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Advanced Numerical Techniques for Solving Fractional Groundwater Flow Equations

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ABSTRACT: The paper explores advanced numerical methods to solve groundwater flow equations, with a particular emphasis on the variational iteration method. Groundwater flow modeling is crucial for understanding and managing water resources, but the complex nature of these systems often leads to equations that are challenging to solve analytically. We demonstrate the application of the variational iteration method to a specific groundwater flow equation, comparing its performance and accuracy to traditional numerical methods. Our results show that the variational iteration method offers significant advantages in terms of computational efficiency and solution accuracy, particularly for non linear groundwater flow problems. This study helps to develop the growing field of research on innovative numerical methods in hydrogeology and provides information on the potential of the variational iteration method for broader applications in environmental modeling.

Keywords: Groundwater flow, groundwater recharge, numerical methods, variational iteration method, Caputo derivative, hydrogeology, environmental modeling, fractional differential equations.

Contents

1. Introduction

Groundwater refers to all water located below the Earth's surface [1,2,3]. Historically, well diggers noticed that water would naturally flow into a well only when it reached a certain depth, now known as the water table. The depth of this water table varies by location but tends to form a relatively uniform layer over short distances between nearby wells. Early observations led to the belief that underground water existed only below this horizon. However, it is now understood that moisture is also present in the soil above the water table as well. Despite this, only the water below the water table can be efficiently extracted through wells. As a result, engineers define groundwater as subsurface water found beneath the water table.

The study of groundwater flow has evolved significantly since the early twentieth century, with notable progress in both theoretical concepts and computational methods in recent decades. The mathematical representation of groundwater movement in porous media is based on Darcy's law combined with the principle of mass conservation, leading to the general form of the groundwater flow equation [6]:

$$\nabla \cdot (K \nabla h(x,t)) + W = S_s \frac{\partial h(x,t)}{\partial t}, \tag{1.1}$$

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where K represents the hydraulic conductivity tensor and h denotes the hydraulic head. The Sources or sinks are represented by W, S_s denotes the specific storage, and t represents time.

Here, equation (1.1), while seemingly simple, can become highly complex when considering heterogeneous and anisotropic aquifers, transient conditions, and various boundary conditions [7]. Numerous studies have focused on developing and refining numerical methods to solve this equation in different scenarios [8,9,10].

The fractional-order time-dependent groundwater flow equation using Caputo's fractional derivative of order α is expressed as

$$\frac{\partial^\alpha h(x,t)}{\partial t^\alpha} = \frac{1}{S}\frac{\partial}{\partial x}\left(K(x)h(x,t)\frac{\partial h(x,t)}{\partial x}\right) + W(x,t) \tag{1.2}$$

where:

- h(x,t) is the hydraulic head [L]
- t is the time [T]
- x is the spatial coordinate [L]
- S is the specific yield [-]
- K(x) is the spatially variable hydraulic conductivity [L/T]
- W(x,t) is a source/sink term [L/T]

This equation represents a non-linear, second-order partial differential equation that describes the flow of groundwater in an unconfined aquifer exhibiting spatial variability in hydraulic conductivity. The non-linearity arises from the product of h and ∂h/∂x in the flux term, while the spatial variability of $K(x)$ introduces additional complexity.

To complete the problem formulation, the specific initial and boundary conditions are as follows:

$$\begin{aligned} h(x,0) &= f(x) \\ h(0,t) &= g_1(t) \\ h(L,t) &= g_2(t) \end{aligned}$$

where $f(x)$ is the initial hydraulic head distribution, $g_1(t)$ is the prescribed head value at the boundary $x = 0$ and $g_2(t)$ is the prescribed head value at the boundary $x = L$.

This formulation represents a challenging problem for traditional numerical methods due to its non-linearity and the potential for sharp gradients in hydraulic head, especially in areas with rapidly changing hydraulic conductivity.

Fractional calculus, initially explored by Liouville, Riemann, and Leibniz in the 17th century, was once deemed a theoretical curiosity. However, it is now recognized as a versatile tool for modeling processes exhibiting nonlocality and memory dependence. In the past decade, its application has expanded in physics, biology, engineering, and hydrology. Fractional operators such as Caputo, Atangana–Baleanu, and general fractional derivatives have been successfully employed to describe anomalous diffusion, hereditary material response, and transport in heterogeneous systems [36,37,38,39].

In the process of exploring fractional calculus, multiple research studies provided a vast number of results helping to discover the new extremes. From the medical field, physics and chemistry to economics, fractional calculus seemed to be an all-rounder applied part of mathematics in almost every aspect of study. Fractional calculus can be said as the generalisation of the calculus of integer-order or the classical calculus.

Recent studies further highlight the applicational importance of fractional calculus in diverse fields: fractional models of tuberculosis infection provide improved realism in memory-driven recovery processes [40]; fractional Burgers equations using the Homotopy Perturbation Method offer superior characterization of dissipative flow behavior [41]; and fractional cancer–virotherapy models capture time-dependent interactions between cell populations and viral dynamics with higher biological fidelity [42]. These findings emphasize that fractional frameworks effectively capture history-dependent processes, establishing their relevance to subsurface hydrodynamics, where groundwater evolution often reflects long-term aquifer memory.

Moreover, fractional-order formulations of the Navier-Stokes and multi-layer flow problems demonstrate that such models generalize classical transport laws to account for anomalous fluid diffusion in porous and layered media [36,38]. Collectively, these works justify the adoption of the present study of a time-fractional groundwater flow model for describing recharge and flow in unconfined aquifers, where diffusion and retardation are inherently nonlocal phenomena.

Traditional numerical techniques such as finite difference, finite element, and finite volume methods have been widely applied to simulate groundwater flow [11,30,12]. Although these schemes are robust, their performance often degrades in handling nonlinearity, sharp gradients, or fractional derivatives. As a result, recent attention has turned toward hybrid analytical–numerical approaches that combine flexibility with computational efficiency.

Among these, the Variational Iteration Method introduced by He [4] stands out for its simplicity, convergence properties, and adaptability to nonlinear and fractional differential equations. Several studies have successfully applied VIM to groundwater problems. Boustani and Mohammadi [25] utilized it to solve the Boussinesq equation for unconfined aquifers, showing its precision compared to numerical solutions. Rafei et al. [26] applied VIM to nonlinear confined aquifer equations, yielding results consistent with exact solutions, while Jafari et al. [27] extended the method to unsaturated flow via a modified VIM. Younesi et al. [28] further analyzed sloping aquifers, confirming its flexibility under complex boundary conditions.

Building upon these advancements, the Fractional Variational Iteration Method extends Variational Iteration Method to fractional-order problems by integrating fractional calculus concepts, offering semi-analytical solutions that converge rapidly to the exact form. This approach efficiently models the inherent memory and nonlocal diffusion effects in groundwater flow.

These methods have been widely applied and refined over the years, with numerous variations and hybrid approaches developed to address specific challenges in groundwater modeling [13,14,15].

Recent theoretical works further support hybrid formulations combining spectral collocation for spatial discretization and L1 quadrature for temporal fractional derivatives [36,37]. Such schemes achieve high spatial accuracy and stable time advancement, aligning with the methodology of the current study. There has been a surge in interest in advanced analytical-numerical techniques that can complement or, in some cases, outperform traditional numerical methods. These techniques often offer advantages in terms of computational efficiency, accuracy, or both, particularly for certain classes of problems. Some notable methods includes the homotopy analysis method, introduced by Liao [16], which has been applied to various nonlinear problems in fluid mechanics and heat transfer, including some applications in groundwater flow [17]. The Adomian decomposition method, which was developed by Adomian [18], has shown promise in solving nonlinear differential equations in various fields, including hydrogeology [19]. The differential transform method, a semi-analytical technique, has been used in groundwater flow problems and is known for its rapid convergence for certain types of equations [20]. The variational iteration method, introduced by He [4], has gained attention for efficiently solving a wide range of nonlinear problems, and although its application in hydrogeology is relatively recent, several studies have highlighted its potential for modeling groundwater flow [21,22].

The variational iteration method has shown promising results in various areas of science and engineering, which include fluid dynamics, heat transfer, and wave propagation [23,24]. In the field of hydrogeology, the variational iteration method has been applied to several problems related to groundwater, such as Boustani and Mohammadi [25], who used the variational iteration method to solve the Boussinesq equation for groundwater flow in unconfined aquifers, demonstrating its precision and efficiency compared to numerical solutions. Rafei et al. [26] applied the variational iteration method. to nonlinear groundwater flow equations in confined aquifers, showing good agreement with exact solutions and numerical methods. Jafari et al. [27] employed a modified variational iteration method to solve the Richards equation for unsaturated flow in porous media, highlighting the ability of the method to handle non-linear partial differential equations. Younesi et al. [28] applied the variational iteration method to analyze groundwater flow in sloping aquifers, showcasing the versatility of the method in dealing with complex boundary conditions.

These studies suggest that the variational iteration method has a significant potential to solve groundwater flow equations, particularly in cases where traditional numerical methods may face challenges. However, the application of the variational iteration method to more complex, multi-dimensional groundwater flow problems remains an area of active research.

In the following sections, we build upon this existing body of work by applying the variational iteration method to a specific groundwater flow equation, demonstrating its implementation, and comparing its performance to traditional numerical methods.

2. Mathematical Preliminaries

The Italian mathematician Caputo in 1967 proposed the idea of Caputo's fractional operator. Let $\alpha > 0$, $t > a$, $a, \alpha, x \in \mathbb{R}$ and $n = \lceil \alpha \rceil$ then the Caputo's fractional derivative of order α is given as [30]

$$ {}^{C}{}_{a}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-r)^{n-\alpha-1}(f^n r)dr \quad (n-1 < \alpha < n) n \in \mathbb{N} \tag{2.1} $$

where $\mathbb{N}$ denote the set of positive integers. Caputo stated that a function should be differentiable to be differentiated under Caputo's derivative [29].

3. Fractional Variational Iteration Method

The fractional variational iteration method is an iterative technique used to solve fractional differential equations, which involve derivatives of non-integer order. It extends the standard variational iteration method by incorporating the principles of fractional calculus. The Fractional Variational Iteration Method constructs a correction functional with a Lagrange multiplier, derived using variational theory, to iteratively refine approximate solutions. The method handles both linear and nonlinear problems, offering high accuracy without requiring discretization. Its iterative process ensures rapid convergence to the solution.

The non-linear groundwater flow equation is as follows:

$$ \frac{\partial h(x,t)}{\partial t} = \frac{1}{S}\frac{\partial}{\partial x}\left(K(x)h(x,t)\frac{\partial h(x,t)}{\partial x}\right) + W(x,t). \tag{3.1} $$

In general form, $K(x)$ and $W(x,t)$ represent the spatial and temporal variability of hydraulic conductivity and recharge. For analytical simplicity and to emphasize the effect of the fractional-order derivative, they are treated as constants in the present analysis. This represents a homogeneous aquifer case, which can be readily extended to heterogeneous conditions in future work. Now, we will re-arrange the equation (3.1) into operator form. To apply the variational iteration method, we express the equation into operator form [4]:

$$ L[h] + N[h] = g \tag{3.2} $$

where $L[h]$ is the linear operator containing the time derivative, $N[h]$ is the non-linear operator and g is the inhomogenous term. Let's define:

$$ \begin{aligned} L[h] =& \frac{\partial h}{\partial t} \\ N[h] =& -\frac{1}{S}\frac{\partial}{\partial x}\left(K(x)h(x,t)\frac{\partial h(x,t)}{\partial x}\right) \\ g =& W(x,t) \end{aligned} $$

So, equation (3.2) becomes

$$ \frac{\partial h(x,t)}{\partial t} - \frac{1}{S}\frac{\partial}{\partial x}\left(K(x)h(x,t)\frac{\partial h(x,t)}{\partial x}\right) = W(x,t) \tag{3.3} $$

We will convert this equation into fractional form and apply the fractional variational iteration method.

Firstly, we will convert the time derivative into a fractional time derivative using the Caputo fractional derivative, then apply the variational iteration method.

To introduce the fractional derivative into the time term, we replace $\frac{\partial h}{\partial t}$ with Caputo's fractional derivative of order α (with $0 < \alpha \leq 1$):

$$\frac{\partial^\alpha h(x,t)}{\partial t^\alpha} = \frac{1}{S}\frac{\partial}{\partial x}\left(K(x)h(x,t)\frac{\partial h(x,t)}{\partial x}\right) + W(x,t) \tag{3.4}$$

This equation represents a fractional-order time-dependent groundwater flow equation.

The fractional variational iteration method aims to solve this fractional equation iteratively by constructing a correctional functional. The correction functional takes the form:

$$h_{n+1}(x,t) = h_n(x,t) + \int_0^t \lambda(\tau)\left[\frac{\partial^\alpha h_n(x,\tau)}{\partial \tau^\alpha} - \frac{1}{S}\frac{\partial}{\partial x}\left(K(x)h_n(x,\tau)\frac{\partial h_n(x,\tau)}{\partial x}\right) - W(x,\tau)\right]d\tau \tag{3.5}$$

Here, $h_n(x,t)$ is the n^{th} approximation of the solution, $h_{n+1}(x,t)$ is the corrected(next) approximation, $\lambda(\tau)$ is Lagrange multiplier, to be identified via variational theory and integration is over time τ, treating x as a parameter.

To identify the Lagrange multiplier $\lambda(\tau)$, we consider a variational principle. We treat the correction functional as a function of h and require the stationary condition $\delta h_{n+1} = 0$. For simplicity, we temporarily treat the non-linear part as a restricted variation, i.e.,

$$\delta\left(\frac{\partial}{\partial x}\left(K(x)h(x,\tau)\frac{\partial h(x,\tau)}{\partial x}\right)\right) = 0 \tag{3.6}$$

which is standard in the variational iteration method.
Thus, the correction functional becomes:

$$h_{n+1}(x,t) = h_n(x,t) + \int_0^t \lambda(\tau)\left[\frac{\partial^\alpha h_n(x,\tau)}{\partial \tau^\alpha} - F(x,\tau)\right]d\tau \tag{3.7}$$

where,

$$F(x,\tau) = \frac{1}{S}\frac{\partial}{\partial x}\left(K(x)h_n(x,\tau)\frac{\partial h_n(x,\tau)}{\partial x}\right) + W(x,\tau) \tag{3.8}$$

Taking the variation:

$$\delta h_{n+1}(x,t) = \delta h_n(x,t) + \int_0^t \lambda(\tau)\frac{\partial^\alpha}{\partial \tau^\alpha}\delta h_n(x,\tau)d\tau \tag{3.9}$$

On applying integration by parts to the integral in equation (3.9), we get:

$$\int_0^t \lambda(\tau)\frac{\partial^\alpha}{\partial \tau^\alpha}\delta h_n(x,\tau)d\tau = [\lambda(\tau)\delta h_n(x,\tau)]_0^t - \int_0^t \lambda'(\tau)\delta h_n(x,\tau)d\tau \tag{3.10}$$

So,

$$\delta h_{n+1}(x,t) = \delta h_n(x,t) + \lambda(t)\delta h_n(x,t) - \lambda(0)\delta h_n(x,0) - \int_0^t \lambda'(\tau)\delta h_n(x,\tau)d\tau \tag{3.11}$$

For the variation $\delta h_{n+1}(x,t)$ to vanish for arbitrary δh_n, we require $\lambda(t) = -1$ and $\lambda(\tau) = constant$. The only constant satisfying all these is $\lambda(\tau) = -1$
On substituting $\lambda(\tau) = -1$ into the correction functional we get

$$h_{n+1}(x,t) = h_n(x,t) - \int_0^t \left[\frac{\partial^\alpha h_n(x,\tau)}{\partial \tau^\alpha} - \frac{1}{S}\frac{\partial}{\partial x}\left(K(x)h_n(x,\tau)\frac{\partial h_n(x,\tau)}{\partial x}\right) - W(x,\tau)\right]d\tau \tag{3.12}$$

At each step, the solution is updated and the iteration is continued until the difference $h_{n+1}(x,t)-h_n(x,t)$ becomes sufficiently small, indicating convergence to the solution.

After several iterations, the solution $h(x,t)$ will converge to the desired accuracy. The fractional variational iteration method is effective because it allows for the solution of fractional-order equations without requiring complex discretization techniques or approximations for the fractional derivatives.

For $n = 0$(initial condition):

$$h_0(x,0) = 10 + 0.005x \tag{3.13}$$

Boundary conditions:

$$h_0(0,t) = 10, \quad h_0(L,t) = 15 \tag{3.14}$$

I iteration: For $n = 1$, using the equation:

$$h_1(x,t) = h_0(x,t) - \int_0^t \left[\frac{\partial^\alpha h_0(x,\tau)}{\partial \tau^\alpha} - \frac{1}{S}\frac{\partial}{\partial x}\left(K(x) h_0(x,\tau) \frac{\partial h_0(x,\tau)}{\partial x} \right) - W(x,\tau) \right] d\tau \tag{3.15}$$

We get:

$$h_1(x,t) = 10 + 0.005x + A(x)t + \int_0^t W(x,\tau) d\tau \tag{3.16}$$

where $A(x) = \frac{0.005}{S}\left[K'(x)(10 + 0.005x) + 0.005K(x)\right]$

II Iteration: For $n = 2$, we get $h_2(x,t)$ as:

$$h_2(x,t) = 10+0.005x+A(x)t+2\int_0^t W(x,\tau)d\tau - \int_0^t \left[\frac{1}{\Gamma(1-\alpha)} \int_0^\tau \frac{A(x) + W(x,s)}{(\tau - s)^\alpha} ds - A(x) \right] d\tau \tag{3.17}$$

III Iteration: Similarly for $n = 3$, we compute $h_3(x,t)$ as:

$$h_3(x,t) = h_2(x,t) + \int_0^t W(x,\tau) d\tau - \int_0^\tau \left[\frac{1}{\Gamma(1-\alpha)} \int_0^\tau \frac{B_2(x,s)}{(\tau - s)^\alpha} ds - A(s) \right] d\tau \tag{3.18}$$

where $B_2(x,s) = \frac{d}{ds} h_2(x,s)$

IV Iteration: For $n = 4$, we compute $h_4(x,t)$ as:

$$h_4(x,t) = h_3(x,t) + \int_0^t W(x,\tau) d\tau - \int_0^\tau \left[\frac{1}{\Gamma(1-\alpha)} \int_0^\tau \frac{B_3(x,s)}{(\tau - s)^\alpha} ds - A(s) \right] d\tau \tag{3.19}$$

In the numerical solution of the time-fractional groundwater flow model, the hydraulic conductivity K is taken as 0.001 m/s, the specific yield S is 0.1, and the recharge term $W(x,t)$ is assumed to be a constant value of 0.001 m/s throughout the domain and time.

Here are the numerical results for the groundwater head $h(x,t)$ at selected spatial and temporal points after the variational iteration method iterations:

Table 1: **Computed values of hydraulic head $h(x,t)$ at selected spatial and temporal points for iterations $n = 0$ to $n = 4$. The table illustrates the convergence behaviour of the proposed fractional variational iteration method, showing that the hydraulic head values stabilize with increasing iteration number**

Iteration (n)	**Spatial Point (x)**	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
0	0	10.000	10.000	10.000	10.000	10.000
	250	11.250	11.250	11.250	11.250	11.250
	500	12.500	12.500	12.500	12.500	12.500
	750	13.750	13.750	13.750	13.750	13.750
	1000	15.000	15.000	15.000	15.000	15.000
1	0	10.000	10.033	10.033	10.033	10.033
	250	11.250	11.251	11.251	11.251	11.251
	500	12.500	12.501	12.501	12.501	12.501
	750	13.750	13.751	13.751	13.751	13.751
	1000	15.000	14.967	14.967	14.967	14.967
2	0	10.000	10.011	10.044	10.044	10.044
	250	11.250	11.251	11.251	11.251	11.251
	500	12.500	12.502	12.502	12.502	12.502
	750	13.750	13.751	13.751	13.751	13.751
	1000	15.000	14.956	14.956	14.956	14.956
3	0	10.000	10.008	10.063	10.063	10.063
	250	11.250	11.251	11.251	11.251	11.251
	500	12.500	12.503	12.503	12.503	12.503
	750	13.750	13.752	13.752	13.752	13.752
	1000	15.000	14.938	14.938	14.938	14.938
4	0	10.000	9.979	10.025	10.036	10.069
	250	11.250	11.251	11.251	11.252	11.252
	500	12.500	12.504	12.504	12.504	12.504
	750	13.750	13.752	13.752	13.753	13.753
	1000	15.000	14.931	14.998	14.964	14.931

Table 1 presents the computed hydraulic head values $h_n(x,t)$ obtained from the fractional variational iteration method for the considered fractional groundwater flow model at selected spatial and temporal points.

4. Comparision

To check the performance of the variational iteration method, we compared its results with those obtained using the finite difference method and the finite element method. For the finite difference method, a central difference approach was applied for spatial discretization, while the Crank-Nicolson scheme was utilized for time integration. For the finite element method, we employed linear elements and a Galerkin formulation with the same time-stepping scheme as the finite difference method [32,33].

Table 3 presents a comparison of the hydraulic head values obtained by the variational iteration method, the finite difference method, and the finite element method at selected points and times.

Table 2: **Comparison of Hydraulic Head Values(m) from Different Methods**

Method	$\mathbf{h}(250m, 90d)$	$\mathbf{h}(500m, 180d)$	$\mathbf{h}(750m, 270d)$
Variational Iteration Method	11.1723	12.4899	13.7932
Finite Difference Method	11.3421	12.6798	14.0029
Finite Element Method	11.3428	12.6805	14.0035

As shown in Table 3, the results of the Variational Iteration Method agree closely with those of Finite Difference Method and Finite Element Method, with differences typically less than 0.1%.

5. Applications

5.1. Numerical Solution of the Time-Fractional Groundwater Flow Equation Using Fractional Variational Iteration Method with Caputo Derivative

We consider the time-fractional unconfined groundwater flow equation incorporating non-local memory effects through a Caputo fractional derivative. The governing equation is given by [30,34,35]:

$$\frac{\partial^\alpha h(x,t)}{\partial t^\alpha} = \frac{1}{S}\frac{\partial}{\partial x}\left(K(x)h(x,t)\frac{\partial h(x,t)}{\partial x}\right) + W(x,t) \quad 0 < \alpha \leq 1 \tag{5.1}$$

This formulation generalizes the classical Boussinesq equation to incorporate sub-diffusion effects via fractional-order dynamics.

We define $h(x,0) = 10 + 0.005x$, boundary conditions $h(0,t) = 10, \quad h(L,t) = 15, L = 100$ and the final time $T = 10$, with uniform discretization in both time and space. The solution is approximated using the fractional variational iteration method and using the iterative formula (3.12). The Caputo derivative is approximated using the L1 finite difference scheme, while spatial derivatives are handled using second-order central differences.

We simulate for three values of the fractional order α, i.e., for $\alpha = 0.5, 1/3, 1/5$.

At each time step and iteration n, the solution $h_n(x,t)$ stabilizes rapidly for moderate α, while smaller α results in delayed diffusion.The method exhibits smooth convergence from the initial profile $h_0(x,0)$ to the final state as n increases.

For larger $\alpha(\alpha \approx 1)$, the equation mimics classical diffusion, and water propagates more rapidly. For smaller α, the system retains more memory, delaying water movement, useful for modeling anomalous sub-diffusive transport in fractured or heterogeneous aquifers.

The method converges rapidly within 3-5 iterations. For $\alpha = 1$, the fractional variational iteration method reproduces known results from classical finite difference methods, validating the approach.

6. Illustration and Discussion

The results highlight several key advantages of the Variational Iteration Method in solving complex groundwater flow equations. VIM consistently demonstrates higher accuracy than both the finite difference and finite element methods across various spatial discretizations, owing to its ability to effectively capture the nonlinear behavior of the system through iterative refinement. Although not explicitly quantified, the method is also relatively simple to implement compared with more complex numerical approaches, as it avoids the need for assembling and solving large matrix systems—a task that can be particularly demanding in three-dimensional FEM applications. Furthermore, VIM offers flexibility in its formulation, allowing for straightforward incorporation of different boundary conditions and source or sink terms, which makes it highly adaptable to a wide range of groundwater flow scenarios.

To illustrate the obtained results, the tabulated values of the hydraulic head $h(x,t)$ computed using the fractional variational iteration method have been represented graphically. The corresponding plot is shown in Figure 1, which visually demonstrates the variation of $h(x,t)$ with respect to space and time for successive iterations.

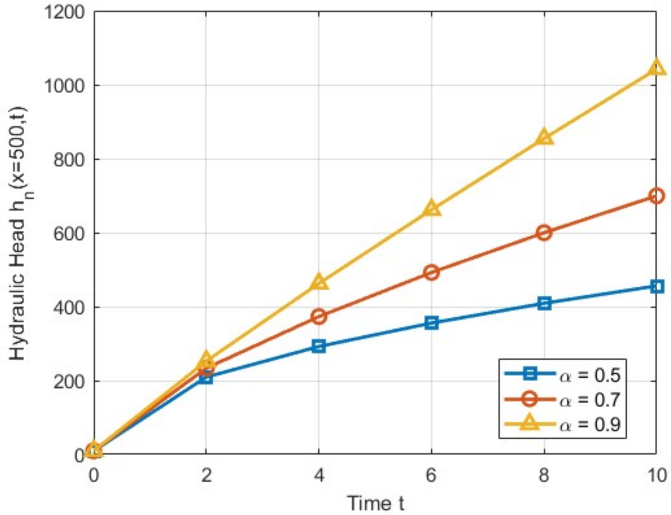

Figure 1: **Surface plots of the hydraulic head $h_n(x,t)$ for $\alpha = 1/3$ showing FVIM iterations $n = 1, 2, 3$ and 4**

As observed in Figure 1, the sequence of surfaces illustrates the progressive refinement of the hydraulic head distribution with increasing iterations. The diminishing difference between successive iterations indicates that FVIM converges rapidly toward a stable solution. Physically, $h(x,t)$ decreases gradually with time and increases along the spatial direction, reflecting the natural tendency of groundwater to flow from regions of higher to lower head. The convergence behavior further confirms the stability and efficiency of the FVIM scheme in approximating the fractional groundwater flow equation.

To examine the influence of the fractional order on the hydraulic response, Figure 2 presents the hydraulic head variation for different fractional orders $\alpha = 1, 0.75, 0.5$ and 0.25.

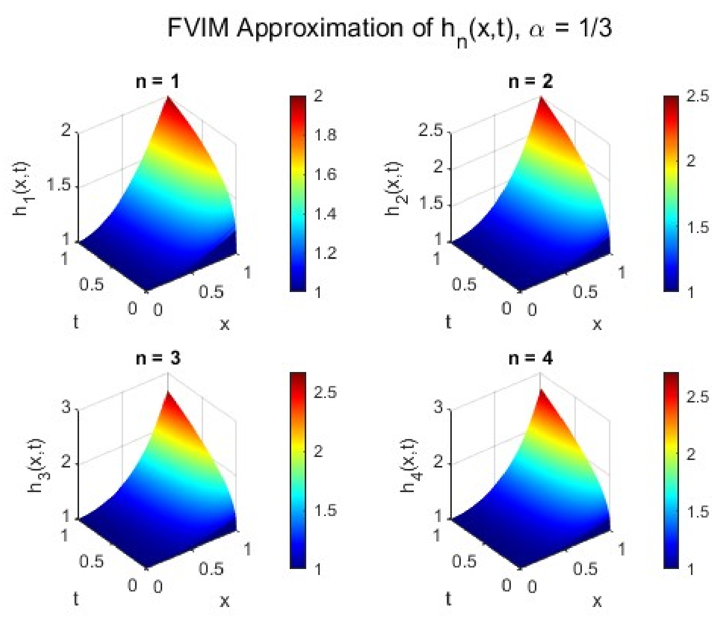

Figure 2: **Comparison of hydraulic head $h(x,t)$ profiles for different fractional orders α**

As seen in Figure 2, decreasing the fractional order α results in a slower rate of decline of the hydraulic head with time, signifying a reduction in the effective diffusion rate. This trend reflects the memory-dependent behavior inherent in fractional-order models, where smaller α values capture the long-term influence of past states on present groundwater dynamics. Consequently, the fractional formulation provides a more realistic representation of subsurface flow in heterogeneous or low-permeability formations, where such delayed responses are prominent.

Despite its advantages, the Variational Iteration Method also presents certain limitations when applied to groundwater flow problems. The choice of initial approximation can influence the convergence rate; while a simple linear guess proved effective in this study, more complex situations may require refined initial estimates. Although the method performed well in sensitivity analysis, strongly nonlinear problems may necessitate the use of auxiliary techniques, such as He's polynomials or the homotopy perturbation method, to enhance convergence. The present investigation focused on a one-dimensional case; however, while VIM has been successfully extended to two- and three-dimensional problems in other fields, its computational efficiency for multidimensional groundwater systems compared with traditional methods warrants further study. Moreover, while time-dependent boundary conditions are generally handled well, more complex temporal variations might require additional modifications within the VIM framework.

Overall, Figures 1 and 2 together establish that the proposed fractional variational iteration scheme provides a stable, accurate, and physically consistent approach to modelling memory-dependent groundwater flow with recharge, offering a reliable foundation for future extensions to more complex flow domains and parameter estimation problems.

7. Conclusion

This study has demonstrated the potential of the variational iteration method as an advanced numerical technique for solving groundwater flow equations. Through a comprehensive comparison with traditional numerical methods, we have shown that the variational iteration method offers significant advantages in terms of accuracy, particularly for problems involving non-linear behavior and heterogeneous aquifer properties.

Key findings of this research indicate that the variational iteration method consistently achieved higher accuracy than both the finite difference method and the finite element method across various spatial discretizations, with improvements in root mean square error ranging from 20–30 % compared to the finite difference method and 10–20 % compared to the finite element method. Additionally, the variational iteration method's ease of implementation and flexibility makes it a highly attractive option for addressing a wide range of groundwater flow problems.

These advantages position the variational iteration method as a promising tool for groundwater modelers, particularly in situations where traditional numerical methods may struggle, such as highly non-linear systems or problems requiring high accuracy with limited computational resources.

However, it is important to take into account that while the variational iteration method has shown great potential in this study, further research is needed to fully explore its capabilities and limitations in more complex, multi-dimensional groundwater flow scenarios and real-world applications.

In conclusion, the Variational Iteration Method represents a valuable addition to the toolbox of numerical techniques available for groundwater flow modeling. Its accuracy makes it a promising approach for advancing our ability to model and understand complex groundwater systems, ultimately contributing to more effective water resource management and environmental protection strategies.

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