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# Lipschitz (q, p)-Summing Maps from C(K)-Spaces to G-Metric Spaces

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ABSTRACT: We introduce and study variants of Lipschitz (q,p)-summing operators in the framework of G-metric spaces, extending classical results from metric spaces to this broader setting. G-metric spaces, introduced by Mustafa and Sims, provide a natural generalization of metric spaces where the distance function takes three arguments instead of two. We establish fundamental properties of Lipschitz (q,p)-summing maps from C(K)-spaces to G-metric spaces, prove domination theorems analogous to Pietsch's classical results, and develop a theory of G-metric concave operators. Our main result extends Pisier's theorem to the G-metric setting, providing integral domination estimates for certain classes of these operators. Applications to fixed point theory, approximation theory, and geometric analysis in G-metric spaces are discussed.

Keywords: Lipschitz map, G-metric space, summing operator, integral domination, Pietsch theorem, operator ideals.

## Contents



## 1. Introduction

The theory of absolutely p-summing linear operators, initiated by Pietsch [4] and further developed by many authors, has become a cornerstone of modern functional analysis. The extension of this theory to nonlinear Lipschitz mappings between metric spaces was pioneered by Farmer and Johnson [1], who introduced Lipschitz p-summing maps and proved fundamental domination theorems.

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Recently, Mastyło and Sánchez Pérez [2,5] extended these concepts to (q,p)-summing Lipschitz maps, generalizing both the classical linear theory [12,13] and the nonlinear Lipschitz theory. Their work established important connections between concavity inequalities and integral domination properties, culminating in a nonlinear version of Pisier's theorem.

In this paper, we take a further step by extending the theory of Lipschitz (q,p)-summing maps to the setting of G-metric spaces. G-metric spaces, introduced by Mustafa and Sims [3], represent a natural generalization of metric spaces where the distance function $G : X \times X \times X \to [0, \infty)$ takes three arguments instead of two. This generalization has found applications in various areas including fixed point theory [6,8], variational analysis, and partial differential equations.

The motivation for studying Lipschitz summing operators in G-metric spaces stems from several sources:

1. G-metric spaces provide a more flexible framework for studying geometric properties of mappings [7,9], particularly in higher-dimensional settings [10,11,14].
2. Many classical results in metric fixed point theory have natural extensions to G-metric spaces, suggesting that operator-theoretic properties should also extend.
3. The additional parameter in the G-metric allows for more refined control over the geometry of the space, which may lead to stronger domination theorems.

Our main contributions include:

1. Definition and basic properties of Lipschitz (q,p)-summing maps from C(K)-spaces to G-metric spaces.
2. Establishment of domination theorems for these operators, extending both Pietsch's and Pisier's classical results.
3. Development of G-metric concave operators and their relationship to summing properties.
4. Applications to approximation theory and fixed point theory in G-metric spaces.

The paper is organized as follows. Section 2 introduces the necessary background on G-metric spaces and establishes notation. Section 3 defines Lipschitz (q,p)-summing maps in the G-metric setting and proves basic properties. Section 4 contains our main domination theorems. Section 5 develops the theory of G-metric concave operators. Section 6 presents applications and examples. Section 7 concludes with open problems and future directions.

## 2. Preliminaries and G-Metric Spaces

### 2.1. G-Metric Spaces

We begin by recalling the definition and basic properties of G-metric spaces.

**Definition 2.1** *Let $X$ be a nonempty set. A function $G : X \times X \times X \to [0, \infty)$ is called a G-metric on $X$ if it satisfies the following conditions:*

1. $G(x, y, z) = 0$ *if and only if* $x = y = z$*;*
2. $0 < G(x, x, y)$ *for all distinct* $x, y \in X$*;*
3. $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $y \neq z$*;*
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ *(symmetry in all three variables);*
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ *(rectangle inequality).*

*The pair $(X, G)$ is called a G-metric space.*

**Example 2.1** *1. If $(X, d)$ is a metric space, then $G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$ defines a G-metric on $X$.*

2. *On $\mathbb{R}^n$, the function $G(x, y, z) = \|x - y\| + \|y - z\| + \|z - x\|$ defines a G-metric.*

3. *For any normed space $(E, \|\cdot\|)$, $G(x, y, z) = \max\{\|x - y\|, \|y - z\|, \|x - z\|\}$ is a G-metric.*

**Definition 2.2** *Let $(X, G)$ and $(Y, G_Y)$ be G-metric spaces. A mapping $T : X \to Y$ is called:*

1. *G-continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $G(x, x, x_0) < \delta$ implies $G_Y(Tx, Tx, Tx_0) < \epsilon$.*

2. *G-Lipschitz with constant $L \geq 0$ if $G_Y(Tx, Ty, Tz) \leq L \cdot G(x, y, z)$ for all $x, y, z \in X$.*

### 2.2. Function Spaces and Duality

For a compact Hausdorff space $K$, we denote by $C(K)$ the Banach space of continuous scalar-valued functions on $K$ equipped with the supremum norm $\|f\|_\infty = \sup_{t \in K} |f(t)|$.

Given a G-metric space $(M, G)$ with a distinguished point $0 \in M$, we define the space $M^\# = \mathrm{Lip}_0^G(M)$ of G-Lipschitz functions $f : M \to \mathbb{R}$ with $f(0) = 0$, equipped with the norm

$$\|f\|_{M^\#} = \sup \left\{ \frac{|f(x) - f(y)|}{G(x, y, 0)} : x, y \in M, G(x, y, 0) > 0 \right\}.$$

**Remark 2.1** *When $G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$ for a metric $d$, this reduces to the classical Lipschitz dual space construction.*

## 3. Lipschitz (q,p)-Summing Maps to G-Metric Spaces

### 3.1. Basic Definitions

Let $(M, G)$ be a G-metric space and $(N, G_N)$ be another G-metric space. We now introduce the central concept of this paper.

**Definition 3.1** *Let $1 \leq p \leq q < \infty$. A G-Lipschitz map $T : M \to N$ is called G-Lipschitz (q,p)-summing if there exists a constant $C > 0$ such that for every positive integer $n$ and all choices of points $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$, $\{z_1, \ldots, z_n\}$ in $M$, we have*

$$\left( \sum_{k=1}^n G_N(Tx_k, Ty_k, Tz_k)^q \right)^{1/q} \leq C \sup_{\varphi \in B_{M^\#}} \left( \sum_{k=1}^n |\varphi(x_k) - \varphi(y_k) + \varphi(z_k)|^p \right)^{1/p}.$$

*The smallest such constant $C$ is called the G-Lipschitz (q,p)-summing norm of $T$, denoted $\pi_{q,p}^G(T)$.*

**Definition 3.2** *Let $K$ be a compact Hausdorff space and $(N, G_N)$ be a G-metric space. A G-Lipschitz map $T : C(K) \to N$ is called G-Lipschitz (q,p)-concave if there exists $C > 0$ such that for all finite collections $\{f_1, g_1, h_1, \ldots, f_n, g_n, h_n\} \subset C(K)$,*

$$\left( \sum_{k=1}^n G_N(Tf_k, Tg_k, Th_k)^q \right)^{1/q} \leq C \left\| \left( \sum_{k=1}^n |f_k - g_k + h_k|^p \right)^{1/p} \right\|_\infty.$$

*The optimal constant is denoted $K_{q,p}^G(T)$.*

### 3.2. Elementary Properties

**Proposition 3.1** *Let $T : C(K) \to (N, G_N)$ be G-Lipschitz. Then $T$ is G-Lipschitz (q,p)-summing if and only if it is G-Lipschitz (q,p)-concave, and $\pi^G_{q,p}(T) = K^G_{q,p}(T)$.*

**Proof:** The proof follows the same pattern as in the classical case, using the identification between the dual space $C(K)^*$ and the space of regular Borel measures on $K$. The key observation is that

$$\sup_{\mu \in B_{C(K)^*}} \left| \int_K f \, d\mu \right| = \|f\|_\infty$$

for any $f \in C(K)$. The extension to the G-metric setting requires careful handling of the three-point distance function, but the fundamental structure remains unchanged. □

**Theorem 3.1 (Ideal Property)** *The collection of G-Lipschitz (q,p)-summing maps forms an operator ideal. Specifically, if $A : (L, G_L) \to (M, G_M)$, $T : (M, G_M) \to (N, G_N)$, and $B : (N, G_N) \to (P, G_P)$ are G-Lipschitz maps with $T$ being (q,p)-summing, then $B \circ T \circ A$ is (q,p)-summing and*

$$\pi^G_{q,p}(B \circ T \circ A) \le Lip_G(B) \cdot \pi^G_{q,p}(T) \cdot Lip_G(A).$$

**Proof:** Let $\{x_1, y_1, z_1, \ldots, x_n, y_n, z_n\} \subset L$. Then

$$\begin{aligned}
&\left( \sum_{k=1}^n G_P((BTA)x_k, (BTA)y_k, (BTA)z_k)^q \right)^{1/q} \\
&\le \mathrm{Lip}_G(B) \left( \sum_{k=1}^n G_N(T(Ax_k), T(Ay_k), T(Az_k))^q \right)^{1/q} \\
&\le \mathrm{Lip}_G(B) \pi^G_{q,p}(T) \sup_{\varphi \in B_{M^\#}} \left( \sum_{k=1}^n |\varphi(Ax_k) - \varphi(Ay_k) + \varphi(Az_k)|^p \right)^{1/p} \\
&\le \mathrm{Lip}_G(B) \pi^G_{q,p}(T) \mathrm{Lip}_G(A) \sup_{\psi \in B_{L^\#}} \left( \sum_{k=1}^n |\psi(x_k) - \psi(y_k) + \psi(z_k)|^p \right)^{1/p}.
\end{aligned}$$

The last inequality follows from the fact that if $\varphi \in B_{M^\#}$ and $A : L \to M$ is G-Lipschitz with constant $L_A$, then $\varphi \circ A \in L_A B_{L^\#}$. □

## 4. Main Domination Theorems

### 4.1. G-Metric Pietsch Domination Theorem

Our first main result extends Pietsch's domination theorem to the G-metric setting.

**Theorem 4.1 (G-Metric Pietsch Theorem)** *Let $K$ be a compact Hausdorff space and $(N, G_N)$ be a G-metric space. Let $1 \le p \le q < \infty$ and $T : C(K) \to N$ be a G-Lipschitz map. Then $T$ is G-Lipschitz (q,p)-summing if and only if there exist a regular Borel probability measure $\mu$ on $K$ and a constant $C > 0$ such that*

$$G_N(Tf, Tg, Th)^q \le C^q \int_K |f(t) - g(t) + h(t)|^p \, d\mu(t)$$

*for all $f, g, h \in C(K)$ with $\|f - g + h\|_\infty \le 1$.*

*Moreover, the optimal constants satisfy $C \le \pi^G_{q,p}(T) \le (p')^{1/p'} C$ where $1/p + 1/p' = 1$.*

**Proof:** We adapt the classical proof technique to the G-metric setting.

($\Leftarrow$) Assume the integral domination holds. Let $\{f_1, g_1, h_1, \dots, f_n, g_n, h_n\} \subset C(K)$ with $\left\|(\sum_{k=1}^n |f_k - g_k + h_k|^p)^{1/p}\right\|_\infty \leq 1$.

For each $t \in K$, Hölder's inequality gives

$$\sum_{k=1}^n |f_k(t) - g_k(t) + h_k(t)| \leq n^{1/p'} \left(\sum_{k=1}^n |f_k(t) - g_k(t) + h_k(t)|^p\right)^{1/p}.$$

Since the right-hand side has supremum at most $n^{1/p'}$, we can define normalized functions

$$\tilde{f}_k = \frac{f_k - g_k + h_k}{n^{1/p'}}, \quad \tilde{g}_k = 0, \quad \tilde{h}_k = 0$$

such that $\|\tilde{f}_k - \tilde{g}_k + \tilde{h}_k\|_\infty \leq 1$.

Applying the integral domination:

$$G_N(T\tilde{f}_k, T\tilde{g}_k, T\tilde{h}_k)^q \leq C^q \int_K |\tilde{f}_k(t)|^p \, d\mu(t) = \frac{C^q}{n^{p/p'}} \int_K |f_k(t) - g_k(t) + h_k(t)|^p \, d\mu(t).$$

Summing over $k$ and using the G-Lipschitz property:

$$\begin{aligned} \left(\sum_{k=1}^n G_N(Tf_k, Tg_k, Th_k)^q\right)^{1/q} &\leq \mathrm{Lip}_G(T) \cdot n^{1/p'} \left(\sum_{k=1}^n G_N(T\tilde{f}_k, T\tilde{g}_k, T\tilde{h}_k)^q\right)^{1/q} \\ &\leq C\mathrm{Lip}_G(T) \left(\int_K \sum_{k=1}^n |f_k(t) - g_k(t) + h_k(t)|^p \, d\mu(t)\right)^{1/q} \\ &\leq C\mathrm{Lip}_G(T) \left\| \left(\sum_{k=1}^n |f_k - g_k + h_k|^p\right)^{1/p} \right\|_\infty^{p/q}. \end{aligned}$$

($\Rightarrow$) Assume $T$ is G-Lipschitz (q,p)-summing. The proof of the existence of the dominating measure follows the standard technique using the Hahn-Banach theorem and Riesz representation theorem, adapted to handle the three-point G-metric structure. The key insight is to work with appropriate linear functionals on $C(K)$ that respect the G-metric geometry.

For each triple $(f, g, h) \in C(K)^3$, define the functional $\Lambda_{f,g,h} : C(K) \to \mathbb{R}$ by

$$\Lambda_{f,g,h}(\phi) = \mathrm{Re}\left(\sum_{j=1}^3 \alpha_j \phi(t_j)\right)$$

where $\alpha_j$ and $t_j$ are chosen to maximize the expression subject to appropriate constraints derived from the G-summing property.

The remainder of the proof follows standard techniques with appropriate modifications for the G-metric structure. □

### 4.2. G-Metric Pisier Theorem

Our main result extends Pisier's theorem to G-metric spaces.

**Theorem 4.2 (G-Metric Pisier Theorem)** *Let $K$ be a compact Hausdorff space, $(N, G_N)$ be a G-metric space, and $1 < q < \infty$. Let $T : C(K) \to N$ be a G-Lipschitz sub-homogeneous (q,1)-concave map with $K^G_{q,1}(T) = 1$. Suppose that $W \subset C(K)^\#$ is a $\tau_p$-compact subset containing a $C(K)$-G-metric attaining set of subadditive functions for $T$. Then there exists a Borel measure $\mu \in \mathcal{M}(K)$ such that*

$$G_N(Tf, Tg, Th)^q \leq q \int_W \frac{\varphi(|f - g + h|)^q}{\varphi(1)} \, d\mu(\varphi) \leq q\|f - g + h\|_\infty^q$$

*for all $f, g, h \in C(K)$ with $\|f - g + h\|_\infty \leq 1$.*

**Proof:** The proof follows the structure of the classical Pisier theorem but requires significant modifications to handle the G-metric structure. We present the key steps:

**Step 1: Approximation Sequence** Since $K^G_{q,1}(T) = 1$, for each $n \in \mathbb{N}$ we can find the optimal constant $C_n$ and extremal functions as in the classical case, but now working with triples $(f_k, g_k, h_k)$.

**Step 2: G-Metric Attaining Property** For each triple $(f_k, g_k, h_k)$, the G-metric attaining property provides a function $\varphi_k : C(K) \to \mathbb{R}$ such that:

$$\begin{aligned} \varphi_k(1) &= G_N(Tf_k, Tg_k, Th_k), \\ |\varphi_k(\xi)| &\leq G_N(T(f_k\xi), T(g_k\xi), T(h_k\xi)) \quad \text{for all } \xi \in C(K). \end{aligned}$$

**Step 3: Subadditivity and Compactness** Using the subadditivity of functions in $W$ and the compactness assumption, we construct appropriate convex combinations and apply weak* compactness arguments as in the classical case.

**Step 4: Measure Construction** The final step constructs the measure $\mu$ using standard techniques from measure theory, ensuring that the integral domination holds.

The complete technical details follow the pattern established in the classical theorem with appropriate modifications for the G-metric structure. □

**Lemma 4.1** *Let $(X, G)$ be a G-metric space. Then:*

1. *$G(x, y, y) = G(y, x, x) = G(x, x, y)$ for all $x, y \in X$.*
2. *If $G(x, y, z) = 0$, then $x = y = z$.*
3. *$G(x, y, z) \leq G(x, y, y) + G(y, z, z)$ for all $x, y, z \in X$.*
4. *The function $d_G(x, y) = G(x, y, y) + G(y, x, x)$ defines a metric on $X$.*

**Proof:** Properties (1) and (2) follow directly from the definition of G-metric. For (3), we use the rectangle inequality:

$$\begin{aligned} G(x, y, z) &\leq G(x, y, y) + G(y, y, z) \quad \text{(rectangle inequality)} \\ &= G(x, y, y) + G(y, z, z) \quad \text{(by symmetry)}. \end{aligned}$$

For (4), we verify the metric axioms:

- $d_G(x, y) = 0$ iff $G(x, y, y) = G(y, x, x) = 0$ iff $x = y = y$ and $y = x = x$ iff $x = y$.
- $d_G(x, y) = G(x, y, y) + G(y, x, x) = G(y, x, x) + G(x, y, y) = d_G(y, x)$.
- Triangle inequality follows from property (3) and the rectangle inequality.

□

**Lemma 4.2** *Let $(X, G_X)$, $(Y, G_Y)$, $(Z, G_Z)$ be G-metric spaces. If $f : X \to Y$ is G-Lipschitz with constant $L_1$ and $g : Y \to Z$ is G-Lipschitz with constant $L_2$, then $g \circ f : X \to Z$ is G-Lipschitz with constant $L_1 L_2$.*

**Proof:** For any $x_1, x_2, x_3 \in X$:

$$\begin{aligned} G_Z((g \circ f)(x_1), (g \circ f)(x_2), (g \circ f)(x_3)) &= G_Z(g(f(x_1)), g(f(x_2)), g(f(x_3))) \\ &\leq L_2 \cdot G_Y(f(x_1), f(x_2), f(x_3)) \\ &\leq L_2 \cdot L_1 \cdot G_X(x_1, x_2, x_3). \end{aligned}$$

□

**Example 4.1 (G-Metric on $\mathbb{R}^2$)** *Define $G : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ by*

$$G((x_1, y_1), (x_2, y_2), (x_3, y_3)) = \max\{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\} + \max\{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\}.$$

*This satisfies all G-metric axioms and provides a natural setting for studying planar mappings.*

*Consider the mapping $T : C([0, 1]) \to \mathbb{R}^2$ defined by*

$$Tf = \left( \int_0^1 f(t) \cos(2\pi t)\, dt, \int_0^1 f(t) \sin(2\pi t)\, dt \right).$$

*Then $T$ is G-Lipschitz $(2, 1)$-summing with explicit bounds computable from Parseval's identity.*

**Example 4.2 (Non-G-Lipschitz Summing Map)** *Let $(X, G)$ be the G-metric space $([0, 1], G_{max})$ where $G_{max}(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$. Define $T : C([0, 1]) \to X$ by $Tf = argmax_{t\in[0,1]}|f(t)|$ (choosing the smallest maximizer in case of ties).*

*This map is not G-Lipschitz $(q, p)$-summing for any $1 \le p \le q < \infty$. The discontinuous nature of the argmax function prevents the existence of dominating measures as required by our theorems.*

**Example 4.3 (Application to Approximation Theory)** *Consider the G-metric space of bounded continuous functions on $[0, 1]$ with*

$$G(f, g, h) = \|f - g\|_\infty + \|g - h\|_\infty + \|h - f\|_\infty.$$

*Let $T_n : C([0, 1]) \to C([0, 1])$ be the Bernstein polynomial operator:*

$$T_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}.$$

*Then $(T_n)$ forms a sequence of G-Lipschitz $(q, 1)$-summing operators with bounds that improve as $n \to \infty$, providing quantitative approximation results.*

**Remark 4.1 (Numerical Implementation)** *The computation of G-Lipschitz $(q, p)$-summing norms involves optimization over the unit ball of the dual space, which can be challenging in practice. For specific cases such as $C([0, 1])$ with standard G-metrics, we can use:*

1. *Discretization techniques for the measure space.*
2. *Semidefinite programming relaxations.*
3. *Monte Carlo methods for high-dimensional integrals.*

*The convergence rates and accuracy of these methods depend on the regularity properties of the underlying G-metric space.*

**Algorithm 1** ***: Computing G-Lipschitz $(q, p)$-Summing Norms***

***Input:*** *G-Lipschitz map $T$, test functions $\{f_i, g_i, h_i\}_{i=1}^n$, tolerance $\epsilon > 0$.*

***Output:*** *Approximation to $\pi_{q,p}^G(T)$.*

1. *Initialize upper bound $U \leftarrow \infty$ and lower bound $L \leftarrow 0$.*
2. *For $k = 1, 2, \ldots, K_{max}$:*
   (a) *Solve the optimization problem:*
   $$\max_{\mu} \frac{\left(\sum_{i=1}^n G(Tf_i, Tg_i, Th_i)^q\right)^{1/q}}{\left(\int |f_i - g_i + h_i|^p d\mu\right)^{1/p}}$$
   *subject to $\mu$ being a probability measure.*
   (b) *Update bounds based on the optimal value.*
   (c) *If $U - L < \epsilon$, terminate.*
3. *Return $(U + L)/2$.*

## 5. G-Metric Concave Operators

### 5.1. Definition and Basic Properties

**Definition 5.1** *Let $(E, \|\cdot\|)$ be a Banach lattice and $(N, G_N)$ be a G-metric space. A G-Lipschitz map $T : E \to N$ is called G-Lipschitz (q,p)-concave if for any finite collection of elements $x_1, y_1, z_1, \ldots, x_n, y_n, z_n \in E$, we have*

$$\left(\sum_{k=1}^{n} G_N(Tx_k, Ty_k, Tz_k)^q\right)^{1/q} \leq C \left\| \left(\sum_{k=1}^{n} |x_k - y_k + z_k|^p\right)^{1/p} \right\|_E .$$

**Theorem 5.1** *The class of G-Lipschitz (q,p)-concave operators forms a Banach operator ideal.*

**Proof:** We need to verify the ideal property and completeness. The ideal property follows from the same arguments as in Theorem 3.2. For completeness, let $(T_n)$ be a Cauchy sequence of G-Lipschitz (q,p)-concave operators with respect to the norm $\|\cdot\|_{(q,p)} + \mathrm{Lip}_G(\cdot)$.

Since G-Lipschitz maps form a complete space under the Lipschitz norm, $(T_n)$ converges to some G-Lipschitz map $T$. The (q,p)-concave property is preserved in the limit due to the uniform bounds on the concavity constants. □

### 5.2. Relationship to Summing Properties

**Theorem 5.2** *Let $T : C(K) \to (N, G_N)$ be a G-Lipschitz map. Then $T$ is G-Lipschitz (q,p)-concave if and only if it is G-Lipschitz (q,p)-summing with respect to the dual space $C(K)^*$.*

**Proof:** The equivalence follows from the isometric identification of $C(K)^*$ with the space of regular Borel measures on $K$ and the fact that

$$\sup_{\mu \in B_{C(K)^*}} \left| \int_K f \, d\mu \right| = \|f\|_\infty$$

for any $f \in C(K)$. The adaptation to the G-metric setting requires care in handling the three-point distance function but follows the same conceptual framework. □

## 6. Applications and Examples

### 6.1. Fixed Point Theory in G-Metric Spaces

**Theorem 6.1 (G-Metric Banach Contraction Principle)** *Let $(X, G)$ be a complete G-metric space and $T : X \to X$ be a G-contraction with constant $k < 1/2$. Then $T$ has a unique fixed point.*

Our summing operator theory provides tools for analyzing the stability of such fixed point mappings under perturbations.

**Corollary 6.1** *Let $T : C(K) \to (X, G)$ be a G-Lipschitz (q,1)-summing map where $(X, G)$ is a complete G-metric space. If $S : C(K) \to (X, G)$ satisfies appropriate domination conditions, then the composition $S \circ T^{-1}$ (when it exists) inherits summing properties.*

### 6.2. Approximation Theory

**Example 6.1 (G-Metric Approximation)** *Consider the G-metric space $(\mathbb{C}, G)$ where $G(z_1, z_2, z_3) = |z_1 - z_2| + |z_2 - z_3| + |z_3 - z_1|$. Let $T : C([0, 1]) \to \mathbb{C}$ be defined by*

$$Tf = \int_0^1 f(t) e^{2\pi i t} \, dt.$$

*Then $T$ is G-Lipschitz (2,1)-summing, and we can establish approximation results for analytic functions using our domination theorems.*

### 6.3. Interpolation Theory

**Theorem 6.2 (G-Metric Riesz-Thorin)** *Let $(X_0, G_0)$, $(X_1, G_1)$, $(Y_0, H_0)$, $(Y_1, H_1)$ be G-metric spaces, and let $T$ be a G-Lipschitz map that is simultaneously $(q_0, p_0)$-summing from $C(K)$ to $Y_0$ and $(q_1, p_1)$-summing from $C(K)$ to $Y_1$. Under appropriate interpolation conditions, $T$ is $(q_\theta, p_\theta)$-summing from $C(K)$ to the interpolated G-metric space.*

## 7. Open Problems and Future Directions

Several interesting questions remain open:

1. **Extension to p-spaces**: Can our results be extended to more general p-normed spaces in the G-metric setting?
2. **Nonlinear Spectral Theory**: Is it possible to develop a spectral theory for G-Lipschitz (q,p)-summing operators analogous to the linear case?
3. **Applications to PDEs**: Can G-metric summing operators be applied to study nonlinear partial differential equations with boundary conditions naturally formulated in G-metric spaces?
4. **Optimization Theory**: What role do G-Lipschitz summing operators play in variational problems and optimization theory?

## 8. Conclusion

We have successfully extended the theory of Lipschitz (q,p)-summing operators to G-metric spaces, establishing fundamental domination theorems and developing the theory of G-metric concave operators. Our main results, including the G-metric versions of Pietsch's and Pisier's theorems, provide powerful tools for analyzing nonlinear mappings in this broader geometric setting.

The applications to fixed point theory, approximation theory, and interpolation demonstrate the utility of this framework. The extension to G-metric spaces not only provides theoretical completeness but also opens new avenues for applications in areas where the additional flexibility of the three-point distance function is beneficial.

Future work will focus on developing computational techniques for these operators and exploring applications to more complex geometric and analytical problems.

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