over every element of B, part (a) follows. □

Definition 3 *Let n be a positive integer and let A be a set of positive integers. Denote by $Q_A(n)$, the number of partitions of n with distinct parts from the set A.*

Now we have some q-series identities.

Theorem 4 *We have*

(a)

$$(-q)_\infty = 1 + \sum_{m=1}^{\infty} q^m(-q)_{m-1}. \tag{2.8}$$

(b)

$$\frac{1}{(-q)_\infty} = 1 - \sum_{m=1}^{\infty} \frac{q^m}{(-q^m)_\infty}. \tag{2.9}$$

(c)

$$(-q)_\infty = 1 + \sum_{m=1}^{\infty} q^m(-q^{m+1})_\infty. \tag{2.10}$$

(d)

$$\frac{1}{(-q)_\infty} = 1 - \sum_{m=1}^{\infty} \frac{q^m}{(-q)_{m-1}}. \tag{2.11}$$

2.3. Equinumerous Results

The derivations of the generating functions given in Theorem 1 and Theorem 3 gives the following equinumerous results.

Theorem 5 *(a) The number of partitions of n with parts from the set A having largest part $b \in A$ is equal to the number of partitions of $n-b$ with parts from the set $A \cap \{1,2,\ldots b\}$.*

(b) The number of partitions of n with parts from the set A having least part $b \in A$ equals the number of partitions of $n-b$ with parts from the set $A \cap \{k, k+1, \cdots, \}$.

The following results are discussed in the introduction part. Here we obtain a proof of non-bijective nature.

Theorem 6 *For $n \geq 2$, we have*

(a) The number of partitions of n with exactly one largest part is equal to the number of partitions of $n-1$.

(b) The number of partitions of n with exactly one least part is equal to the number of partitions of $n+1$ with parts greater than one.

Let $L(n)$ denotes the number of partitions of n with exactly one largest part. Then we have

$$\sum_{n=2}^{\infty} L(n)q^n = \frac{q^2}{(1-q)} + \frac{q^3}{(1-q)(1-q^2)} + \frac{q^4}{(1-q)(1-q^2)(1-q^3)} + \cdots .$$

Dividing by q and adding one we have

$$\begin{aligned} 1 + \sum_{n=2}^{\infty} L(n)q^{n-1} &= 1 + \sum_{n=1}^{\infty} L(n+1)q^n \\ &= 1 + \frac{q}{1-q} + \frac{q^2}{(1-q)(1-q^2)} + \frac{q^3}{(1-q)(1-q^2)(1-q^3)} + \cdots \\ &= \sum_{n=0}^{\infty} p(n)q^n. \end{aligned}$$

Equating the coefficients of q^n gives part (a).

Let $l(n)$ denotes the number of partitions of n with exactly one least part. Then we have

$$\sum_{n=1}^{\infty} l(n)q^n = \frac{q}{(1-q^2)(1-q^3)\cdots} + \frac{q^2}{(1-q^3)(1-q^4)\cdots} + \cdots .$$

This gives

$$\begin{aligned} q\sum_{n=1}^{\infty} l(n)q^n + 1 + q\prod_{m=1}^{\infty}(1-q^m)^{-1} &= 1 + \frac{q}{(1-q)(1-q^2)\cdots} + \frac{q^2}{(1-q^2)(1-q^3)\cdots} + \cdots \\ &= \prod_{n=1}^{\infty}(1-q^m)^{-1}, \end{aligned}$$

which on simplification gives

$$\begin{aligned} 1 + \sum_{n=2}^{\infty} l(n-1)q^n &= \prod_{n=1}^{\infty}(1-q^m)^{-1}(1-q) \\ &= \frac{1}{(1-q^2)(1-q^3)\cdots}. \end{aligned}$$

Since the right extreme is the generating function of the number of partitions of n with parts greater than one, part (b) follows.

□

Further exploration of the generating function gives the following result.

Theorem 7 *For $n \geq 2$, we have*

(a) The number of partitions of n into distinct parts is equal to the number of partitions of $n-1$ in which the largest part may repeat once, and all other parts are distinct.

(b) The number of partitions of n into distinct parts is equal to the number of partitions of $n-1$, denoted as $\lambda_1 + \lambda_2 + ... + \lambda_s$, into distinct parts satisfying the condition $\lambda_s - 1 - \lambda_2 \geq 2$.

Using Theorem 4, part (a), we obtain:

$$(-q)_\infty = 1 + \sum_{m=1}^{\infty} q^m (q)_{m-1} = 1 + q \sum_{m=1}^{\infty} q^{m-1} (q)_{m-1}.$$

The result of part (a) follows by comparing the coefficients of q^n on both sides of the preceding equation.
Using the part (c) of Theorem 4, we get

$$(-q)_\infty = 1 + \sum_{m=1}^{\infty} q^m (q)_{m+1} = 1 + q \sum_{m=1}^{\infty} q^{m-1} (q)_{m+1}.$$

Since the right-hand side represents the generating function for the number of partitions of n, expressed as $\lambda_1 + \lambda_2 + ... + \lambda_s$, into distinct parts satisfying the condition $\lambda_s - 1 - \lambda_2 \geq 2$, part (b) of our theorem holds. □ Margaret Archibald et al. presented the generating function for the number of partitions of n such that the largest part appears exactly k times (see Theorem 2.3 in [2]) by

$$\frac{q^k}{\prod_{j \geq k} (1 - q^j)}.$$

This can be written as

$$q \frac{q^{k-1}}{\prod_{j \geq k} (1 - q^j)}.$$

Then considering (b) of Theorem 1 we have the following result.

Theorem 8 *The number of partitions of n such that largest part appears exactly k times equals the number of partitions of $n-1$ with $k-1$ as the least part appearing exactly once.*

In corollary 2.5 of [2], the generating function for the total multiplicity of the largest part in all partitions of n is given by

$$\sum_{j \geq 1} \frac{jt^j}{\prod_{i \geq j}} (1 - t^i) = \frac{1}{\prod_{i \geq 1} (1 - t^i)} \sum_{j \geq 1} jt^j \prod_{i=1}^{j-1} (1 - t^i).$$

By comparing the left-hand side of the above expression with the generating function for $L_k(n, A, B)$, we obtain the following result.

Theorem 9 *The total multiplicity of the largest parts in all partitions of n is equal to the number of partitions of $n-1$ plus the sum of least parts in partitions of $n-1$ in which least part appearing exactly once plus the number of least parts in partitions of $n-1$ in which least part appearing exactly once.*

2.4. In Euler's Perspective

The first notable equinumerous result in the theory of partitions was established by Euler [12], who proved that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts. He demonstrated this result using the following identity, which involves a careful cancellation of terms:

$$\prod_{i=1}^{\infty} (1 + q^i) = \prod_{j=1}^{\infty} \frac{1}{1 - q^{2j-1}}.$$

In what follows, we introduce a class of integer sets that facilitates a structured approach to such cancellations. This enables us to derive new equinumerous results.

Definition 4 *Let $A \subseteq \mathbb{N}$. We say that A is a* Power-of-2-Indivisible Set *if, for every $a \in A$ and every $k \geq 1$, we have $2^k a \notin A$.*

Definition 5 *Let A be a Power-of-2-Indivisible set. We define $2^{\mathbb{N}}$-closure of A, denoted $cl_{2^{\mathbb{N}}}(A)$, as follows*

$$cl_{2^{\mathbb{N}}}(A) = \cup_{k=0}^{\infty} 2^k A.$$

Now we obtain a generalization of Euler's assertion by involving the sets of the above definitions.

Theorem 10 *Let A be a Power-of-2-Indivisible set. Then the number of partitions of n with distinct parts from the set $cl_{2^{\mathbb{N}}}(A)$ is equal to the number of partitions of n with parts from the set A.*

We observe that

$$\begin{aligned}\prod_{a\in cl_{2^{\mathbb{N}}}(A)}(1+x^a) &= \prod_{a\in cl_{2^{\mathbb{N}}}(A)}\frac{(1+x^a)(1-x^a)}{(1-x^a)}\\ &= \prod_{a\in cl_{2^{\mathbb{N}}}(A)}\frac{(1-x^{2a})}{(1-x^a)}\\ &= \frac{\prod_{a\in 2cl_{2^{\mathbb{N}}}(A)}(1-x^a)}{\prod_{a\in A}(1-x^a)\prod_{a\in 2cl_{2^{\mathbb{N}}}(A)}(1-x^a)}\\ &= \prod_{a\in A}\frac{1}{1-x^a}.\end{aligned}$$

The result follows. □

Remark 1

1. *The choice of A as the set of odd positive integers establishes Euler's assertion.*
2. *Setting $A = \{1\}$ yields the following well-known q-series identity:*

$$\prod_{n=0}^{\infty}(1+q^{2^n}) = \frac{1}{1-q},$$

which is equivalent to the uniqueness of the base-2 representation of positive integers.

In the following result, we derive an equinumerous result while incorporating considerations on the least part.

Theorem 11 *Let A be a Power-of-2-Indivisible set. The number of partitions of n with distinct parts from $cl_{2^{\mathbb{N}}}(A)$ and least part $b \in A$ is equal to the number of partitions of $n-b$ with parts from the set $A \cap \{b+1, \ldots\}$.*

The following equalities establishes the result:

$$\begin{aligned}q^b \times \prod_{\substack{a\in cl_{2^{\mathbb{N}}}(A), a>b\\ b\in A}}(1+q^a) &= q^b \times \prod_{\substack{a\in cl_{2^{\mathbb{N}}}(A), a>b\\ b\in A}}\frac{(1+q^a)(1-q^a)}{(1-q^a)}\\ &= q^b \times \frac{\prod_{t\in 2cl_{2^{\mathbb{N}}}(A), \frac{t}{2}>b, b\in A}(1-q^t)}{\prod_{a\in A, a>b}(1-q^a)\prod_{s\in 2cl_{2^{\mathbb{N}}}(A), \frac{s}{2}>b}(1-q^s)}\\ &= q^b \times \frac{1}{\prod_{a\in A, a>b}(1-q^a)}.\end{aligned}$$

□

3. Estimates and Congruences

Let $p_A(n)$ denote the number of partitions of n with parts belonging to A. If A is a finite set of positive integers, say $\{a_1, \ldots, a_k\}$ with $\gcd(a_1, \ldots, a_k) = 1$, there is a well-known estimate for $p_A(n)$:

$$p_A(n) \sim \frac{n^{k-1}}{(a_1 \cdots a_k)(k-1)!}.$$

Many proofs have been obtained for this result, refer [15,17,11,20,14,10]. However, fewer results of this type are known when A is an infinite set. In the special case $A = \mathbb{N}$, the following estimate was provided by Ramanujan [18]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

In view of Theorem 10, we obtain an estimate for the number of partitions of n with distinct parts from the set $cl_{2^{\mathbb{N}}}(A)$ (which is an infinite set) when A is a finite Power-of-2-Indivisible set.

Theorem 12 *Let A be a finite Power-of-2-Indivisible set with $\gcd(A) = 1$. Let $Q_{cl_{2^{\mathbb{N}}}(A)}(n)$ be the number of partitions of n with distinct parts from the set $cl_{2^{\mathbb{N}}}(A)$. We have*

$$Q_{cl_{2^{\mathbb{N}}}(A)} \sim \frac{n^{|A|-1}}{\left(\prod_{a \in A} a\right)(|A|-1)!}.$$

As a consequence of Theorem 11 we have the following estimate.

Theorem 13 *Let A be a finite Power-of-2-Indivisible set. Define $B = A \cap \{b+1, b+2, \ldots\}$. We have*

$$ld(n, cl_{2^{\mathbb{N}}}(A), \{b\}) \sim \frac{n^{|B|-1}}{\left(\prod_{b \in B} b\right)(|B|-1)!}.$$

The origin of the concept of quasipolynomials remains unclear; however, we provide below the definition of a quasipolynomial as stated in [21].

Definition 6 *An arithmetical function f is said to be a quasipolynomial with quasiperiod α (a positive integer) if, for each residue r in the range $0, 1, \ldots, \alpha - 1$, the expression $f(\alpha t + r)$ is a polynomial in t. The integer α is referred to as a quasiperiod of f. Each polynomial $f(\alpha t + r)$ is called a constituent polynomial of f.*

The function $p_A(n)$ is known to be a quasi-polynomial with a quasiperiod of $\prod_{a \in A} a$ when A is a finite set of relatively prime positive integers. Moreover, each constituent polynomial of $p_A(n)$ is a polynomial of degree $|A| - 1$ This assertion has been established independently by E. T. Bell [4], E. M. Wright [22], O. J. Rodseth et al. [19], and David Christopher et al. [10].

As a consequence of Theorem 10 and Theorem 13, it follows that the functions $Q_{cl_{2^{\mathbb{N}}}(A)}(n)$ and $ld(n, cl_{2^{\mathbb{N}}}(A), \{b\})$ exhibit quasi-polynomial structure when A is a finite Power-of-2-Indivisible set. This structural property can be leveraged to derive congruences for $Q_{cl_{2^{\mathbb{N}}}(A)}(n)$ and $ld(n, cl_{2^{\mathbb{N}}}(A), \{b\})$ in specific instances of A. We conclude this paper by presenting a few such illustrations.

To derive congruence relations from the polynomial structure, we require the following two results. The first follows from the facts that: (i) a polynomial vanishes when the order of the forward difference exceeds its degree, and (ii) $p \mid \binom{p}{k}$ for a prime p and $1 \leq k \leq p-1$.

Lemma 1 *Let $f(x)$ be a polynomial of degree k with rational coefficients such that $f(t)$ is an integer for every integer $t \geq 0$ and let p be a prime number greater than k. If r is a non-negative integer such that $f(r) \equiv 0 \pmod{p}$ then we have $f(mp + r) \equiv 0 \pmod{p}$ for every positive integer m.*

Lemma 2 *Let $f(x)$ be a polynomial with integer coefficients and let $m \geq 2$ be a positive integer. If r is a positive integer such that $f(r) \equiv 0 \pmod{m}$ then, we have $f(mn + r) \equiv 0 \pmod{m}$ for every positive integer n.*

Consider the set $A = \{1, 3, 5\}$ which is clearly a Power-of-2-Indivisible set. Also we have $B = cl_{2^{\mathbb{N}}}(A) = \{2^n : n \in \mathbb{N}\} \cup \{3 \times 2^n : n \in \mathbb{N}\} \cup \{5 \times 2^n : n \in \mathbb{N}\}$. Here $p_A(n)$ is a quasipolynomial with quasiperiod 15. The degree of each constituent polynomial is 2. The choice of primes 3, 5 and 7 gives the following result as an application of Lemma 1, since $p_A(5) \equiv 0 \pmod 3$, $p_A(8) \equiv 0 \pmod 5$ and $p_A(10) \equiv 0 \pmod 7$.

Theorem 14 *For $B = \{2^n : n \in \mathbb{N}\} \cup \{3 \times 2^n : n \in \mathbb{N}\} \cup \{5 \times 2^n : n \in \mathbb{N}\}$, we have*

1. $Q_B(15m + 5) \equiv 0 \pmod 3$;

2. $Q_B(15m + 8) \equiv 0 \pmod 5$;

3. $Q_B(15m + 10) \equiv 0 \pmod 7$.

Since $p_A(9) \equiv 0 \pmod 6$, $p_A(11) \equiv 0 \pmod 8$ and $p_A(14) \equiv 0 \pmod{11}$, we have the following result as an application of Lemma 2.

Theorem 15 *For $B = \{2^n : n \in \mathbb{N}\} \cup \{3 \times 2^n : n \in \mathbb{N}\} \cup \{5 \times 2^n : n \in \mathbb{N}\}$, we have*

1. $Q_B(6n + 9) \equiv 0 \pmod 6$;

2. $Q_B(8n + 11) \equiv 0 \pmod 8$;

3. $Q_B(11n + 14) \equiv 0 \pmod{11}$.

Now consider the set $A' = \{1, 3, 5, 7\}$. Then in view of Theorem 11, we have $ld(n, cl_{2^{\mathbb{N}}}(A'), \{3\}) = p_{\{5,7\}}(n - 3)$. The values $p_{\{5,7\}}(75) \equiv 0 \pmod 3$ and $p_{\{5,7\}}(140) \equiv 0 \pmod 5$ gives the following result as an application of Lemma 1.

Theorem 16 *For $m \geq 2$ and $B = \{t2^n : t = 1, 3, 5, 7, n \in \mathbb{N}\}$, we have*

1. $ld(35m + 8, B, \{3\}) \equiv 0 \pmod 3$;
2. $ld(35m + 3, B, \{3\}) \equiv 0 \pmod 5$.

An application of Lemma 2 gives the following result.

Theorem 17 *For $B = \{t2^n : t = 1, 3, 5, 7, n \in \mathbb{N}\}$ we have*

1. $ld(3m, B, \{3\}) \equiv 0 \pmod 3$ *for every* $m \geq 26$;
2. $ld(5m + 3, B, \{3\}) \equiv 0 \pmod 5$ *for every* $m \geq 29$.

4. Concluding Remarks

In this paper, we investigated several classes of partition functions arising from q-analogues and restricted combinatorial conditions, establishing new identities and relations among them. By exploring suitable generating functions and analytic transformations, we were able to connect these quantities with classical partition functions, including the number of partitions into distinct parts.

Our results clarify how certain parameters affect the behavior of the associated counting functions, showing that, in specific cases, they coincide with well-known partition functions. This provides a unifying perspective that helps to explain previous observations in the literature and corrects some misconceptions regarding the independence of these quantities.

Beyond their intrinsic combinatorial interest, the identities obtained here suggest possible extensions in different directions. For instance, analogous constructions may be developed for partitions subject to congruence conditions or weighted restrictions, as well as for multivariate or higher-dimensional generating functions. Another promising direction is the investigation of asymptotic properties of the functions considered, which may shed further light on their growth and structural behavior.

We hope that the approach adopted in this work contributes to a deeper understanding of the interplay between q-series, partition theory, and combinatorial identities, and that it stimulates further research in this active area of mathematics.

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A. David Christopher,
Department of Mathematics,
The American College, Madurai, Tamil Nadu
India.
E-mail address: davidchristopher@americancollege.edu.in

and

Mateus Alegri,
Department of Mathematics (DMAI),
Universidade Federal de Sergipe,
Brasil.
E-mail address: malegri@academico.ufs.br