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Skew generalized quasi-cyclic codes over a non-chain ring $F_q + \mathfrak{v}F_q$

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ABSTRACT: For a prime p, let F_q be the finite field of order $q = p^d$. This paper presents the study on skew generalized quasi-cyclic (SGQC) codes of length n over the non-chain ring $F_q + \mathfrak{v}F_q$, where $\mathfrak{v}^2 = \mathfrak{v}$. Here, first, we prove the dual of an SGQC code of length n is also an SGQC code of the same length and derive a necessary and sufficient condition for the existence of a self-dual code. Then, we discuss the 1-generator polynomial and the ρ-generator polynomial for these codes. Further, we determine the dimension and BCH-type bound for the 1-generator case. As a by-product, with the help of MAGMA software, we provide a few examples of SGQC codes with improved parameters compared to those given in the existing literature and obtain some 2-generator optimal and near-optimal SGQC codes of index 2.

Key Words: Skew cyclic codes, gray map, generator polynomial, idempotent generator, skew generalized quasi-cyclic codes.

Contents

1. Introduction

In the theory of error-correcting codes, linear codes over finite fields play a crucial role in many error-correction schemes. Initial works of linear codes are based on the binary field, and researchers have been continuing their study on it due to the ease of their practical implementation. Later, this study was extended to codes over finite fields and rings, keeping broader aspects and optimal codes in mind. In the 1990s, some studies on cyclic and self-dual cyclic codes over the ring $\mathbb{Z}_4$ have been reported in [14,9,20], whereas [6] studied the same family of codes over the ring $F_2 + uF_2$. In 2000, Abualrub and Siap [4] studied cyclic codes over the rings $\mathbb{Z}_2 + u\mathbb{Z}_2$ and $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$, whereas Zhu et al. studied cyclic codes over $F_2 + vF_2$ [27]. Later on, cyclic, quasi-cyclic (QC), and generalized quasi-cyclic (GQC) codes over finite commutative rings have been studied by introducing different Gray maps in [10,26,22,23,18].

On the other hand, in 2007, Boucher et al. [7] introduced cyclic codes over finite noncommutative rings (skew polynomial rings) and presented θ-cyclic codes over $F_q[z, \theta]$ with restrictions on their length, where F_q is the finite field and θ is an automorphism over $\mathbb{F}_q$. They have obtained some codes that improved upon previously best-known linear codes. Meanwhile, in 2011, Siap et al. [25] studied skew

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cyclic (SC) codes of arbitrary length over the field F_q. After that, some other works on SC codes over rings have been seen in [3,13,15,8,17].

Recall that skew quasi-cyclic (SC) codes of length nl and index l over a finite field F_q are linear codes where the SC shift of each codeword by l positions is another codeword. It is noted that the SC codes of index $l = 1$ are well-known SC codes of length n. Also, it has been shown that the class of SC codes significantly contributes to the class of linear codes over finite fields and rings [2,19,18,5]. Later, the notion of SGQC codes over finite fields was introduced by Gao et al. [11] and studied with the restriction that the order of the automorphism divides the length of codes. Also, based on the structural properties of SGQC codes, Abualrub et al. [1] gave some good skew l-GQC codes and constructed some asymmetric quantum codes over the finite field F_4. Recently, Seneviratne and Abualrub [24] studied SGQC codes of arbitrary length over the field F_q and obtained many new linear codes.

The above works inspired and gave us a space to consider and study the SGQC code over the finite non-chain ring $F_q + \mathfrak{v}F_q$ where $\mathfrak{v}^2 = \mathfrak{v}$. By considering the automorphism θ_t as $\theta_t : \mathtt{a} + \mathfrak{v}\mathtt{b} \mapsto \mathtt{a}^{p^t} + \ \mathtt{v}\mathtt{b}^{p^t}$, we establish the algebraic structure of these codes. Since the class of SGQC codes is much larger than the class of SC codes, it opens the door to search for better codes in this class. Here, we present 1-generator and the ρ-generator SGQC codes. Further, we show that 1-generator idempotent polynomial exists for SGQC code over F_q and $F_q + \mathfrak{v}F_q$, respectively. We arrange our paper as follows. Section 2 recalls some known results concerning the skew polynomial ring $S[\mathtt{z}; \theta_t]$, where $S = F_q + \mathfrak{v}F_q$, and SC code. Section 3 provides the algebraic structure of SGQC codes and their duals over the finite non-chain ring S. Section 4 discusses the duality of SGQC code under certain conditions on the code length, while Section 5 presents the generator for these codes. Additionally, we introduce an idempotent generator polynomial over F_q and S for SGQC codes and list some 1-generator and 2-generator polynomial parameters, among them some are optimal, near-optimal codes. Also, some parameters are better than those given in existing literature [11,19] over $F_3 + \mathfrak{v}F_3$, $F_4 + \mathfrak{v}F_4$, and $F_9 + \mathfrak{v}F_9$, respectively. Section 6 presents the conclusions and outlines some challenging problems.

List of Symbols and Abbreviations

Throughout the paper, we use some symbols and abbreviations, which are listed below.

p : A prime number
d : A positive integer
q : A prime power number p^d
$\mathbb{F}_q$: Field with q elements
θ_t : An automorphism map over F_q
SC : Skew cyclic
σ : Skew cyclic shift operator
QC : Quasi-cyclic
GQC : Generalized quasi-cyclic
SQC : Skew quasi-cyclic
SGQC : Skew generalized quasi-cyclic
ϕ : Gray map
ρ : A positive integer
ψ_j : Conjugation map
$|_r$: Is the right divisor of
$gcrd$: Greatest common right divisor
$gcld$: Greatest common left divisor

2. Preliminaries

In this section, we recall the basic definitions and results of linear and SC codes over a finite non-chain ring that we will use throughout our work.

Let F_q be a finite field containing q elements, and $S = F_q[\mathfrak{v}]/\langle \mathfrak{v}^2 - \mathfrak{v}\rangle = \{\mathtt{a}+\mathfrak{v}\mathtt{b} : \mathtt{a}, \mathtt{b} \in F_q \text{ and } \mathfrak{v}^2 = \mathfrak{v}\}$, where $q = p^d$. The ring S has order q^2, is non-chain, and possesses exactly two maximal ideals, namely $\langle 1 - \mathfrak{v}\rangle$ and $\langle \mathfrak{v}\rangle$. For more details on this ring, we refer [21]. An S-submodule C of S^n is called a *linear code* of length n over S, and its elements are called *codewords*. We define a F_q-linear *Gray map* as

$$\phi : S \mapsto F_q^2 \text{ given by } \phi(\mathtt{a} + \mathfrak{v}\mathtt{b}) = (\mathtt{a}, \mathtt{a} + \mathtt{b}).$$

This linear map ϕ can be extended component-wise from S^n to F_q^{2n} given by

$$\phi(s_1, s_2, \dots, s_n) = (\mathtt{a}_1, \dots, \mathtt{a}_n, \mathtt{a}_1 + \mathtt{b}_1, \dots, \mathtt{a}_n + \mathtt{b}_n),$$

where $s_i = \mathtt{a}_i + \mathfrak{v}\mathtt{b}_i \in S$, for all $i = 1, 2, \dots, n$. The number of nonzero elements in $c \in F_q^n$ is called the *Hamming weight* and is denoted by $w_H(c)$. For any pair of words $c = (c_0, c_1, \dots, c_n), c' = (c'_0, c'_1, \dots, c'_n) \in F_q^n$, the *Hamming distance* between c and c' is defined as $d(c, c') = |\{i = c_i \neq c'_i\}|$. One can easily verify that $d(c, c') = w_H(c - c')$. The minimum distance between each pair of codewords of C over F_q is said to be the distance of C and is denoted by $d(C)$. The number of nonzero elements in $\phi(s)$ is called the *Lee weight* of an element $s \in S$ and is denoted by $w_L(s)$ i.e., $w_L(s) = w_H(\phi(s))$. The *Lee distance* of any pair of codewords of a linear code C over S is defined by $d_L(s_1, s_2) = w_L(s_1 - s_2)$, where $s_1, s_2 \in S^n$. Note that the Gray map ϕ induces an isometry between linear codes over S^n (*Lee distance*) and their images in F_q^{2n} (*Hamming distance*) and preserves orthogonality.

Now, we define some operations on linear codes, similar to those given in [13, Definition 1]. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are two linear codes over the field F_q. Then we define two operations $\oplus$ and $\otimes$ on $\mathcal{X}$ and $\mathcal{Y}$ given by

$$\mathcal{X} \oplus \mathcal{Y} = \{\mathtt{x} + \mathtt{z} : \mathtt{x} \in \mathcal{X}, \mathtt{z} \in \mathcal{Y}\}, \text{ and}$$
$$\mathcal{X} \otimes \mathcal{Y} = \{(\mathtt{x}, \mathtt{z}) : \mathtt{x} \in \mathcal{X}, \mathtt{z} \in \mathcal{Y}\}.$$

Let C be a linear code of length n over the ring S. Define

$$C_1 := \{c \in F_q^n : c + \mathfrak{v}s \in C \text{ for some } s \in F_q^n\}, \text{ and}$$
$$C_2 := \{c + s \in F_q^n : c + \mathfrak{v}s \in C\}.$$

Clearly, C_1 and C_2 are linear codes over F_q, and from Corollary 1 [13], linear code C over S can be written as $C = (1 - \mathfrak{v})C_1 \oplus \mathfrak{v}C_2$.

Let θ_t be an automorphism defined on S given by $\mathtt{a}^{p^t} + \mathfrak{v}\mathtt{b}^{p^t}$, where $\mathtt{a} + \mathfrak{v}\mathtt{b} \in S$. Clearly, θ_t acts on F_q as follows:

$$\theta_t : F_q \mapsto F_q$$
$$\mathtt{a} \mapsto \mathtt{a}^{p^t}.$$

Definition 2.1 *Consider the set*

$$S[\boldsymbol{z}; \theta_t] := \left\{a(\boldsymbol{z}) = a_0 + a_1\boldsymbol{z} + \dots + a_{n-1}\boldsymbol{z}^{n-1} : a_i \in S \textit{ for all } i = 0, 1, 2, \dots, n-1\right\}.$$

Then $S[\boldsymbol{z}; \theta_t]$ is a ring under the usual addition of polynomials, and the multiplication is defined under the rule $(a\boldsymbol{z}^i)(b\boldsymbol{z}^j) = a\theta_t^i(b)\boldsymbol{z}^{i+j}$ for all $a, b \in S$. This ring is known as the skew polynomial ring.

Clearly, $S[\mathtt{z}; \theta_t]$ is a noncommutative ring unless θ_t is the identity automorphism. Therefore, before establishing the structure of codes, we have to specify the existence of left/right divisibility. Recall that for $a(\mathtt{z}), b(\mathtt{z}) \in S[\mathtt{z}; \theta_t]$ with $a(\mathtt{z}) \neq 0$, $a(\mathtt{z}) \mid_r b(\mathtt{z})$ if there exists $c(\mathtt{z}) \in S[\mathtt{z}; \theta_t]$ such that $b(\mathtt{z}) = c(\mathtt{z})a(\mathtt{z})$.

Theorem 2.1 *[19, Theorem 2.4] Let $a(\boldsymbol{z})$ and $b(\boldsymbol{z})$ be two polynomials in $S[\boldsymbol{z};\theta_t]$ with leading coefficient of $b(\boldsymbol{z})$ is a unit, then there exists $q(\boldsymbol{z})$ and $p(\boldsymbol{z})$ such that $a(\boldsymbol{z}) = q(\boldsymbol{z})b(\boldsymbol{z}) + p(\boldsymbol{z})$, where $p(\boldsymbol{z}) = 0$ or $\deg p(\boldsymbol{z}) < \deg a(\boldsymbol{z})$.*

Definition 2.2 ***Greatest Common Right Divisor**: A polynomial $s(\boldsymbol{z})$ is the* gcrd *of $a(\boldsymbol{z})$ and $b(\boldsymbol{z})$ in $S[\boldsymbol{z};\theta_t]$ if*

- *$s(\boldsymbol{z})$ is the right divisor of both $a(\boldsymbol{z})$ and $b(\boldsymbol{z})$, and*
- *for every common right divisor $s'(\boldsymbol{z})$ of $a(\boldsymbol{z})$ and $b(\boldsymbol{z})$, it holds that $s'(\boldsymbol{z}) \mid_r s(\boldsymbol{z})$.*

Similarly, we can define gcld. Obviously, to construct the SGQC codes, we first look at the structure of SC codes on S so that we can develop the improved versions of the results obtained in [13].

Definition 2.3 *Suppose C is a subset of S^n, then C is said to be a SC code of length n if C satisfies the following:*

1. *C is an S-submodule of S^n;*
2. *$\sigma(c) = (\theta_t(c_{n-1}), \theta_t(c_0), \dots, \theta_t(c_{n-2})) \in C$ whenever $c = (c_0, c_1, \dots, c_{n-1}) \in C$.*

Here, σ is called the SC shift operator.

Let $S_n = \dfrac{S[\mathrm{z};\theta_t]}{\langle \mathrm{z}^n - 1\rangle}$, $s(\mathrm{z}) + (\mathrm{z}^n - 1)$ be an element of S_n, and $a(\mathrm{z}) \in S[\mathrm{z};\theta_t]$. Define multiplication from left as

$$a(\mathrm{z}) * (s(\mathrm{z}) + \langle \mathrm{z}^n - 1\rangle) = a(\mathrm{z}) * s(\mathrm{z}) + \langle \mathrm{z}^n - 1\rangle \text{ for all } a(\mathrm{z}) \in S[\mathrm{z};\theta_t]. \tag{2.1}$$

Clearly, multiplication on S_n is well defined. Then, under left multiplication defined in Equation (2.1), S_n is a left $S[\mathrm{z};\theta_t]$-module. Also, the SC codes in $S[\mathrm{z};\theta_t]$ is a left $S[\mathrm{z};\theta_t]$-submodule of the left module S_n. Now, we recall some results of the SC code from [13] that we will use later.

Lemma 2.1 *Every left(or right) S_n-submodule of S_n is principally generated.*

Theorem 2.2 *Let $C_1, C_2 \subseteq F_q^n$ be SC codes of length n with monic generator $s_1(\boldsymbol{z})$, $s_2(\boldsymbol{z}) \in F_q[\boldsymbol{z};\theta_t]$ ($s_1(\boldsymbol{z}), s_2(\boldsymbol{z}) \mid_r (\boldsymbol{z}^n - 1)$). Then the code $C = (1-\mathfrak{v})C_1 \oplus \mathfrak{v}C_2 \subseteq S^n$ is a SC code with generator $s(\boldsymbol{z}) = (1-\mathfrak{v})s_1(\boldsymbol{z}) + \mathfrak{v}s_2(\boldsymbol{z}) \in S[\boldsymbol{z};\theta_t]$ which satisfies $s(\boldsymbol{z}) \mid_r (\boldsymbol{z}^n - 1)$. Equivalently, $C = \langle s(\boldsymbol{z})\rangle$.*

Two polynomials $s_1(\mathrm{z}), s_2(\mathrm{z}) \in S[\mathrm{z};\theta_t]$ are said to be right coprime if there exist polynomials $u_1(\mathrm{z})$ and $u_2(\mathrm{z})$ in $S[\mathrm{z};\theta_t]$ such that $u_1(\mathrm{z})v_1(\mathrm{z}) + u_2(\mathrm{z})v_2(\mathrm{z}) = 1$. The left coprime can be defined similarly. This notion allows for expressing a code through an alternative generator, as presented in the following result.

Lemma 2.2 *[19, Lemma 2.6] Suppose C is an SC code of length n over S with $C = \langle a(\boldsymbol{z})\rangle$ such that $\boldsymbol{z}^n - 1 = s'(\boldsymbol{z})s(\boldsymbol{z})$. Then, every alternative generator of code C has the form $C = \langle v(\boldsymbol{z})a(\boldsymbol{z})\rangle$, where $s'(\boldsymbol{z})$ is the right coprime to $v(\boldsymbol{z})$.*

3. Algebraic Structure of SGQC Codes

In this section, we study the structural properties of the SGQC code over the ring S. We generalize Definition 2 [24] of the SC shift over the ring S. Toward this, first recall the definition of the SGQC codes.

Definition 3.1 *Suppose S is a non-chain ring, and θ_t is an automorphism of S with $|\theta_t| = m_t$. Throughout, Let $t_1, t_2, \dots, t_l$ be positive integers and $N = t_1 + t_2 + \cdots + t_l$. A subset C of $\boldsymbol{S} = S^{t_1} \times S^{t_2} \times \cdots \times S^{t_l}$ is called an SGQC code of block length $(t_1, t_2, \dots, t_l)$ and index l, if C holds the following criteria:*

1. *C is a S-submodule of $\boldsymbol{S}$, and*
2. *if $c = (c_1, c_2, \dots, c_l)$, then $\sigma_l(c) = (\sigma(c_1), \sigma(c_2), \dots, \sigma(c_l)) \in C$, where $c_i = (c_{i1}, c_{i2}, \dots, c_{it_i}) \in S^{t_i}$, for all $i = 1, 2, \dots, l$.*

Hence, SGQC codes of length N and index l over S are closed under the shift σ_l. If each $t_i's$ is equal, then SGQC codes are SQC codes over S. If $l = 1$, SGQC codes are SC codes over S.

Let $a = (a_1, a_2, \dots, a_l) \in \mathbf{S}$, where

$$a_j = (a_{j,0} + \mathfrak{v}a'_{j,0}, a_{j,1} + \mathfrak{v}a'_{j,1}, \dots, a_{j,t_j-1} + \mathfrak{v}a'_{j,t_j-1}) \text{ for } j = 1, 2, \dots, l.$$

For any vector $a_j \in S^{t_j}$, we associate the vector with the polynomial $a_j(\mathtt{z}) = (a_{j,0}+\mathfrak{v}a'_{j,0})+(a_{j,1}+\mathfrak{v}a'_{j,1})\mathtt{z}+ \cdots + (a_{j,t_j-1} + \mathfrak{v}a'_{j,t_j-1})\mathtt{z}^{t_j-1} = \sum_{i=0}^{t_j-1}(a_{j,i} + \mathfrak{v}a'_{j,i})\mathtt{z}^{t_j-1}$ in the left $S[\mathtt{z};\theta_t]$-module $S_{t_j} = \dfrac{S[\mathtt{z};\theta_t]}{\langle \mathtt{z}^{t_j} - 1\rangle}$.

The ring $\mathbf{S}' = S_{t_1} \times S_{t_2} \times \cdots \times S_{t_l}$ is a left $S[\mathtt{z};\theta_t]$-module with left multiplication given by

$$s(\mathtt{z}).a(\mathtt{z}) = (s(\mathtt{z}).a_1(\mathtt{z}), s(\mathtt{z}).a_2(\mathtt{z}), \dots, s(\mathtt{z}).a_l(\mathtt{z})), \tag{3.1}$$

where $s(\mathtt{z}) \in S[\mathtt{z};\theta_t]$ and $a(\mathtt{z}) = (a_1(\mathtt{z}), a_2(\mathtt{z}), \dots, a_l(\mathtt{z})) \in \mathbf{S}'$.
Suppose $a = (a_1, a_2, \dots, a_l)$ is an element of $\mathbf{S}$. Then, the map

$$\mu : \mathbf{S} \to \mathbf{S}' \text{ defined by } \mu(a) = (a_1(\mathtt{z}), a_2(\mathtt{z}), \dots, a_l(\mathtt{z})) = a(\mathtt{z}).$$

It defines a one-to-one correspondence. Hence, a codeword in vector form $c = (c_1, c_2, \dots, c_l) \in C$ will be of the form of a polynomial $c(\mathtt{z}) = (c_1(\mathtt{z}), c_2(\mathtt{z}), \dots, c_l(\mathtt{z}))$ in the set $\mathbf{S}'$.

Lemma 3.1 *C is an SGQC code of block length $(t_1, t_2, \dots, t_l)$ and index l if and only if C is a left $S[\mathtt{z};\theta_t]$-submodule of $\mathbf{S}'$.*

The Lemma 3.1 is valid for every block length $(t_1, t_2, \dots, t_l)$ and m_t, where $|\theta_t| = m_t$. Therefore, there is no need to impose the condition that m_t/t_i for all $i = 1, 2, \dots, l$.

Theorem 3.1 *Let C be a linear code over S of length N. If $C = (1-\mathfrak{v})C_1 \oplus \mathfrak{v}C_2$, where C_1 and C_2 are linear codes of length N over F_q, then C is an SGQC code over S if and only if C_1 and C_2 are SGQC codes of block length $(t_1, t_2, \dots, t_l)$ and index l over F_q.*

Proof: Suppose C_1 and C_2 are SGQC codes of block length $(t_1, t_2, \dots, t_l)$ and index l over F_q. We aim to show that C is an SGQC over S. Let $s = (s_1, s_2, \dots, s_l) \in C$, where each $s_i = (a_i + \mathfrak{v}b_i) \in S$. Pick $a = (a_1, a_2, \dots, a_l)$, where $a_i = (a_{i,1}, a_{i,2}, \dots, a_{i,t_i})$, and $b = (b_1, b_2, \dots, b_l)$, where $b_i = (b_{i,1}, b_{i,2}, \dots, b_{i,t_i})$ for all $i = 1, \dots, l$. Then $a \in C_1$ and $a + b \in C_2$ which implies that

$$\begin{aligned}\sigma_l(a) &= \big(\sigma(a_1), \sigma(a_2), \dots, \sigma(a_l)\big) \in C_1\\ &= \big(\theta_t(a_{1,t_1}), \theta_t(a_{1,1}), \dots, \theta_t(a_{1,t_1-1}), \dots, \theta_t(a_{l,t_l}), \theta_t(a_{l,1}), \dots, \theta_t(a_{l,t_l-1})\big)\\ &= \Big((a_{1,t_1})^{p^t}, (a_{1,1})^{p^t}, \dots, a_{1,t_1-1})^{p^t}, \dots, (a_{l,t_l})^{p^t}, (a_{l,1})^{p^t}, \dots, (a_{l,t_l-1})^{p^t}\Big)\end{aligned}$$

and similarly,

$$\begin{aligned}&\sigma_l(a+b)\\ &= \big(\sigma(a_1+b_1), \sigma(a_2+b_2), \dots, \sigma(a_l+b_l)\big) \in C_2\\ &= \begin{pmatrix}\theta_t(a_{1,t_1}+b_{1,t_1}), \theta_t(a_{1,1}+b_{1,1}), \dots, \theta_t(a_{1,t_1-1}+b_{1,t_1-1}),\\ \dots, \theta_t(a_{l,t_l}+b_{l,t_l}), \theta_t(a_{l,1}+b_{l,1}), \dots, \theta_t(a_{l,t_l-1}+b_{l,t_l-1})\end{pmatrix}\\ &= \begin{pmatrix}(a_{1,t_1})^{p^t}+(b_{1,t_1})^{p^t}, (a_{1,1})^{p^t}+(b_{1,1})^{p^t}, \dots, (a_{1,t_1-1})^{p^t}+(b_{1,t_1-1})^{p^t},\\ \dots, (a_{l,t_l})^{p^t}+(b_{l,t_l})^{p^t}, (a_{l,1})^{p^t}+(b_{l,1})^{p^t}, \dots, (a_{l,t_l-1})^{p^t}+(b_{l,t_l-1})^{p^t}\end{pmatrix}.\end{aligned}$$

Then $\sigma_l(s) = (1-\mathfrak{v})\sigma_l(a) + \mathfrak{v}\sigma_l(a+b) \in C$, which implies that C is an SGQC code of length $N = t_1 + t_2 + \cdots + t_l$ and index l over S.
Conversely, assume C is an SGQC code over S of length $N = t_1 + t_2 + \cdots + t_l$ and index l. For any

$a = (a_1, a_2, \dots, a_l) \in C_1$ and $b = (b_1, b_2, \dots, b_l) \in C_2$, and if we consider $s_i = a_i + \mathfrak{v}(-a_i + b_i)$, for all $i = 1, \dots, l$, then $s = (s_1, s_2, \dots, s_l) \in C$. Since C is an SGQC code over S, we have

$$\begin{aligned}
&\sigma_l(s)\\
&= \big(\sigma(s_1), \sigma(s_2), \dots, \sigma(s_l)\big) \in C\\
&= \big(\sigma(a_1) + \mathfrak{v}\sigma(-a_1 + b_1), \sigma(a_2) + \mathfrak{v}\sigma(-a_2 + b_2), \dots, \sigma(a_l) + \mathfrak{v}\sigma(-a_l + b_l)\big)\\
&= \begin{pmatrix} \theta_t(a_{1,t_1})^{p^t} + \mathfrak{v}(-a_{1,t_1} + b_{1,t_1})^{p^t}, \theta_t(a_{1,1})^{p^t} + \mathfrak{v}(-a_{1,1} + b_{1,1})^{p^t}, \dots, \\ \theta_t(a_{1,t_1-1})^{p^t} + v(-a_{1,t_1-1} + b_{1,t_1-1})^{p^t}, \dots, \theta_t(a_{l,t_l})^{p^t} + \mathfrak{v}(-a_{l,t_l} + b_{l,t_l})^{p^t}, \\ \theta_t(a_{l,1})^{p^t} + \mathfrak{v}(-a_{l,1} + b_{l,1})^{p^t}, \dots, \theta_t(a_{l,t_1-1})^{p^t} + \mathfrak{v}(-a_{l,t_l-1} + b_{l,t_l-1})^{p^t} \end{pmatrix}.
\end{aligned}$$

Therefore, $\phi(\sigma_l(s)) = (\sigma_l(a), \sigma_l(b)) \in C_1 \otimes C_2$, and hence $\sigma_l(a) \in C_1$ and $\sigma_l(b) \in C_2$. Thus, C_1 and C_2 are SGQC codes over F_q. □

4. Duality of SGQC codes

Here, we present the study of the dual of an SGQC code over S of block length $(t_1, t_2, \dots, t_l)$ and index l. First, we recall some basic definitions.

Suppose $a = (a_1, a_2, \dots, a_l)$ and $b = (b_1, b_2, \dots, b_l)$ are two elements of $\mathbf{S} = S^{t_1} \times S^{t_2} \times \cdots \times S^{t_l}$, where

$$\begin{aligned}
a_i &= (a_{i,0} + \mathfrak{v}a'_{i,0}, a_{i,1} + \mathfrak{v}a'_{i,1}, \dots, a_{i,t_i-1} + \mathfrak{v}a'_{i,t_i-1}), \text{ and}\\
b_i &= (b_{i,0} + \mathfrak{v}b'_{i,0}, b_{i,1} + \mathfrak{v}b'_{i,1}, \dots, b_{i,t_i-1} + \mathfrak{v}b'_{i,t_i-1}) \in S^{t_i}.
\end{aligned}$$

The usual inner product of a and b is defined as

$$\langle a, b\rangle = \sum_{i=1}^{l} a_i.b_i = \sum_{i=1}^{l}\sum_{j=0}^{t_i-1}(a_{i,j} + \mathfrak{v}a'_{i,j}).(b_{i,j} + \mathfrak{v}b'_{i,j}).$$

Suppose C is an SGQC code of block length $(t_1, t_2, \dots, t_l)$ and index l. The dual of C is denoted by $C^\perp$ and is defined as $C^\perp = \{a = (a_1, a_2, \dots, a_l) \in \mathbf{S} : \langle a, c\rangle = \sum_{i=1}^{l} a_i c_i = 0, \text{ for each } c = (c_1, c_2, \dots, c_l) \in C\}$.

Theorem 4.1 *Let C be an SGQC code of block length $(t_1, t_2, \dots, t_l)$ and index l over S, then the dual of C is an SGQC code of length N and index l.*

Proof: Let

$$\begin{aligned}
s &= \big(s_1, s_2, \dots, s_l\big) \in C^\perp\\
&= \begin{pmatrix} s_{1,0} + \mathfrak{v}s'_{1,0}, s_{1,1} + \mathfrak{v}s'_{1,1}, \dots, s_{1,t_1-1} + \mathfrak{v}s'_{1,t_1-1} \\ s_{2,0} + \mathfrak{v}s'_{2,0}, s_{2,1} + \mathfrak{v}s'_{2,1}, \dots, s_{2,t_2-1} + \mathfrak{v}s'_{2,t_2-1} \\ \dots, \quad \dots, \quad ,\dots \quad ,\dots \\ s_{l,0} + \mathfrak{v}s'_{l,0}, s_{l,1} + \mathfrak{v}s'_{l,1}, \dots, s_{l,t_l-1} + \mathfrak{v}s'_{l,t_l-1} \end{pmatrix}.
\end{aligned}$$

Our objective is to show $\sigma_l(s) \in C^\perp$.
i.e,

$$\begin{aligned}
&\sigma_l(s)\\
&= (\sigma(s_1), \sigma(s_2), \dots, \sigma(s_l))\\
&= \begin{pmatrix} (\theta_t(s_{1,t_1-1} + \mathfrak{v}s'_{1,t_1-1}), \theta_t(s_{1,0} + \mathfrak{v}s'_{1,0}), \dots, \theta_t(s_{1,t_1-2} + \mathfrak{v}s'_{1,t_1-2}) \\ \theta_t(s_{2,t_2-1} + \mathfrak{v}s'_{2,t_2-1}), \theta_t(s_{2,0} + \mathfrak{v}s'_{2,0}), \dots, \theta_t(s_{2,t_2-2} + \mathfrak{v}s'_{2,t_2-2}) \\ \dots, \quad ,\dots \quad ,\dots \quad ,\dots \\ \theta_t(s_{l,t_l-1} + \mathfrak{v}s'_{l,t_l-1}), \theta_t(s_{l,0} + \mathfrak{v}s'_{l,0}), \dots, \theta_t(s_{l,t_l-2} + \mathfrak{v}s'_{l,t_l-2}) \end{pmatrix}.
\end{aligned}$$

Now, for any $c=(c_1,c_2,\dots,c_l)\in C$, for all $i=1,2,\dots,l$

$$c_i=\begin{pmatrix}(c_{1,0}+\mathfrak{v}c'_{1,0},c_{1,1}+\mathfrak{v}c'_{1,1},\dots,c_{1,t_1-1}+\mathfrak{v}c'_{1,t_1-1}\\ c_{2,0}+\mathfrak{v}c'_{2,0},c_{2,1}+\mathfrak{v}c'_{2,1},\dots,c_{2,t_2-1}+\mathfrak{v}c'_{2,t_2-1}\\ \dots \quad \dots \quad \dots \quad \dots\\ c_{l,0}+\mathfrak{v}c'_{l,0},c_{l,1}+\mathfrak{v}c'_{l,1},\dots,c_{l,t_l-1}+\mathfrak{v}c'_{l,t_l-1})\end{pmatrix}.$$

We have to show that $\langle\sigma_l(s),c\rangle=0$.
Now,

$$\begin{aligned}&\langle\sigma_l(s),c\rangle\\ =&\begin{pmatrix}\theta_t(s_{1,t_1-1}+\mathfrak{v}s'_{1,t_1-1})(c_{1,0}+\mathfrak{v}c'_{1,0})+\theta_t(s_{1,0}+\mathfrak{v}s'_{1,0})(c_{1,1}+\mathfrak{v}c'_{1,1})+\\ dots+\theta_t(s_{1,t_1-2}+\mathfrak{v}s'_{1,t_1-2})(c_{1,t_1-1}+\mathfrak{v}c'_{1,t_1-1})+\dots+\theta_t(s_{l,t_l-1}+\\ \mathfrak{v}s'_{l,t_l-1})(c_{l,0}+\mathfrak{v}c'_{l,0})+\dots+\theta_t(s_{l,t_l-2}+\mathfrak{v}s'_{l,t_l-2})(c_{l,t_l-1}+\mathfrak{v}c'_{l,t_l-1})\end{pmatrix}.\end{aligned}$$

As C is an SGQC code, then $\sigma_l^k(c)\in C$ for any positive integer k. Now, suppose that

$$M=\ \mathrm{lcm}(m_t,t_1,t_2,\dots,t_l),$$

where m_t is the order of automorphism θ_t, then

$$\sigma_l^M(Y)=Y,\quad \text{for any}\quad Y\in\mathbf{S}=S^{t_1}\times S^{t_2}\times\dots\times S^{t_l}.$$

Hence, $\theta_t^M(y)=y,$ for any $y\in Y$, and $\theta_t^{M-1}(y)=\theta_t^{-1}(y)$. So,

$$\begin{aligned}&\sigma_l^{M-1}(c)\\ =&\begin{pmatrix}\theta_t^{-1}(c_{1,1}+\mathfrak{v}c'_{1,1}),\theta_t^{-1}(c_{1,2}+\mathfrak{v}c'_{1,2}),\dots,\theta_t^{-1}(c_{1,t_l-1}+\mathfrak{v}c'_{1,t_l-1}),\\ \theta_t^{-1}(c_{1,0}+\mathfrak{v}c'_{1,0}),\dots,\theta_t^{-1}(c_{l,1}+\mathfrak{v}c'_{l,1}),\\ \theta_t^{-1}(c_{l,2}+\mathfrak{v}c'_{l,2}),\dots,\theta_t^{-1}(c_{l,t_l-1}+\mathfrak{v}c'_{l,t_l-1}),\theta_t^{-1}(c_{l,0}+\mathfrak{v}c'_{l,0})\end{pmatrix}.\end{aligned}$$

Now, as C is an SGQC code, we have

$$\begin{aligned}&\langle s,\sigma_l^{M-1}(c)\rangle\\ =&(s_{1,0}+\mathfrak{v}s'_{1,0}).(\theta_t^{-1}(c_{1,1}+\mathfrak{v}c'_{1,1})+\dots+(s_{1,t_1-1}+\mathfrak{v}s'_{1,t_1-1}).\theta_t^{-1}(c_{1,0}+\mathfrak{v}c'_{1,0})+\dots+(s_{l,0}+\mathfrak{v}s'_{l,0}).\theta_t^{-1}(c_{l,1}+\mathfrak{v}c'_{l,1})\\ &+\dots+(s_{l,t_l-1}+\mathfrak{v}s'_{l,t_l-1}).\theta_t^{-1}(c_{l,0}+\mathfrak{v}c'_{l,0})=0.\end{aligned}$$

On applying θ_t to both sides of the above equation, we get

$$\begin{aligned}&\theta_t(s_{1,0}+\mathfrak{v}s'_{1,0}).(c_{1,1}+\mathfrak{v}c'_{1,1})+\dots+\theta_t(s_{1,t_1-1}+\mathfrak{v}s'_{1,t_1-1}).(c_{1,0}+\mathfrak{v}c'_{1,0})\\ &+\dots+\theta_t(s_{l,0}+\mathfrak{v}s'_{l,0}).(c_{l,1}+\mathfrak{v}c'_{l,1})+\dots+\theta_t(s_{l,t_l-1}+\mathfrak{v}s'_{l,t_l-1}).(c_{l,0}+\mathfrak{v}c'_{l,0})=0.\end{aligned}$$

We get $\langle\sigma_l(s),c\rangle=0$ by arranging the above expression. Thus, $\sigma_l(s)\in C^\perp$. Hence, $C^\perp$ is an SGQC code of length $N=t_1+t_2+\dots+t_l$ and index l. □

From Theorems 3.1, 4.1, the following corollary follows easily.

Corollary 4.1 *An SGQC code C over S is self-dual if and only if its constituent codes C_1 and C_2 are self-dual SGQC codes over F_q.*

In Section 3, we defined the map $\mu:\mathbf{S}\to\mathbf{S}'$, which defines a one-to-one correspondence between SGQC codes over S of block length $(t_1,t_2,\dots,t_l)$ and index l and linear codes over $\mathbf{S}'=S_{t_1}\times S_{t_2}\times\dots\times S_{t_l}$ of length N. Here, we consider the above map μ as a polynomial representation of the SGQC codes, and then consider the **Hermitian** inner product. Following the definition in [5, Section 6], define a "conjugation" map ψ_j on S_{t_j} by

$$\psi_j(a\mathtt{z}^i)=\theta_t^{-i}(a)\mathtt{z}^{t_j-i};0\le i\le t_j-1\text{ and }j=1,2,\dots,l,$$

and the map is extended to all elements of S_{t_j} by linearity of addition.

Definition 4.1 *Suppose $a(\boldsymbol{z}) = (a_1(\boldsymbol{z}), a_2(\boldsymbol{z}), \ldots, a_l(\boldsymbol{z}))$, and $b(\boldsymbol{z}) = (b_1(\boldsymbol{z}), b_2(\boldsymbol{z}), \ldots, b_l(\boldsymbol{z}))$ of $\boldsymbol{S'}$, then the Hermitian inner product is given by*

$$a(\boldsymbol{z}) * b(\boldsymbol{z}) = \sum_{i=1}^{l} a_i(\boldsymbol{z}) \cdot \psi_j(b_i(\boldsymbol{z})).$$

Suppose that the order of θ_t is m_t and m_t/t_j : for all $j = 1, 2, \ldots, l$. Therefore, $\theta_t^{m_t} = 1 = \theta_t^{t_j}$. The following result is the generalization of Proposition 3.2 [18].

Theorem 4.2 *Let $s = (s_{t_1}, s_{t_2}, \ldots, s_{t_l})$, $w = (w_{t_1}, w_{t_2}, \ldots, w_{t_l}) \in \boldsymbol{S}$, and suppose $s(\boldsymbol{z})$ and $w(\boldsymbol{z}) \in \boldsymbol{S'}$ denote the polynomial representation of s and w, respectively. Then $\sigma_l^r(s) \cdot w = 0$: for $0 \leq r \leq t_j - 1$, and for all $j = 1, 2, \ldots, l$ if and only if $s(\boldsymbol{z}) * w(\boldsymbol{z}) = 0$.*

Proof: First, we assume that $s(\mathbf{z}) * w(\mathbf{z}) = 0$. Then

$$\begin{aligned}
0 &= \sum_{j=1}^{l} s_{t_j}(\mathbf{z}) \cdot \psi_j(w_{t_j}(\mathbf{z})), \\
&= \sum_{j=1}^{l} \left(\sum_{n=0}^{t_j-1} s_{n,j} + \mathfrak{v} s'_{n,j} \mathbf{z}^n \right) \cdot \psi_j \left(\sum_{k=0}^{t_j-1} w_{k,j} + \mathfrak{v} w'_{k,j} \mathbf{z}^k \right), \\
&= \sum_{j=1}^{l} \left(\sum_{n=0}^{t_j-1} s_{n,j} + \mathfrak{v} s'_{n,j} \mathbf{z}^n \right) \cdot \left(\sum_{k=0}^{t_j-1} \theta_t^{-k} (w_{k,j} + \mathfrak{v} w'_{k,j}) \mathbf{z}^{t_j-k} \right), \\
&= \sum_{m=0}^{t_j-1} \left(\sum_{j=1}^{l} \sum_{i=0}^{t_j-1} ((s_{i+m,j} + \mathfrak{v} s'_{i+m,j} \right) \theta_t^m (w_{i,j} + \mathfrak{v} w'_{i,j}))) \mathbf{z}^m,
\end{aligned}$$

where the subscript $i + t_j - 1$ is taken modulo t_j, for all $j = 1, 2, \ldots, l$. By comparing the coefficients of $\mathbf{z}^{t_j-1}$ for all $j = 1, 2, \ldots, l$, on both sides, we get

$$\begin{aligned}
0 &= \sum_{j=1}^{l} \sum_{i=0}^{t_j-1} \left((s_{i+m,j} + \mathfrak{v} s'_{i+m,j}) \theta_t^m (w_{i,j} + \mathfrak{v} w'_{i,j}) \right), \text{ for } 0 \leq m \leq t_j - 1, \\
&= \theta_t^m (\sigma_l^{n_j - m}(s).w), \text{ for all } j = 1, 2, \ldots, l.
\end{aligned}$$

Hence, it implies that $\sigma_l^{n_j-m}(s).w = 0$ for all $0 \leq m \leq t_j - 1$ which is equivalent to $\sigma_l^r(s) \cdot w = 0$ for all $0 \leq r \leq t_j - 1$. □

Corollary 4.2 *Suppose C is an* `SGQC` *code of block length $(t_1, t_2, \ldots, t_l)$ and index l over S. Then*

$$C^{\perp} = \{w(\boldsymbol{z}) \in \boldsymbol{S'} : s(\boldsymbol{z}) * w(\boldsymbol{z}) = 0, \text{ for all } s(\boldsymbol{z}) \in C\}.$$

Theorem 4.3 *Let C be an* `SGQC` *code of block length $(t_1, t_2, \ldots, t_l)$ and index l over S. Then $\mu(C^{\perp}) = \mu(C)^{\perp}$, where the dual in $\boldsymbol{S}$ and $\boldsymbol{S'}$ is obtained with respect to Euclidean and Hermitian inner product, respectively.*

Proof: Suppose C is an `SGQC` code and $s \in C$, then $\sigma_l^k(s) \in C$. From Theorem 4.2, we have $\sigma_l^k(s) \cdot w = 0; 0 \leq k \leq t_j - 1$, for all $j = 1, 2, \ldots, l$. which implies that $\mu(w) \in \mu(C^{\perp})$. Now, once again from Theorem 4.2, if $\sigma_l^k(s) \cdot w = 0$, then

$$s(\mathbf{z}) * w(\mathbf{z}) = 0, \text{ and } \mu(s) * \mu(w) = 0.$$

As $\mu(s) \in \mu(C)$, we get $\mu(w) \in \mu(C)^{\perp}$. Therefore, $\mu(C^{\perp}) \subseteq \mu(C)^{\perp}$.

On the contrary, assume $w(\mathbf{z}) = \mu(w) \in \mu(C)^{\perp}$, then there exists $s(\mathbf{z}) = \mu(s)$ in $\mu(C)$ such that $s(\mathbf{z}) * w(\mathbf{z}) = 0 = \mu(s) * \mu(w)$. From previous Theorem 4.2, $\sigma_l^k(s) \cdot w = 0$. As $s \in C$ and C is an `SGQC` code, then $\sigma_l^k(s) \in C$. Therefore, $w \in C$ that means $\mu(w) \in \mu(C^{\perp})$. Thus, $\mu(C) \subseteq \mu(C)^{\perp}$ and as a result, we get $\mu(C) = \mu(C)^{\perp}$. □

Corollary 4.3 *Suppose C is an SGQC code of block length $(t_1, t_2, \ldots, t_l)$ and index l over S. Then C is self-dual over S under the usual inner product if and only if $\mu(C)$ is self-dual over S_{t_j} for all $j = 1, 2, \ldots, l$, under the Hermitian inner product.*

5. Generator set for SGQC codes

In this section, we extend the results derived in Section 4 of Gao et al. [11] to obtain the generator set for SGQC codes and establish a BCH-type bound on their minimum Hamming distance.

A SGQC code C of block length $(t_1, t_2, \ldots, t_l)$ and index l over S is called a ρ-generator code if ρ is the least positive integer for which there exist codewords

$$a_j(\mathbf{z}) = (a_{j1}(\mathbf{z}), a_{j2}(\mathbf{z}), \ldots, a_{jl}(\mathbf{z})), \text{ for } 1 \leq j \leq \rho$$

such that $C = s_1(\mathbf{z})a_1(\mathbf{z}) + s_2(\mathbf{z})a_2(\mathbf{z}) + \cdots + s_\rho(\mathbf{z})a_\rho(\mathbf{z})$ for some $s_1(\mathbf{z}), s_2(\mathbf{z}), \ldots, s_\rho(\mathbf{z})$ in $S[\mathbf{z}; \theta_t]$. The set $\{a_1(\mathbf{z}), a_2(\mathbf{z}), \ldots, a_\rho(\mathbf{z})\}$ is called a generating set for SGQC code C.

5.1. 1-generator polynomial SGQC codes

Definition 5.1 *If an SGQC code C generated by a single element $s(\boldsymbol{z}) = (s_1(\boldsymbol{z}), s_2(\boldsymbol{z}), \ldots, s_l(\boldsymbol{z}))$, where $s_i(\boldsymbol{z}) \in S_{t_i}$, for all $i = 1, 2, \ldots, l$, then C is called a 1-generator SGQC code. Clearly, it has the form $C = \{a(\boldsymbol{z})s(\boldsymbol{z}) = (a(\boldsymbol{z})s_1(\boldsymbol{z}), a(\boldsymbol{z})s_2(\boldsymbol{z}), \ldots, a(\boldsymbol{z})s_l(\boldsymbol{z})) : a(\boldsymbol{z}) \in S[\boldsymbol{z}; \theta_t]\}$. The monic polynomial $f(\boldsymbol{z})$ of a minimum degree satisfying $s(\boldsymbol{z})f(\boldsymbol{z}) = 0$ is called the parity check polynomial of C.*

Suppose C is an SGQC code of 1-generator of block length $(t_1, \ldots, t_l)$ and length $N = t_1+t_2+\cdots+t_l$ with the generator polynomial $s(\mathbf{z}) = (s_1(\mathbf{z}), s_2(\mathbf{z}), \ldots, s_l(\mathbf{z}))$, where $s_i(\mathbf{z}) \in S_{t_i} = \dfrac{S[\mathbf{z}; \theta_t]}{\langle \mathbf{z}^{t_i} - 1\rangle}$, *and* $i = 1, 2, \ldots, l$. Define the map

$$\Psi_i : \mathbf{S}' \to S_{t_i}, \text{ given by } (a_1(\mathbf{z}), a_2(\mathbf{z}), \ldots, a_l(\mathbf{z})) \to a_i(\mathbf{z}).$$

It is a well-defined module homomorphism. Also, $\Psi_i(C) = C_i$. Since C is a 1-generator SGQC code over S, C is a left $S[\mathbf{z}; \theta_t]$-submodule of $\mathbf{S}'$. Hence, C_i is a left S-submodule of S_{t_i}. i.e. C_i is an SC code of length t_i. By Theorem 2.2, we have $C_i = \langle s_i(\mathbf{z})\rangle$, where $s_i(\mathbf{z}) = (1 - \mathfrak{v})s_i^1(\mathbf{z}) + \mathfrak{v}s_i^2(\mathbf{z})$, and $s_i(\mathbf{z}) \mid_r (\mathbf{z}^{t_i} - 1)$, such that $\mathbf{z}^{t_i} - 1 = s_i'(\mathbf{z})s_i(\mathbf{z})$ for $1 \leq i \leq l$. And by the Lemma 2.2, any generator of C_i has the form $\langle q_i(\mathbf{z})s_i(\mathbf{z})\rangle$, where $s_i'(\mathbf{z})$ and $q_i(\mathbf{z})$ are right coprime. From the above discussion, we summarize this in the following theorem.

Theorem 5.1 *Suppose C is a 1-generator SGQC code of block length $(t_1, t_2, \ldots, t_l)$ and index l over S, then 1-generator polynomial of C can be taken of the form*

$$V(\boldsymbol{z}) = (q_1(\boldsymbol{z})s_1(\boldsymbol{z}), q_2(\boldsymbol{z})s_2(\boldsymbol{z}), \ldots, q_l(\boldsymbol{z})s_l(\boldsymbol{z})),$$

where $s_i(\boldsymbol{z}) \mid_r (\boldsymbol{z}^{t_i} - 1)$ and $gcrd(q_i(\boldsymbol{z}), (\boldsymbol{z}^{t_i} - 1)/s_i(\boldsymbol{z})) = 1$.

In [11], Theorem 4.2 discusses the parity check polynomial for 1-generator SGQCcode over the field F_q. We extend this over the ring S.

Theorem 5.2 *Let C be a 1-generartor SGQC code of block length $(t_1, t_2, \ldots, t_l)$ and index l , generated by $s(\boldsymbol{z}) = (s_1(\boldsymbol{z}), s_2(\boldsymbol{z}), \ldots, s_l(\boldsymbol{z})) \in \boldsymbol{S}'$. Assume $f(\boldsymbol{z})$ is the parity check polynomial of the 1-generator SGQC code C. Then*

1. $f(\boldsymbol{z}) = lclm\{\dfrac{\boldsymbol{z}^{t_i} - 1}{gcld(s_i(\boldsymbol{z}), \boldsymbol{z}^{t_i} - 1}\}_{i=1,2,\ldots,l}$.
2. *As a submodule over S, dimension of C is* $\deg(f(\boldsymbol{z}))$.

Proof: Let $\mathbf{S}' = S_{t_1} \times S_{t_2} \times \cdots \times S_{t_l}$ be a $S[\mathsf{z};\theta_t]$-module under componentwise addition and scalar multiplication as defined in Equation (3.1). For $1 \leq i \leq l$, define well defined module-homomorphism

$$\Psi_i : \mathbf{S}' \to S_{t_i}, \text{ given by } \psi_i(a_1(\mathsf{z}), a_2(\mathsf{z}), \ldots, a_l(\mathsf{z})) = a_i(\mathsf{z}).$$

It implies that $s_i(\mathsf{z}) = s_i^1(\mathsf{z}) + \mathfrak{v}s_i^2(\mathsf{z}) \in \psi_i(C)$, since $s(\mathsf{z}) \in C$ implies $\alpha(\mathsf{z}) \cdot s(\mathsf{z}) \in C$. So, $\Psi_i(C)$ is a left-submodule of S_{t_i} and, hence a $\mathtt{SC}$ code of length t_i in $\dfrac{S[\mathsf{z};\theta_t]}{\langle \mathsf{z}^{t_i} - 1\rangle}$. It has a parity check polynomial $f_i(\mathsf{z}) = \dfrac{\mathsf{z}^{t_i} - 1}{gcld(s_i(\mathsf{z}), \mathsf{z}^{t_i} - 1)}$. If $\alpha(\mathsf{z}) \cdot s_i(\mathsf{z}) = 0,$ where $\alpha(\mathsf{z}) = \alpha^1(\mathsf{z}) + \mathfrak{v}\alpha^2(\mathsf{z})$ and $s_i(\mathsf{z}) = s_i^1(\mathsf{z}) + \mathfrak{v}s_i^2(\mathsf{z})$, then $f_i(\mathsf{z}) \mid_r \alpha(\mathsf{z})$; it is against the assumption that $f_i(\mathsf{z})$ has the least possible degree. Therefore, statement 1 follows the assumption that $f(\mathsf{z})$ has the minimum possible degree.
For the statement 2, we define a map $\zeta : S[\mathsf{z};\theta_t] \to \mathbf{S}'$ given by

$$\beta(\mathsf{z}) \to \beta(\mathsf{z})(s_1(\mathsf{z}), s_2(\mathsf{z}), \ldots, s_l(\mathsf{z})).$$

It is a $S[\mathsf{z};\theta_t]$-module homomorphism with kernel $(f(\mathsf{z}))$. Hence, $C \cong \dfrac{S[\mathsf{z};\theta_t]}{\langle f(\mathsf{z})\rangle}$ and $\dim \dfrac{S[\mathsf{z};\theta_t]}{\langle f(\mathsf{z})\rangle} = \deg(f(\mathsf{z}))$. Thus, the dimension of C is $\deg(f(\mathsf{z}))$. □

Corollary 5.1 *Suppose C is an SGQC code of block length $(t_1, t_2, \ldots, t_l)$ and index l with the generator $c(z) = (c_1(z), c_2(z), \ldots, c_l(z))$. Suppose $f_i(z) := gcld(f(z), z^{t_i} - 1)$. With the notation of Theorem 5.2, if $\Psi_i(C)$ has generator polynomial $u_i(z)$, then for some polynomial $v_i(z)$ dividing by $f_i(z)$, we have*

$$\frac{(z^{t_i} - 1)}{f_i(z)} \cdot v_i(z) = u_i(z). \tag{5.1}$$

Proof: As f(z) is the parity-check polynomial of C, then

$$\mathbf{0} = f(\mathsf{z})c(\mathsf{z}) = f(\mathsf{z})(c_1(\mathsf{z}), c_2(\mathsf{z}), \ldots, c_l(\mathsf{z})), \tag{5.2}$$

i.e, $f(\mathsf{z}) \cdot c_i(\mathsf{z}) = 0 \pmod{\mathsf{z}^{t_i} - 1}$, for all $1 \leq i \leq l$. Therefore, $c_i(\mathsf{z}) \in \langle \dfrac{\mathsf{z}^{t_i} - 1}{f_i(\mathsf{z})} \rangle$, hence $\psi_i(C) \subseteq \langle \dfrac{\mathsf{z}^{t_i} - 1}{f_i(\mathsf{z})} \rangle$. This implies 5.1 if $u_i(\mathsf{z})$ is the generator polynomial. It is deduced from $f_i(\mathsf{z}) \cdot u_i(\mathsf{z}) = (\mathsf{z}^{t_i} - 1)v_i(\mathsf{z})$ and $f_i(\mathsf{z}) \cdot u_i(\mathsf{z}) = \mathsf{z}^{t_i} - 1$ that $v_i(\mathsf{z})/f_i(\mathsf{z})$. □

In the following examples, we use Theorems 5.1 and 5.2.

Example 5.1 *Let $S[z;\theta]$ be the skew polynomial ring under the Frobenius automorphism θ over $S = F_4 + \mathfrak{v}F_4$. Consider the polynomials $s_1(z) = z^2 + 1 \mid_r (z^4 - 1)$, and $s_2(z) = z^3 + 1 \mid_r (z^6 - 1)$ in $S[z;\theta]$. Let C be a 1-generator SGQC code of block length $(4, 6)$ and index 2, generated by $s(z) = (s_1(z), s_2(z))$. By using theorem 6, we compute*

$$\begin{aligned} f(z) =& lclm\left\{z^2 + 1, z^3 + 1\right\}, \\ =& z^4 + z^3 + z + 1. \end{aligned}$$

The dimension of C is 4, and its generator matrix over S is given as

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Its Gray image is $[20, 4, 8]$ QC code of degree 10 linear code over F_4.

Example 5.2 *Consider $S[\boldsymbol{z};\theta]$, where θ is a Frobenius automorphism over $S = F_9 + \mathfrak{v}F_9$. Take the polynomials $s_1(\boldsymbol{z}) = \boldsymbol{z}^2 + 2 \mid_r (\boldsymbol{z}^4 - 1)$, and $s_2(\boldsymbol{z}) = \boldsymbol{z}^6 + 2\boldsymbol{z}^4 + \boldsymbol{z}^2 + 2 \mid_r (\boldsymbol{z}^8 - 1)$ in $S[\boldsymbol{z};\theta]$. Let C be a 1-generator SGQC code of block length $(4,8)$ and index 2, generated by $s(\boldsymbol{z}) = (s_1(\boldsymbol{z}), s_2(\boldsymbol{z}))$. Using Theorem 6, we obtain $f(\boldsymbol{z}) = \boldsymbol{z}^2 + 1$. The dimension of C is 2, and its generator matrix over $F_9 + \mathfrak{v}F_9$ is given as*

$$\begin{pmatrix} 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \end{pmatrix}.$$

Its Gray image is $[24,2,12]$ SC code of degree 12 linear code over F_9.

In [11], the authors provide the minimum Hamming distance bound for a 1-generator SGQC code in the case where $|\langle\theta_t\rangle| = m_t$ divides each t_i over F_q. Using Proposition 1 along with Corollary 1 of [13] and Theorem 4.1, we give the following bound for 1-generator SGQC codes over S.

Theorem 5.3 *Let C be a 1-generator SGQC code of block length $(t_1, t_2\ , \ldots, t_l)$ and index l over S, such that $C = (1-\mathfrak{v})C_1 \oplus \mathfrak{v}C_2$, where C_1 and C_2 are 1-generator SGQC codes over F_q with generator polynomial $s(\boldsymbol{z}) = (s_1(\boldsymbol{z}), \ldots, s_l(\boldsymbol{z}))$ and $s'(\boldsymbol{z}) = (s'_1(\boldsymbol{z}), \ldots, s'_l(\boldsymbol{z}))$, where $s_i(\boldsymbol{z})$ and $s'_i(\boldsymbol{z})$ are in $\dfrac{F_q[\boldsymbol{z};\theta_t]}{\langle \boldsymbol{z}^{t_1} - 1\rangle} \times \dfrac{F_q[\boldsymbol{z};\theta_t]}{\langle \boldsymbol{z}^{t_2} - 1\rangle} \times \cdots \times \dfrac{F_q[\boldsymbol{z};\theta_t]}{\langle \boldsymbol{z}^{t_l} - 1\rangle}$. If d_1 and d_2 are designed distances of C_1 and C_2 respectively, then designed distance of C $d(C) \geq min\{d_1, d_2\}$.*

We review Theorem 7 of [13] in the below Theorem 5.4, which gives the number of SC codes of a certain length. While following its approach, we introduce different notations suitable for our result.

Theorem 5.4 *Let $(t_i, m_t) = 1$ and $\boldsymbol{z}^{t_i} - 1 = \prod_{j=1}^{r} p_{ij}^{s_{ij}}(\boldsymbol{z})$ where $p_{ij}(\boldsymbol{z}) \in F_q[\boldsymbol{z};\theta_t]$ is irreducible polynomial. Then the number of SC codes of length n over S is $\prod_{j=1}^{r}(s_{ij}+1)^2$.*

With the help of the above Theorem 5.4, we give the number of 1-generator SGQC codes in the below theorem.

Theorem 5.5 *Suppose C is the 1-generator SGQC code of block length $(t_1, \ldots, t_l)$ and length $N = t_1 + \cdots + t_l$ and index l over S generated by $s(\boldsymbol{z}) = (s_1(\boldsymbol{z}), s_2(\boldsymbol{z}), \ldots, s_l(\boldsymbol{z})) \in \boldsymbol{S'}$. Let $(m_t, t_i) = 1$, where $|\theta_t| = m_t$ and $\boldsymbol{z}^{t_i} - 1 = \prod_{j}^{r} v_{ij}^{s_{ij}}(\boldsymbol{z})$, where $v_{ij}(\boldsymbol{z}) \in F_q[\boldsymbol{z};\theta_t]$. Then the number of 1-generator SGQC codes of length N over S is $\prod_{i}^{l}(\prod_{j=1}^{r}(s_{ij}+1)^2)$.*

Proof: Define the map

$$\Psi_i : \mathbf{S'} \to S_{t_i},\ \text{given by}$$
$$(a_1(\mathrm{z}), a_2(\mathrm{z}), \ldots, a_l(\mathrm{z})) \to a_i(\mathrm{z}).$$

It is a well-defined module homomorphism. It implies that $s_i(\mathrm{z}) = s_i^1(\mathrm{z}) + \mathfrak{v}s_i^2(\mathrm{z}) \in \psi_i(C)$, since $s(\mathrm{z}) \in C$ implies $\alpha(\mathrm{z}) \cdot s(\mathrm{z}) \in C$. Thus, $\Psi_i(C)$ is a left-submodule of S_{t_i} and therefore an SC code of length t_i in $\dfrac{S[\mathrm{z};\theta_t]}{\langle \mathrm{z}^{t_i} - 1\rangle}$. Now, $\mathrm{z}^{t_i} - 1 = \prod_{j=1}^{r} v_{ij}^{s_{ij}}(\mathrm{z})$ where $v_{ij}(\mathrm{z}) \in F_q[\mathrm{z};\theta_t]$ is irreducible polynomial for all $i = 1, 2, \ldots, l$. By Theorem 5.4, the number of SC codes of length t_i over S is $\prod_{j=1}^{r}(s_{ij}+1)^2$. Now, taking the cartesian product of $\psi_i(C)$ for all $i = 1, 2, \ldots, l$. i.e, $\psi_1(C) \times \psi_2(C) \times \cdots \times \psi_l(C)$ which is isomorphic to C. Hence, the number of 1-generator SGQC code over S is $\prod_{i}^{l}(\prod_{j=1}^{r}(s_{ij}+1)^2)$. □

In the following result, we present the alternative type of generator set that we employ in our computation due to its simplicity.

Theorem 5.6 *Let $C = (1-\mathfrak{v})C_1 \oplus vC_2$ be a SGQC code of block length $(t_1, \ldots, t_l)$ and length $N = t_1 + t_2 + \cdots + t_l$ and index l over S. Let $f(\boldsymbol{z})$ and $g(\boldsymbol{z})$ be 1-generator polynomials of C_1 and C_2 over F_q, respectively. Then $C = \langle (1-\mathfrak{v})f(\boldsymbol{z}), \mathfrak{v}g(\boldsymbol{z})\rangle$.*

Proof: Since we have 1-generator $\texttt{SGQC}$ codes $C_1 = \langle f(\mathtt{z})\rangle$ and $C_2 = \langle g(\mathtt{z})\rangle$, where

$$f(\mathtt{z}) = (f_1(\mathtt{z}), f_2(\mathtt{z}), \ldots, f_l(\mathtt{z})) \quad \text{and} \quad g(\mathtt{z}) = (g_1(\mathtt{z}), g_2(\mathtt{z}), \ldots, g_l(\mathtt{z})),$$

with each $f_i(\mathtt{z})$ and $g_i(\mathtt{z})$ being right divisors of $\mathtt{z}^{t_i} - 1$, then

$$\begin{aligned} C &= \left\{a(\mathtt{z}) = (1-\mathfrak{v})r(\mathtt{z})f(\mathtt{z}) + vr'(\mathtt{z})g(\mathtt{z}) \mid r(\mathtt{z}), r'(\mathtt{z}) \in F_q[\mathtt{z};\theta_t]\right\} \\ &= \left\{\begin{matrix} a(\mathtt{z}) = \Big((1-\mathfrak{v})r(\mathtt{z})f_1(\mathtt{z}), (1-\mathfrak{v})r(\mathtt{z})f_2(\mathtt{z}), \ldots, (1-\mathfrak{v})r(\mathtt{z}) \\ f_l(\mathtt{z})\Big) + \Big(vr'(\mathtt{z})g_1(\mathtt{z}), \ldots, vr'(\mathtt{z})g_l(\mathtt{z})\Big) \end{matrix}\right\} \\ &= \left\{\begin{matrix} a(\mathtt{z}) = \Big((1-\mathfrak{v})r(\mathtt{z})f_1(\mathtt{z}) + vr'(\mathtt{z})g_1(\mathtt{z}), (1-\mathfrak{v})r(\mathtt{z})f_2(\mathtt{z}) + \\ vr'(\mathtt{z})g_2(\mathtt{z}), \ldots, (1-\mathfrak{v})r(\mathtt{z})f_l(\mathtt{z}) + vr'(\mathtt{z})g_l(\mathtt{z})\Big) \end{matrix}\right\}. \end{aligned}$$

Thus, $C \subseteq \langle (1-\mathfrak{v})(f_1(\mathtt{z}), \ldots, f_l(\mathtt{z})), v(g_1(\mathtt{z}), \ldots, g_l(\mathtt{z}))\rangle \subseteq \mathbf{S}'$. On the other hand, consider

$$(1-\mathfrak{v})k(\mathtt{z})f(\mathtt{z}) + vk'(\mathtt{z})g(\mathtt{z}) \in \langle (1-\mathfrak{v})f(\mathtt{z}), vg(\mathtt{z})\rangle,$$

where $k(\mathtt{z}) = (k_1(\mathtt{z}), k_2(\mathtt{z}), \ldots, k_l(\mathtt{z}))$, $k'(\mathtt{z}) = (k'_1(\mathtt{z}), k'_2(\mathtt{z}), \ldots, k'_l(\mathtt{z})) \in \mathbf{S}'$. Then

$$\begin{aligned} (1-\mathfrak{v})k(\mathtt{z}) &= \{(1-\mathfrak{v})k_1(\mathtt{z}), (1-\mathfrak{v})k_2(\mathtt{z}), \ldots, (1-\mathfrak{v})k_l(\mathtt{z})\} \\ &= \{(1-\mathfrak{v})r_1(\mathtt{z}), (1-\mathfrak{v})r_2(\mathtt{z}), \ldots, (1-\mathfrak{v})r_l(\mathtt{z})\}, \text{ for some } r_1(\mathtt{z}), \ldots, r_l(\mathtt{z}) \in F_q[z, \theta_t], \end{aligned}$$

and

$$\begin{aligned} vk'(\mathtt{z}) &= \{vk'_1(\mathtt{z}), \ldots, vk'_l(\mathtt{z})\} \\ &= \{vr'_1(\mathtt{z}), \ldots, vr'_l(\mathtt{z})\}, \\ &\quad \text{for some } r'_1(\mathtt{z}), \ldots, r'_l(\mathtt{z}) \in F_q[z, \theta_t]. \end{aligned}$$

Therefore, $\langle (1-\mathfrak{v})f(\mathtt{z}), vg(\mathtt{z})\rangle \subseteq C$, which implies that $C = \langle (1-\mathfrak{v})f(\mathtt{z}), vg(\mathtt{z})\rangle$. □

Theorem 5.7 *Suppose C_1 and C_2 are $\texttt{SGQC}$ codes over F_q and $f(\mathtt{z})$, $g(\mathtt{z})$ are 1-generator polynomials of these codes, respectively. Let $C = (1-\mathfrak{v})C_1 \oplus \mathfrak{v}C_2$. Then there is a unique polynomial $h(\mathtt{z}) \in \mathbf{S}'$ such that $C = \langle h(\mathtt{z})\rangle$, and each component of $h(\mathtt{z}) \mid_r (\mathtt{z}^{t_i} - 1)$ where $h(\mathtt{z}) = (1-\mathfrak{v})f(\mathtt{z}) + \mathfrak{v}g(\mathtt{z})$.*

Proof: From Theorem 5.6, we have $C = \langle (1-\mathfrak{v})f(\mathtt{z}), vg(\mathtt{z})\rangle$. Let $h(\mathtt{z}) = (1-\mathfrak{v})f(\mathtt{z}) + \mathfrak{v}g(\mathtt{z}) = ((1-\mathfrak{v})f_1(\mathtt{z}) + \mathfrak{v}g_1(\mathtt{z}), (1-\mathfrak{v})f_2(\mathtt{z}) + \mathfrak{v}g_2(\mathtt{z}), \ldots, (1-\mathfrak{v})f_l(\mathtt{z}) + \mathfrak{v}g_l(\mathtt{z}))$. Clearly, $\langle h(\mathtt{z})\rangle \subseteq C$. Since $(1-\mathfrak{v})f(\mathtt{z}) = (1-\mathfrak{v})h(\mathtt{z})$ and $vg(\mathtt{z}) = vh(\mathtt{z})$ we conclude that $C \subseteq \langle h(\mathtt{z}) = (1-\mathfrak{v})f(\mathtt{z}) + \mathfrak{v}g(\mathtt{z})\rangle$, which implies $C = \langle h(\mathtt{z})\rangle$. We have $f(\mathtt{z}) = (f_1(\mathtt{z}), f_2(\mathtt{z}), \ldots, f_l(\mathtt{z}))$ and $g(\mathtt{z}) = (g_1(\mathtt{z}), g_2(\mathtt{z}), \ldots, g_l(\mathtt{z}))$ as generator polynomial, where each component of $f(\mathtt{z})$ and $g(\mathtt{z})$ is right divisor of $\mathtt{z}^{t_i} - 1$ in $F_q[\mathtt{z};\theta_t]$, for all $i = 1, 2, \ldots, l$. Then for each i, $\exists\ r_i(\mathtt{z})$ and $r'_i(\mathtt{z})$ in $\frac{F_q[\mathtt{z};\theta_t]}{\mathtt{z}^{t_i} - 1}$ such that $\mathtt{z}^{t_i} - 1 = r_i(\mathtt{z})f_i(\mathtt{z}) = r'_i(\mathtt{z})g_i(\mathtt{z})$. Let $r(\mathtt{z}) = (r_1(\mathtt{z}), r_2(\mathtt{z}), \ldots, r_l(\mathtt{z})$ and $r'(\mathtt{z}) = (r'_1(\mathtt{z}), r'_2(\mathtt{z}), \ldots, r'_l(\mathtt{z})$. Now, consider the following expression:

$$\begin{aligned} &[(1-\mathfrak{v})r(\mathtt{z}) + \mathfrak{v}r'(\mathtt{z})]h(\mathtt{z}) \\ =&[(1-\mathfrak{v})r(\mathtt{z}) + \mathfrak{v}r'(\mathtt{z})]((1-\mathfrak{v})f(\mathtt{z}) + \mathfrak{v}g(\mathtt{z})) \\ =&[(1-\mathfrak{v})^2 r_1(\mathtt{z})f_1(\mathtt{z}) + \mathfrak{v}^2 r'_1(\mathtt{z})g_1(\mathtt{z}), (1-\mathfrak{v})^2 r_2(\mathtt{z})f_2(\mathtt{z}) + \mathfrak{v}^2 r'_2(\mathtt{z})g_2(\mathtt{z}) \\ &, \ldots, (1-\mathfrak{v})^2 r_l(\mathtt{z})f_l(\mathtt{z}) + \mathfrak{v}^2 r'_l(\mathtt{z})g_l(\mathtt{z})] \\ =&[(1-\mathfrak{v})r_1(\mathtt{z})f_1(\mathtt{z}) + \mathfrak{v}r'_1(\mathtt{z})g_1(\mathtt{z}), \ldots, (1-\mathfrak{v})r_l(\mathtt{z})f_l(\mathtt{z}) + \mathfrak{v}r'_l(\mathtt{z})g_l(\mathtt{z})] \\ =&[(1-\mathfrak{v})(\mathtt{z}^{t_1} - 1) + \mathfrak{v}(\mathtt{z}^{t_1} - 1), \ldots, (1-\mathfrak{v})(\mathtt{z}^{t_l} - 1) + \mathfrak{v}(\mathtt{z}^{t_l} - 1)] \\ =&[\mathtt{z}^{t_1} - 1, \ldots, \mathtt{z}^{t_l} - 1]. \end{aligned}$$

Thus, $(1-\mathfrak{v})f_i(\mathtt{z}) + \mathfrak{v}g_i(\mathtt{z}) \mid_r (\mathtt{z}^{t_i} - 1)$, for all $i = 1, 2, \ldots, l$. □

In the following examples, we use Theorems 5.3, 5.6 and 5.7.

Example 5.3 *Consider the polynomials $\boldsymbol{z}^8 - 1$ and $\boldsymbol{z}^4 - 1$ are in $F_4[\boldsymbol{z};\theta]$, where θ is the Frobenius automorphism over F_4. The factorization of these polynomials is as follows:*

$$\begin{aligned}\boldsymbol{z}^8 - 1 =&(\boldsymbol{z}^6 + t\boldsymbol{z}^5 + t^2\boldsymbol{z}^4 + \boldsymbol{z}^2 + t\boldsymbol{z} + t^2)(\boldsymbol{z}^2 + t\boldsymbol{z} + t)\\ =&(\boldsymbol{z}^5 + t\boldsymbol{z}^4 + \boldsymbol{z} + t)(\boldsymbol{z}^3 + t^2\boldsymbol{z}^2 + \boldsymbol{z} + t^2)\\ \boldsymbol{z}^4 - 1 =&(\boldsymbol{z}^3 + t^2\boldsymbol{z}^2 + \boldsymbol{z} + t^2)(\boldsymbol{z} + t)\\ =&(\boldsymbol{z}^2 + 1)(\boldsymbol{z}^2 + 1).\end{aligned}$$

Here, t is the generator of multiplicative group of F_4. Consider $C_1 = \langle f_1(\boldsymbol{z}),\ f_2(\boldsymbol{z})\rangle$ and $C_2 = \langle g_1(\boldsymbol{z}),\ g_2(\boldsymbol{z})\rangle$ are 1-generator SGQC codes of block length $(8,4)$ and length 12 with index 2 over F_4 where $f_1(\boldsymbol{z}) = \boldsymbol{z}^4 + t^2\boldsymbol{z}^3 + t^2\boldsymbol{z}^2 + t^2$, $f_2(\boldsymbol{z}) = \boldsymbol{z} + t$, $g_1(\boldsymbol{z}) = \boldsymbol{z}^3 + t^2\boldsymbol{z}^2 + \boldsymbol{z} + t^2$, and $g_2(\boldsymbol{z}) = \boldsymbol{z}^2 + 1$. C_1 is of dimension 6 and C_2 is of dimension 5. The generator matrices of C_1 and C_2 are
$G_1 =$

$$\begin{pmatrix} t & t & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\ 0 & t^2 & t^2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & t & t & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & t^2 & t^2 & 1 & 0 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & t & t & 1 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & t^2 & t^2 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

and G_2 =

$$\begin{pmatrix} t^2 & 1 & t^2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\ 0 & t & 1 & t & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & t^2 & 1 & t^2 & 1 & 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & t & 1 & t & 1 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & t^2 & 1 & t^2 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

*respectively. Then we obtain C_1 as $[12,6,4]$ a linear code and C_2 as linear code $[12,5,4]$ over F_4. Consider the $C = (1-\mathfrak{v})C_1 \oplus \mathfrak{v}C_2$, a 1-generator SGQC code of block length $(8,4)$ and length 12 with index 2 over $F_4 + \mathfrak{v}F_4$. Then its generator matrix is $G = \begin{pmatrix}(1-\mathfrak{v})G_1\\ \mathfrak{v}G_2\end{pmatrix}$ whose gray image is code $[24,11,4]$ over F_4 which is **better than** $[24,10,4]$ given in [19].*

Example 5.4 *The factorization of $\boldsymbol{z}^4 - 1$ and $\boldsymbol{z}^6 - 1$ in $F_9[\boldsymbol{z};\theta]$ is given as follows:*

$$\begin{aligned}\boldsymbol{z}^4 - 1 =&(\boldsymbol{z}^2 + t^2\boldsymbol{z} + t^3)(\boldsymbol{z}^2 + t^6\boldsymbol{z} + t)\\ =&(\boldsymbol{z}^3 + t^5\boldsymbol{z}^2 + 2\boldsymbol{z} + t)(\boldsymbol{z} + t^3)\\ \boldsymbol{z}^6 - 1 =&(\boldsymbol{z}^4 + t^7\boldsymbol{z}^3 + t^3\boldsymbol{z}^2 + t^3\boldsymbol{z} + t^2)(\boldsymbol{z}^2 + t^3\boldsymbol{z} + t^2)\\ =&(\boldsymbol{z}^4 + t^3\boldsymbol{z}^3 + t^3\boldsymbol{z}^2 + t^7\boldsymbol{z} + t^2)(\boldsymbol{z}^2 + t^7\boldsymbol{z} + t^2).\end{aligned}$$

Consider $C_1 = \langle f_1(\boldsymbol{z}),\ f_2(\boldsymbol{z})\rangle$ and $C_2 = \langle g_1(\boldsymbol{z}),\ g_2(\boldsymbol{z})\rangle$ are 1-generator SGQC codes of block length $(4,6)$ and length 10 with index 2 over F_9 having equal dimensions of $k_1 = k_2 = 5$ where $f_1(\boldsymbol{z}) = \boldsymbol{z}^2 + t^6\boldsymbol{z} + t$, $f_2(\boldsymbol{z}) = \boldsymbol{z}^2 + t^3\boldsymbol{z} + t^2$, $g_1(\boldsymbol{z}) = \boldsymbol{z} + t^3$, and $g_2(\boldsymbol{z}) = \boldsymbol{z}^2 + t^7\boldsymbol{z} + t^2$. We obtain C_1 as code $[10,5,4]$ and C_2 as code $[10,5,4]$ over F_9. Consider $C = (1-\mathfrak{v})C_1 \oplus \mathfrak{v}C_2$, a 1-generator SGQC code of block length $(4,6)$ and length 10 of index 2 over $F_9 + \mathfrak{v}F_9$. Then the generator polynomial of C is $\langle (1-\mathfrak{v})f_1(\boldsymbol{z}) + \mathfrak{v}g_1(\boldsymbol{z}),\ (1-\mathfrak{v})f_2(\boldsymbol{z}) + \mathfrak{v}g_2(\boldsymbol{z})\rangle$ and the dimension and minimum distance are $k_1 + k_2 = 10$ and 4, respectively. Thus, the gray image of C is the code $[20,10,4]$ over F_9.

5.2. Idempotent generators of 1-generator SGQC codes

In the case of a commutative ring, if $(n,q) = 1$, where $q = p^d$; d is a positive integer with p being a prime number, there is a unique idempotent generator for each cyclic code of length n over F_q. Moreover, SC codes over F_q have idempotent generators under some restrictions on the length of the code. In this regard, Irfan et al. [13] already identified idempotent generators of SC codes over S. In this subsection, we show that SGQC codes admit idempotent generators under certain length restrictions, and we illustrate the result with examples over both F_q and S.

Theorem 5.8 *[13, Theorem 6] Let C be an SC code length n generated by $C = \langle f(z)\rangle$, $f(z) \mid_r (z^n - 1)$ in $F_q[z;\theta_t]$. Consider $\gcd(n, m_t) = 1$ and $\gcd(n, q) = 1$, where $m_t = |\langle\theta_t\rangle|$, then there exists an idempotent polynomial $s(z) \in \dfrac{F_q[z;\theta_t]}{\langle z^n - 1\rangle}$ such that $C = \langle s(z)\rangle$.*

From Lemma 3.1 and Theorem 5.7, the following theorem identifies an idempotent generator of SGQC code C over F_q.

Theorem 5.9 *Let C be a 1-generator SGQC code over F_q of block length $(t_1, t_2, \dots, t_l)$ and length $N = t_1 + t_2 + \dots + t_l$ with $C = \langle c(z)\rangle = \langle c_1(z), c_2(z), \dots, c_l(z)\rangle$ where $c_i(z) \in F_q[z;\theta]$ is right divisor of $z^{t_i} - 1$ for all $i = 1, \dots, l$. If $(t_i, q) = 1, (t_i, m_t) = 1,$ for all $i = 1, 2, \dots, l$ where $m_t = |\langle\theta_t\rangle|$, then there exists an idempotent polynomial $s(z) = (s_1(z), s_2(z), \dots, s_l(z)) \in \dfrac{F_q[z;\theta_t]}{\langle z^{t_1} - 1\rangle} \times \dfrac{F_q[z;\theta_t]}{\langle z^{t_2} - 1\rangle} \times \dots \times \dfrac{F_q[z;\theta_t]}{\langle z^{t_l} - 1\rangle}$ such that $C = \langle s(z)\rangle$.*

Proof: Suppose C is a 1-generator SGQC code over F_q of block length $(t_1, \dots, t_l)$ and length $N = t_1 + t_2 + \dots + t_l$ with generator polynomial $c(z) = \langle c_1(z), c_2(z), \dots, c_l(z)\rangle$, where $c_i(z) \in F_q[z;\theta]$ is a right divisor of $z^{t_i} - 1$. If $(t_i, q) = 1, (t_i, m) = 1,$ for all $i = 1, 2, \dots, l$, then define the map

$$\Phi_i : \frac{F_q[z;\theta]}{\langle z^{t_1} - 1\rangle} \times \frac{F_q[z;\theta]}{\langle z^{t_2} - 1\rangle} \times \dots \times \frac{F_q[z;\theta]}{\langle z^{t_l} - 1\rangle} \to \frac{F_q[z;\theta]}{\langle z^{t_i} - 1\rangle} \text{ by } (a_1(z), a_2(z), \dots, a_l(z)) \to a_i(z).$$

It is a well defined module homomorphism, and $\Phi_i(C) = C_i$. Since C is an 1-generator SGQC code over $F_q[z;\theta_t]$, C is a left $F_q[z;\theta_t]$-submodule of $\dfrac{F_q[z;\theta]}{\langle z^{t_1} - 1\rangle} \times \dfrac{F_q[z;\theta]}{\langle z^{t_2} - 1\rangle} \times \dots \times \dfrac{F_q[z;\theta]}{\langle z^{t_l} - 1\rangle}$. Therefore, C_i is also a left submodule of $\dfrac{F_q[z;\theta]}{\langle z^{t_i} - 1\rangle}$. From Lemma 3.1, C_i is an SC code of length t_i, which implies $C_i = \langle g_i(z)\rangle$, where $g_i(z) \mid_r (z^{t_i} - 1)$. Now, from Theorem 5.8, there exists an idempotent polynomial $s_i(z) \in \dfrac{F_q[z;\theta]}{\langle z^{t_i} - 1\rangle}$ such that $C_i = \langle e_i(z)\rangle$. Taking cartesian product of $\Phi_i(C)$, where $i = 1, 2, \dots, l$ then $\Phi_1(C) \times \Phi_2(C) \times \dots \times \Phi_l(C) \cong C$. Since $\phi_i(C) = \langle s_i(z)\rangle$, we conclude that $C \cong \langle s_1(z), s_2(z), \dots, s_l(z)\rangle$, where each $s_i(z)$ is an idempotent polynomial. □

Following the above Theorem 5.8 and Lemma 2.2, the following theorem identifies an idempotent generator of the SC code C over S.

Theorem 5.10 *[13, Corollary 8] If $C = (1 - \mathfrak{v})C_1 \oplus \mathfrak{v}C_2$ is an SC code of length n over S and $(n, m_t) = 1$, $(n, q) = 1$, then C_i has an idempotent generator, say $s_i(z)$ for $i = 1, 2$. Moreover, $s(z) = (1 - \mathfrak{v})s_1(z) + \mathfrak{v}s_2(z)$ is an idempotent generator of C, i.e., $C = \langle s(z)\rangle$.*

Theorem 5.11 *Let C be a 1-generator SGQC code over S of block length $(t_1, t_2, \dots, t_l)$ and index l with $C = \langle (u_1(z), u_2(z), \dots, u_l(z))\rangle$, where $u_i(z) \mid_r (z^{t_i} - 1)$ in $S[z;\theta_t]$. If $(t_i, q) = 1$, and $(t_i, m_t) = 1,$ for all $i = 1, 2, \dots, l$, where $m_t = |\langle\theta_t\rangle|$, then there exists an idempotent polynomial $s(z) = ((1 - \mathfrak{v})s_1(z) + \mathfrak{v}s_1'(z), (1 - \mathfrak{v})s_2(z) + \mathfrak{v}s_2'(z), \dots, (1 - \mathfrak{v})s_l(z) + \mathfrak{v}s_l'(z)) \in \mathbf{S}'$ such that $C = \langle s(z)\rangle$.*

Proof: Define the map

$$\Psi_i : \mathbf{S}' \to S_{t_i}, \text{ given by } (a_1(z), a_2(z), \dots, a_l(z)) \to a_i(z).$$

It is a well-defined module homomorphism. Here, $\Psi_i(C)$ is a left-submodule of S_{t_i} and hence $\Psi_i(C)$ is an SC code of length t_i in S_{t_i}. Now, by Theorem 5.9, $\Psi_i(C)$ has an idempotent polynomial, i.e. $\Psi_i(C) = \langle (1 - \mathfrak{v})s_i(z) + \mathfrak{v}s_i'(z)$, where $s_i(z)$ and $s_i'(z)$ are the idempotent polynomial generator of the constituent $\Psi_i(C)$ over F_q. The proof is now similar to Theorem 5.10. Hence, $C \cong \langle (1 - \mathfrak{v})s_1(z) + \mathfrak{v}s_1'(z), (1 - \mathfrak{v})s_2(z) + \mathfrak{v}s_2'(z), \dots, (1 - \mathfrak{v})s_l(z) + \mathfrak{v}s_l'(z)\rangle$. □

Example 5.5 *Let $F_4[\boldsymbol{z};\theta]$ be a skew polynomial ring, where θ is a Frobenius automorphism over F_4. Consider polynomials $g_1(\boldsymbol{z}) = \boldsymbol{z}^2+\boldsymbol{z}+1$ and $g_2(\boldsymbol{z}) = \boldsymbol{z}^4+\boldsymbol{z}^3+\boldsymbol{z}^2+\boldsymbol{z}+1$ in $F_4[\boldsymbol{z};\theta]$ such that $g_1(\boldsymbol{z}) \mid_r (\boldsymbol{z}^3-1)$ and $g_2(\boldsymbol{z}) \mid_r (\boldsymbol{z}^5-1)$, respectively. In addition, $g_1(\boldsymbol{z})$ and $g_2(\boldsymbol{z})$ are idempotent polynomials in $\dfrac{F_4[\boldsymbol{z};\theta]}{\langle \boldsymbol{z}^3-1\rangle}$ and $\dfrac{F_4[\boldsymbol{z};\theta]}{\langle \boldsymbol{z}^5-1\rangle}$, respectively. Let C be a 1-generator SGQC code of block length $(3,5)$ and length 8 of index two generated by $C = \langle g_1(\boldsymbol{z}), g_2(\boldsymbol{z})\rangle$ over F_4. Then, from Theorem 4.2 of [11], parity check polynomial of C is*

$$\begin{aligned} f(\boldsymbol{z}) =& lclm\left\{\frac{\boldsymbol{z}^3-1}{g_1(\boldsymbol{z})}, \frac{\boldsymbol{z}^5-1}{g_2(\boldsymbol{z})}\right\} \\ =& \boldsymbol{z}+1. \end{aligned}$$

Thus, C is an SGQC code of length 8 of index 2 and dimension 1. The generator matrix for C is given by $\begin{pmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{pmatrix}$. Then, the code C as the parameter $[8,1,8]$ is a 1-generator SGQC code over F_4, an optimal linear code over F_4.

Example 5.6 *Using the notation of Example 1, consider $g_3(\boldsymbol{z}) = \boldsymbol{z}^4+\boldsymbol{z}^2+\boldsymbol{z}+1 \mid_r (\boldsymbol{z}^7-1)$ in $F_4[\boldsymbol{z};\theta]$ and an idempotent polynomial in $\dfrac{F_4[\boldsymbol{z};\theta]}{\langle \boldsymbol{z}^7-1\rangle}$. Let C be a 1-generator SGQC codes of block length $(3,5,7)$ and length 15 with index 3, generated by $C = \langle g_1(\boldsymbol{z}), g_2(\boldsymbol{z}), g_3(\boldsymbol{z})\rangle$ over F_4. From Theorem 4.2 of [11], parity check polynomial of C is $\boldsymbol{z}^4+\boldsymbol{z}^3+\boldsymbol{z}^2+1$. Hence, C is a 1-generator SGQC code of length 15 of index 3 and dimension 4. Thus, C having parameters $[15,4,4]$ is a 1-generator SGQC code over F_4.*

5.3. ρ-Generator Polynomial SGQC codes

This subsection introduces a family of ρ-generator polynomials for SGQC codes over the ring S. Here, we develop our method upon the approach introduced by Seneviratne et al. in [24] for the generator polynomial over F_q. Using these generators, we determine the cardinality and dimension of the SGQC codes and derive improved code parameters relative to those available in the existing literature [11,19].

Let C be an SGQC code of block length $(t_1, t_2, \ldots, t_l)$ and index l. Suppose $a(\mathrm{z}) = (a_1(\mathrm{z}) + \mathfrak{v}a'_1(\mathrm{z}), \ldots, a_l(\mathrm{z}) + \mathfrak{v}a'_l(\mathrm{z}))$. Define the sets

$$K_i = \left\{\begin{array}{c} p_i(\mathrm{z}) : \text{a codeword } a(\mathrm{z}) = (a_1(\mathrm{z}) + \mathfrak{v}a'_1(\mathrm{z}), a_2(\mathrm{z}) + \mathfrak{v}a'_2(\mathrm{z}), \ldots, p_i(\mathrm{z}), 0, 0, \ldots, 0) \in C, \\ \text{where } p_i(\mathrm{z}) \in \dfrac{S[\mathrm{z};\theta_t]}{\langle \mathrm{z}^{t_i}-1\rangle} \text{ and } a_{i+1}(\mathrm{z}) + \mathfrak{v}a'_{i+1}(\mathrm{z}) = a_{i+2}(\mathrm{z}) + \mathfrak{v}a'_{i+2}(\mathrm{z}) = \cdots = 0 \end{array}\right\},$$

i.e.,

$$\begin{aligned} K_1 =& \{p_1(\mathrm{z}) : \text{a codeword } a(\mathrm{z}) = (p_1(\mathrm{z}), 0, 0, \ldots, 0) \in C\}, \\ K_2 =& \{p_2(\mathrm{z}) : \text{a codeword } \ a(\mathrm{z}) = (a_1(\mathrm{z}) + \mathfrak{v}a'_1(\mathrm{z}), p_2(\mathrm{z}), 0, \ldots, 0) \in C\} \\ & \text{and} \\ K_l =& \left\{\begin{array}{l} p_l(\mathrm{z}) : \text{a codeword } \ a(\mathrm{z}) = (a_1(\mathrm{z}) + \mathfrak{v}a'_1(\mathrm{z}), a_2(\mathrm{z}) + \mathfrak{v}a'_2(\mathrm{z}), \\ \qquad\qquad\qquad\qquad \ldots, p_l(\mathrm{z})) \in C \end{array}\right\}. \end{aligned}$$

As $(0,0,\ldots,0) \in C$, K_i is the nonvoid set for all $i = 1,2,\ldots,l$.

Lemma 5.1 *The above set K_i is a left submodule of $\dfrac{S[\boldsymbol{z};\theta_t]}{\langle \boldsymbol{z}^{t_i}-1\rangle}$ for all $i = 1,2,\ldots,l$.*

Proof: Suppose $p_i(\mathrm{z}), q_i(\mathrm{z}) \in K_i$ and $s(\mathrm{z}) \in S[\mathrm{z};\theta_t]$, then there exist

$$\begin{aligned} a(\mathrm{z}) &= (a_1(\mathrm{z}) + \mathfrak{v}a'_1(\mathrm{z}), a_2(\mathrm{z}) + \mathfrak{v}a'_2(\mathrm{z}), \ldots, p_i(\mathrm{z}), 0, \ldots, 0) \text{ and} \\ b(\mathrm{z}) &= (b_1(\mathrm{z}) + \mathfrak{v}b'_1(\mathrm{z}), b_2(\mathrm{z}) + \mathfrak{v}b'_2(\mathrm{z}), \ldots, q_i(\mathrm{z}), 0, \ldots, 0) \in C. \end{aligned}$$

Since C is a left submodule of $S_{t_1} \times S_{t_2} \times \cdots \times S_{t_l}$, then $a(\mathtt{z}) + b(\mathtt{z}) = (a_1(\mathtt{z}) + \mathfrak{v}a_1'(\mathtt{z}) + (b_1(\mathtt{z}) + \mathfrak{v}b_1'((\mathtt{z})), a_2(\mathtt{z}) + \mathfrak{v}a_2'(\mathtt{z}) + (b_2(\mathtt{z}) + \mathfrak{v}b_2'((\mathtt{z}), \dots, p_i(\mathtt{z}) + q_i(\mathtt{z}), 0, \dots, 0)$, and $s(\mathtt{z})a(\mathtt{z}) = (s(\mathtt{z})(a_1(\mathtt{z}) + \mathfrak{v}a_1'(\mathtt{z})), s(\mathtt{z})(a_2(\mathtt{z}) + \mathfrak{v}a_2'(\mathtt{z})), \dots, s(\mathtt{z})p_i(\mathtt{z})\ , 0, \dots, 0)$ are codewords in C, where $s(z) \in S[z;\theta_t]$. Thus, $p_i(\mathtt{z}) + q_i(\mathtt{z})$ and $s(\mathtt{z})p_i(\mathtt{z})$ are elements in K_i, which implies that K_i is a left submodule of $\frac{S[\mathtt{z};\theta_t]}{\langle \mathtt{z}^{t_i} - 1\rangle}$. □

Note: From Corollary 2.1, each K_i is principally generated, i.e., $K_i = \langle f_i(\mathtt{z})\rangle$, where $f_i(\mathtt{z}) \mid_r (\mathtt{z}^{t_i} - 1)$.

Lemma 5.2 *Let C be an* SGQC *code of block length $(t_1, t_2, \dots, t_l)$ and index l and $a(\mathtt{z}) = (a_1(\mathtt{z}) + \mathfrak{v}a_1'(\mathtt{z}), a_2(\mathtt{z}) + \mathfrak{v}a_2'(\mathtt{z}), \dots, a_l(\mathtt{z}) + \mathfrak{v}a_l'(\mathtt{z})) \in C$. Then the sets*

$$L = \{(h_1(\mathtt{z}), h_2(\mathtt{z}), \dots, h_{l-1}(\mathtt{z}) : \text{ a codeword } (h_1(\mathtt{z}), h_2(\mathtt{z}), \dots, h_{l-1}(\mathtt{z}), h_l(\mathtt{z})) \in C\}$$

and

$$M = \{(h_2(\mathtt{z}), \dots, h_l(\mathtt{z}) : \text{a codeword } (h_1(\mathtt{z}), h_2(\mathtt{z}), \dots, h_{l-1}(\mathtt{z}), h_l(\mathtt{z})) \in C, \text{ where } h_i(\mathtt{z}) = h_i(\mathtt{z}) + \mathfrak{v}h_i'(\mathtt{z})\}$$

are left submodules of $S_{t_1} \times S_{t_2} \times \cdots \times S_{t_{l-1}}$ and $S_{t_2} \times S_{t_3} \times \cdots \times S_{t_l}$, respectively.

Proof: The proof proceeds in the same manner as that of Lemma 5.1. □

Using the above-defined notation, we give the set of ρ-generator polynomials of the SGQC code in the subsequent theorem.

Theorem 5.12 *Let C be an* SGQC *code of block length $(t_1, t_2, \dots, t_l)$ and index l. Then*

$$C = \left\langle \begin{matrix} ((1-\mathfrak{v})f_1(\mathtt{z}) + \mathfrak{v}f_1'(\mathtt{z}), 0, \dots, 0), (p_{21}(\mathtt{z}), (1-\mathfrak{v})f_2(\mathtt{z}) + \mathfrak{v}f_2'(\mathtt{z}), 0, \dots, 0), (p_{31}(\mathtt{z}), p_{32}(\mathtt{z}), \\ (1-\mathfrak{v})f_3(\mathtt{z}) + \mathfrak{v}f_3'(\mathtt{z}), 0, \dots, 0), \dots, (p_{l1}(\mathtt{z}), p_{l2}(\mathtt{z}), \dots, p_{l(l-1)}(\mathtt{z}), (1-\mathfrak{v})f_l(\mathtt{z}) + \mathfrak{v}f_l'(\mathtt{z})) \end{matrix} \right\rangle$$

, where for each $i = 1, 2, \dots, l$, $(1-\mathfrak{v})f_i(\mathtt{z}) + \mathfrak{v}f_i'(\mathtt{z}) \mid_r (\mathtt{z}^{t_i} - 1)$.

Proof: We will prove it inductively. Suppose the index $l = 1$, then $C = \langle f_{t_1}(\mathtt{z})\rangle = \langle (1-\mathfrak{v})f_1(\mathtt{z}) + \mathfrak{v}f_1'(\mathtt{z})\rangle$, i.e., C is a SC code which is principally generated in $\frac{S[\mathtt{z};\theta_t]}{\langle \mathtt{z}^{t_1} - 1\rangle}$. Assume that the statement holds for all $i < l$. Let C be an SGQC code as stated in the statement of index l and $a(\mathtt{z}) = (a_1(\mathtt{z}) + \mathfrak{v}a_1'(\mathtt{z}), a_2(\mathtt{z}) + \mathfrak{v}a_2'(\mathtt{z}), \dots, a_l(\mathtt{z}) + \mathfrak{v}a_l'(\mathtt{z})) \in C$, where $a_i(\mathtt{z}) + \mathfrak{v}a_i'(\mathtt{z}) \in \frac{S[\mathtt{z};\theta_t]}{\langle \mathtt{z}^{t_i} - 1\rangle}$. From the definition of set K_l, we have $a_l(\mathtt{z}) + \mathfrak{v}a_l'(\mathtt{z}) \in K_l = \langle (1-\mathfrak{v})f_l(\mathtt{z}) + \mathfrak{v}f_l'(\mathtt{z})\rangle$ and $a_l(\mathtt{z}) + \mathfrak{v}a_l'(\mathtt{z}) = (q_l(\mathtt{z}) + \mathfrak{v}q_l'(\mathtt{z}))((1-\mathfrak{v})f_l(\mathtt{z}) + \mathfrak{v}f_l'(\mathtt{z})) = q_{t_l}(\mathtt{z})f_{t_l}(\mathtt{z})$ and since $f_{t_l}(\mathtt{z}) \in K_l$, there exists a codeword $(p_{l1}(\mathtt{z}), p_{l2}(\mathtt{z}), \dots, p_{l(l-1)}(\mathtt{z}), f_{t_l}(\mathtt{z}))$ in C. Thus,

$$\begin{aligned} a(\mathtt{z}) &= \Big(a_1(\mathtt{z}) + \mathfrak{v}a_1'(\mathtt{z}), a_2(\mathtt{z}) + \mathfrak{v}a_2'(\mathtt{z}), \dots, q_{t_l}(\mathtt{z})f_{t_l}(\mathtt{z})\Big) \\ &= q_{t_l}(\mathtt{z}) \left(\begin{matrix} (p_{l1}(\mathtt{z}), p_{l2}(\mathtt{z}), \dots, p_{l(l-1)}(\mathtt{z}), f_{t_l}(\mathtt{z})) + (a_1(\mathtt{z}) + \mathfrak{v}a_1'(\mathtt{z}) - q_{t_l}(\mathtt{z})p_{l1}(\mathtt{z}), a_2(\mathtt{z}) + \mathfrak{v}a_2'(\mathtt{z}) - q_{t_l}(\mathtt{z})p_{l2}(\mathtt{z}), \\ \dots a_{l-1}(\mathtt{z}) + \mathfrak{v}a_{l-1}'(\mathtt{z}) - q_{t_l}(\mathtt{z})p_{l(l-1)}(\mathtt{z}), 0) \end{matrix} \right). \end{aligned}$$

Since

$$\begin{matrix} \big(q_{t_l}(\mathtt{z})(p_{l1}(\mathtt{z}), p_{l2}(\mathtt{z}), \dots, p_{l(l-1)}(\mathtt{z}), f_{t_l}(\mathtt{z}))\big) \in C, \text{ and} \\ \left(\begin{matrix} a_1(\mathtt{z}) + \mathfrak{v}a_1'(\mathtt{z}) - q_{t_l}(\mathtt{z})p_{l1}(\mathtt{z}), a_2(\mathtt{z}) + \mathfrak{v}a_2'(\mathtt{z}) - q_{t_l}(\mathtt{z})p_{l2}(\mathtt{z}) \\ , \dots, a_{l-1}(\mathtt{z}) + \mathfrak{v}a_{l-1}'(\mathtt{z}) - q_{t_l}(\mathtt{z})p_{l(l-1)}(\mathtt{z}), 0 \end{matrix} \right) \in C. \end{matrix}$$

By Lemma 5.2,

$$(a_1(\mathtt{z}) + \mathfrak{v}a_1'(\mathtt{z}) - q_{t_l}(\mathtt{z})p_{l1}(\mathtt{z}), a_2(\mathtt{z}) + \mathfrak{v}a_2'(\mathtt{z}) - q_{t_l}(\mathtt{z})p_{l2}(\mathtt{z}), \dots, a_{l-1}(\mathtt{z}) + \mathfrak{v}a_{l-1}'(\mathtt{z}) - q_{t_l}(\mathtt{z})p_{l(l-1)}(\mathtt{z})) \in L.$$

As L is a left-submodule of $S_{t_1} \times S_{t_2} \times \cdots \times S_{t_{l-1}}$ and by the inductive hypothesis, we have

$$L = \left\langle \begin{matrix} ((1-\mathfrak{v})f_1(\mathtt{z}) + \mathfrak{v}f_1'(\mathtt{z}), 0, \dots, 0), (p_{21}(\mathtt{z}), (1-\mathfrak{v})f_2(\mathtt{z}) + \mathfrak{v}f_2'(\mathtt{z}), 0, \dots, 0), (p_{31}(\mathtt{z}), p_{32}(\mathtt{z}), (1-\mathfrak{v})f_3(\mathtt{z}) \\ +\mathfrak{v}f_3'(\mathtt{z}), 0, \dots, 0), \dots, (p_{(l-1)1}(\mathtt{z}), p_{(l-1)2}(\mathtt{z}), \dots, p_{(l-1)(l-1)}(\mathtt{z}), f_{t_{(l-1)}}(\mathtt{z})) \end{matrix} \right\rangle$$

, where $f_{t_i}(\mathtt{z}) \mid_r (\mathtt{z}^{t_i} - 1)$, for all $i = 1, 2, \dots, l-1$. Thus,

$$\begin{gathered}(a_1(\mathsf{z}) + \mathfrak{v}a_1'(\mathsf{z}) - q_{t_l}(\mathsf{z})p_{l1}(\mathsf{z}), a_2(\mathsf{z}) + \mathfrak{v}a_2'(\mathsf{z}) - q_{t_l}(\mathsf{z})p_{l2}(\mathsf{z}), \dots, a_{l-1}(\mathsf{z}) + \mathfrak{v}a_{l-1}'(\mathsf{z}) - q_{t_l}(\mathsf{z})p_{l(l-1)}(\mathsf{z})) \\ = c_1(\mathsf{z})((1-\mathfrak{v})f_1(\mathsf{z}) + \mathfrak{v}f_1'(\mathsf{z}), 0, \dots, 0) + c_2(\mathsf{z})(p_{21}(\mathsf{z}), (1-\mathfrak{v})f_2(\mathsf{z}) + \mathfrak{v}f_2'(\mathsf{z}), 0, \dots, 0) + \dots + \\ c_{l-1}(\mathsf{z})(p_{(l-1)1}(\mathsf{z}), p_{(l-1)2}(\mathsf{z}), \dots, p_{(l-1)(l-2)}(\mathsf{z}), f_{t_{l-1}}(\mathsf{z})),\end{gathered}$$

and

$$\begin{aligned} a(\mathsf{z}) &= (a_1(\mathsf{z}) + \mathfrak{v}a_1'(\mathsf{z}), a_2(\mathsf{z}) + \mathfrak{v}a_2'(\mathsf{z}), \dots, q_{t_l}(\mathsf{z})f_{t_l}(\mathsf{z})) \\ &= \begin{pmatrix} (q_{t_l}(\mathsf{z})(p_{l1}(\mathsf{z}), p_{l2}(\mathsf{z}), \dots, p_{l(l-1)}(\mathsf{z}), f_{t_l}(\mathsf{z})) + (a_1(\mathsf{z}) + \mathfrak{v}a_1'(\mathsf{z}) \\ -q_{t_l}(\mathsf{z})p_{l1}(\mathsf{z}), a_2(\mathsf{z}) + \mathfrak{v}a_2'(\mathsf{z}) - q_{t_l}(\mathsf{z})p_{l2}(\mathsf{z}), \dots, \\ a_{l-1}(\mathsf{z}) + \mathfrak{v}a_{l-1}'(\mathsf{z}) - q_{t_l}(\mathsf{z})p_{l(l-1)}(\mathsf{z}), 0) \end{pmatrix} \\ &= \begin{pmatrix} (q_{t_l}(\mathsf{z})(p_{l1}(\mathsf{z}), p_{l2}(\mathsf{z}), \dots, p_{l(l-1)}(\mathsf{z}), f_{t_l}(\mathsf{z})) + c_1(\mathsf{z})((1-\mathfrak{v}) \\ f_1(\mathsf{z}) + \mathfrak{v}f_1'(\mathsf{z}), 0, \dots, 0) + \dots + c_{l-1}(\mathsf{z})(p_{(l-1)1}(\mathsf{z}), p_{(l-1)2}(\mathsf{z}) \\ , \dots, p_{(l-1)(l-2)}(\mathsf{z}), f_{t_{l-1}}(\mathsf{z})) \end{pmatrix}. \end{aligned}$$

Therefore,

$$C = \left\langle \begin{matrix} ((1-\mathfrak{v})f_1(\mathsf{z}) + \mathfrak{v}f_1'(\mathsf{z}), 0, \dots, 0), (p_{21}(\mathsf{z}), (1-\mathfrak{v})f_2(\mathsf{z}) + \\ \mathfrak{v}f_2'(\mathsf{z}), 0, \dots, 0), (p_{31}(\mathsf{z}), p_{32}(\mathsf{z}), (1-\mathfrak{v})f_3(\mathsf{z}) + \mathfrak{v}f_3'(\mathsf{z}), 0, \dots \\ , 0), \dots, (p_{l1}(\mathsf{z}), p_{l2}(\mathsf{z}), \dots, p_{l(l-1)}(\mathsf{z}), (1-\mathfrak{v})f_l(\mathsf{z}) + \mathfrak{v}f_l'(\mathsf{z})) \end{matrix} \right\rangle,$$

where $(1-\mathfrak{v})f_i(\mathsf{z}) + \mathfrak{v}f_i'(\mathsf{z}) \mid_r (\mathsf{z}^{t_i} - 1)$, for all $i = 1, 2, \dots, l$. □

Theorem 5.13 *Let C be an SGQC code of block length $(t_1, t_2, \dots, t_l)$ and index l given by*

$$C = \left\langle \begin{matrix} (f_{t_1}(z), 0, \dots, 0), (p_{21}(z), f_{t_2}(z), 0, \dots, 0), (p_{31}(z), p_{32}(z), f_{t_3}(z), 0, \dots, 0), \dots, \\ (p_{l1}(z), p_{l2}(z), \dots, p_{l(l-1)}(z), (f_{t_l}(z)) \end{matrix} \right\rangle,$$

where $(1-\mathfrak{v})f_i(z) + \mathfrak{v}f_i'(z) = f_{t_i}(z) \mid_r (z^{t_i} - 1)$ for all $i = 1, 2, \dots, l$. Then

1. $\deg p_{ij}(z) < \deg f_{t_j}(z)$ *for all $i = 2, \dots, l$, and $j = 1, 2, \dots l-1$ with $i > j$.*
2. *If $(z^{t_i} - 1) = q_{t_i}(z)f_{t_i}(z)$, then $q_{t_i}(z)p_{(i)(i-1)}(z) \in \langle f_{t_{i-1}}(z) \rangle$ and $q_{t_i}(z)p_{(i)(i-1)}(z) = s_{t_i}(z)f_{t_{i-1}}(z)$, for all $i = 2, 3, \dots, l$.*

Proof: The first part can be proven inductively. So, we left out the first part. For the second part, we observe that

$$\begin{aligned} &q_{t_i}(\mathsf{z})(p_{i1}(\mathsf{z}), p_{i2}(\mathsf{z}), \dots, p_{i(i-1)}(\mathsf{z}), f_{t_i}(\mathsf{z}), 0, \dots, 0) \\ &= (q_{t_i}(\mathsf{z})p_{i1}(\mathsf{z}), q_{t_i}(\mathsf{z})p_{i2}(\mathsf{z}), \dots, q_{t_i}(\mathsf{z})p_{i(i-1)}(\mathsf{z}), 0, \dots, 0). \end{aligned}$$

This implies $q_{t_i}(\mathsf{z})p_{i(i-1)}(\mathsf{z}) \in K_{i-1} = \langle f_{t_{i-1}}(\mathsf{z}) \rangle$. Hence, $q_{t_i}(\mathsf{z})p_{i(i-1)}(\mathsf{z}) = s_{t_i}(\mathsf{z})f_{t_{i-1}}(\mathsf{z})$ for all $i = 2, \dots, l$. □

The subsequent theorem establishes the cardinality and dimension of the SGQC codes based on the structural properties of generators.

Theorem 5.14 *Let C be an SGQC code of block length $(t_1, t_2, \dots, t_l)$ and index l. If*

$$C = \left\langle \begin{matrix} (f_{t_1}(z), 0, \dots, 0), (p_{21}(z), f_{t_2}(z), 0, \dots, 0), (p_{31}(z), p_{32}(z), f_{t_3}(z), 0, \dots, 0), \\ \dots, (p_{l1}(z), p_{l2}(z), \dots, p_{l(l-1)}(z), (f_{t_l}(z)) \end{matrix} \right\rangle,$$

where $(1-\mathfrak{v})f_i(z) + \mathfrak{v}f_i'(z) = f_{t_i}(z) \mid_r (z^{t_i} - 1)$, for all $i = 1, 2, \dots, l$. Then $rank(C) = \deg(q_{t_1}(z)) + \deg(q_{t_2}(z)) + \dots + \deg(q_{t_l}(z))$ and $|C| = q^{2\deg(q_{t_1}(z))}q^{2\deg(q_{t_2}(z))} \dots q^{2\deg(q_{t_l}(z))}$, with keeping the same notation as in Theorem 5.13.

Proof: The proof follows by applying the Principle of Mathematical Induction on index l and using Theorem 5.13 and Lemma 5.2. $\square$

Example 5.7 *Consider the polynomial factorization in $S[\boldsymbol{z};\theta]$, where θ is a Frobenius automorphism. We take the polynomial $\boldsymbol{z}^8-1$ over $S=F_4+\mathfrak{v}F_4$. We have the following factorizations:*

$$\begin{aligned}\boldsymbol{z}^8-1 =&(\boldsymbol{z}^5+(\mathfrak{v}+t^2)\boldsymbol{z}^4+\boldsymbol{z}^3+(\mathfrak{v}+t)\boldsymbol{z}^2+1)(\boldsymbol{z}^3+(\mathfrak{v}+t)\boldsymbol{z}^2+1)\\ =&(\boldsymbol{z}^5+(\mathfrak{v}+t)\boldsymbol{z}^4+\boldsymbol{z}^3+(\mathfrak{v}+t^2)\boldsymbol{z}^2+1)(\boldsymbol{z}^3+(\mathfrak{v}+t^2)\boldsymbol{z}^2+1)\\ =&(\boldsymbol{z}^5+(t^2\mathfrak{v}+1)\boldsymbol{z}^4+(\mathfrak{v}+1)\boldsymbol{z}^3+(t^2\mathfrak{v}+t^2)\boldsymbol{z}^2+\mathfrak{v}\boldsymbol{z}+t)*(\boldsymbol{z}^3+(t\mathfrak{v}+1)\boldsymbol{z}^2+\mathfrak{v}\boldsymbol{z}+t^2) \quad \textit{and so on.}\end{aligned}$$

Next, consider the factorization of $\boldsymbol{z}^6-1$ over $S=F_9+\mathfrak{v}F_9$. We have

$$\begin{aligned}\boldsymbol{z}^6-1 =&(\boldsymbol{z}^4+(t^5v+2)\boldsymbol{z}^3+(tv+1)z+2)(\boldsymbol{z}^2+(tv+1)z+1)\\ =&(\boldsymbol{z}^3+(t^2v+2)\boldsymbol{z}^2+(tv+t)z+2)(\boldsymbol{z}^3+(t^2v+1)\boldsymbol{z}^2+(tv+t)z+1)\\ =&(\boldsymbol{z}^3+(2v+t)\boldsymbol{z}^2+(t^2v+t^5)z+2)(\boldsymbol{z}^3+(v+t^7)\boldsymbol{z}^2+(t^2v+t^5)z+1)\\ =&(\boldsymbol{z}^3+v\boldsymbol{z}^2+2vz+2)(\boldsymbol{z}^3+2v\boldsymbol{z}^2+2vz+1) \quad \textit{and so on.}\end{aligned}$$

A fundamental objective in coding theory is the construction of codes with optimal parameters, such as the maximum possible minimum distance for a given length and dimension or improved code rates. For finite fields, the reference point is Grassl's table [12] of the best-known linear codes. That code table has continuously been updated with new codes appearing in the literature by different researchers.

In Tables 1, 2 and 3, we present the parameters of Gray images of 2-generator SGQC codes of index 2 over S where we consider $q=3$, $q=9$ and $q=4$, respectively. We have considered the Frobenius automorphism for each code. We write the coefficients of the generator polynomial in ascending order of the degree of the indeterminate; for example, the polynomial $f_1(\mathtt{z})=\mathtt{z}^3+(\mathfrak{v}+t)\mathtt{z}^2+\mathtt{z}+(\mathfrak{v}+t)$ is represented by $(\mathfrak{v}+t)1(\mathfrak{v}+t)1$.

6. Conclusion

In this work, we have studied the structure of SGQC codes over S without any restriction on length. Here, we have derived 1-generator and multi-generator polynomial codes and their corresponding dimensions. Further, it has examined the 1-generator idempotent polynomial over F_q and S, and provided some suitable examples. Moreover, we have also established a lower bound on the minimum Hamming distance for the 1-generator SGQC codes and demonstrated that they produce improved parameters compared to those reported in the existing literature [11,19].

However, to determine the minimum distances and generator polynomials of $C^\perp$ in terms of the generator polynomial of C for the ρ-generator polynomial codes is still open. Also, the construction of quantum codes based on these codes would be a promising area of research.

7. Tables

Table 1: The 2-generator SGQC codes over $F_3+\mathfrak{v}F_3$

t_1,t_2,N	Generator polynomials	$\phi(C)$
$6,1,7$	$f_{t_1}=(2\mathfrak{v}+2)2(\mathfrak{v}+1)1,\ p_{21}=(\mathfrak{v}+1)(\mathfrak{v}+2)1,\ f_{t_2}=1$	$[14,4,7]*$
$6,2,8$	$f_{t_1}=(2\mathfrak{v}+2)2(\mathfrak{v}+1)1,\ p_{21}=(2\mathfrak{v}+2)0(\mathfrak{v}+1),\ f_{t_2}=(2\mathfrak{v}+2)1$	$[16,4,7]$
$6,2,8$	$f_{t_1}=(2\mathfrak{v}+2)2(\mathfrak{v}+1)1,\ p_{21}=1\mathfrak{v}(2\mathfrak{v}+2),\ f_{t_2}=1$	$[16,5,6]$
$8,1,9$	$f_{t_1}=(2\mathfrak{v}+2)(2\mathfrak{v}+1)(2\mathfrak{v}+2)1,\ p_{21}=(2\mathfrak{v}+2)(\mathfrak{v})2,\ f_{t_2}=1$	$[18,6,7]$
$6,4,10$	$f_{t_1}=(2\mathfrak{v}+2)2(\mathfrak{v}+1)1,\ p_{21}=(\mathfrak{v}+1)(\mathfrak{v}+2)1,\ f_{t_2}=1111$	$[20,4,7]$
$6,9,15$	$f_{t_1}=(2\mathfrak{v}+2)2(\mathfrak{v}+1)1,\ p_{21}=(\mathfrak{v}+1)(\mathfrak{v}+2)1,\ f_{t_2}=21021021$	$[30,5,7]$

$*$: denotes the **near-optimal** code.

@: better parameter than $[12,3,6]$ given in the [11].

Table 2: The 2-generator SGQC codes over $F_9 + \mathfrak{v}F_9$

t_1, t_2, N	Generator polynomials	$\phi(C)$
$6, 2, 8$	$f_{t_1} = (2\mathfrak{v}+2)2(\mathfrak{v}+1)1$, $p_{21} = (2\mathfrak{v}+2)0(\mathfrak{v}+1)$, $f_{t_2} = (2\mathfrak{v}+2)1$	$[16, 4, 7]$
$6, 1, 7$	$f_{t_1} = 2(\mathfrak{v}+2)(2\mathfrak{v}+1)1$, $p_{21} = 1(2\mathfrak{v}+2)1$, $f_{t_2} = 1$	$[14, 4, 6]$
$4, 2, 6$	$f_{t_1} = (t^7)2(t^3)1$, $p_{21} = 1(t^7)(t^5)$, $f_{t_2} = 1$	$[12, 3, 7]^{@}$
$5, 1, 6$	$f_{t_1} = 21$, $p_{21} = (2\mathfrak{v}+2)$, $f_{t_2} = 1$	$[12, 5, 4]$

Table 3: The 2-generator SGQC codes over $F_4 + \mathfrak{v}F_4$

t_1, t_2, N	Generator polynomials	$\phi(C)$
$10, 2, 12$	$f_{t_1} = (t^2\mathfrak{v}+1)1(t^2\mathfrak{v}+1)(t^2\mathfrak{v}+1)1(t^2\mathfrak{v}+1)1(t^2\mathfrak{v}+1)1(t^2\mathfrak{v}+1)$, $p_{21} = 101010101$, $f_{t_2} = (t\mathfrak{v}+1)1$	$[24, 2, 14]^{@}$
$4, 8, 12$	$f_{t_1} = t1t1$, $p_{21} = t^2 1$, $f_{t_2} = 11$	$[24, 8, 6]$
$4, 2, 6$	$f_{t_1} = (t^2\mathfrak{v}+t)1(t^2\mathfrak{v}+t)1$, $p_{21} = (\mathfrak{v}+t^2)t^2(t^2\mathfrak{v}+t)$, $f_{t_2} = 1$	$[12, 3, 8]**$
$10, 2, 12$	$f_{t_1} = (t^2\mathfrak{v}+t)(t^2\mathfrak{v}+t^2)1(\mathfrak{v}+t^2)(t^2\mathfrak{v}+1)1$, $p_{21} = 11t1$, $f_{t_2} = (t^2+\mathfrak{v}t)1$	$[24, 6, 10]^{@}$
$8, 4, 12$	$f_{t_1} = (\mathfrak{v}+t)1(\mathfrak{v}+t)1$, $p_{21} = (\mathfrak{v}+t)(\mathfrak{v}+t)1$, $f_{t_2} = 1$	$[24, 9, 4]$
$8, 1, 9$	$f_{t_1} = (\mathfrak{v}+t)1(\mathfrak{v}+t)1$, $p_{21} = (\mathfrak{v}+t)(\mathfrak{v}+t)1$, $f_{t_2} = 1$	$[18, 6, 7]$
$4, 2, 6$	$f_{t_1} = (t^2\mathfrak{v}+1)(t\mathfrak{v})1$,$p_{21} = 1(t^2\mathfrak{v})$, $f_{t_2} = 1$	$[12, 4, 5]$
$6, 1, 7$	$f_{t_1} = (\mathfrak{v}+t)(\mathfrak{v}+t)11$, $p_{21} = (\mathfrak{v}+t^2)1(\mathfrak{v}+t^2)$, $f_{t_2} = 1$	$[14, 4, 7]$
$7, 1, 8$	$f_{t_1} = 10111$, $p_{21} = 1101$, $f_{t_2} = 1$	$[16, 4, 8]^{@}$

@: better parameters than $[24, 2, 12]$, $[24, 6, 8]$ and $[16, 4, 6]$, respectively given in the [19].
$**$: denotes the **optimal** code.

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Declarations

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References

1. T. Abualrub, M.F. Ezerman, P. Seneviratne and P. Solé, *Skew generalized quasi-cyclic codes*, TWMS J. Pure Appl. Math. 9(2), 123-134, (2018).

2. T. Abualrub, A. Ghrayeb, N. Aydin and I. Siap, *On the construction of skew quasi-cyclic codes*, IEEE Trans. Inform. Theory 56(5), 2081-2090, (2010).

3. T. Abualrub and P. Seneviratne, *Skew codes over rings*, Proc. in IMECS, Hong Kong, II (2010).

4. T. Abualrub and I. Siap, *Cyclic codes over the rings $Z_2 + uZ_2$ and $Z_2 + uZ_2 + u^2Z_2$*, Des. Codes Cryptogr. 42, 273-287, (2007).

5. M. Bhaintwal, *Skew quasi-cyclic codes over Galois rings*, Des. Codes Cryptogr. 62, 85-101, (2012).

6. A. Bonnecaze and P. Udaya, *Cyclic codes and self-dual codes over $F_2 + uF_2$*, IEEE Trans. Inform. Theory 45(4), 1250-1255, (1999).

7. D. Boucher, W. Geiselman and F. Ulmer, *Skew cyclic codes*, Appl. Algebra Engrg. Comm. Comput. 18, 379-389, (2007).

8. D. Boucher, P. Solé and F. Ulmer, *Skew constacyclic codes over Galois rings*, Adv. Math. Commun. 2(3), 273-292, (2008).

9. S.T. Dougherty, P. Gaborit, M. Harada, A. Munemasa and P. Solé, *Type IV self-dual codes over rings*, IEEE Trans. Inform. Theory 45(7), 2345-2360, (1999).

10. M. Esmaeili and S. Yari, *Generalized quasi-cyclic codes: structural properties and code construction*, Appl. Algebra Engrg. Comm. Comput. 20, 159-173, (2009).

11. J. Gao, L. Shen and F.W. Fu, *A Chinese remainder theorem approach to skew generalized quasi-cyclic codes over finite fields*, Cryptogr. Commun. 8, 51-66, (2016).

12. M. Grassl, *Table of bounds on linear codes*, Available at http://www.codetables.de. (Accessed 27th March 2025).

13. F. Gursoy, I. Siap and B. Yildiz, *Construction of skew cyclic codes over* $\mathbb{F}_q + v\mathbb{F}_q$, Adv. Math. Commun. 8(3), 313-322, (2014).

14. A.R. Hammons, Jr., P.V. Kumar, R. Calderbank, N.J.A. Sloane and P. Solé, *The* $\mathbb{Z}_4$*-linearity of Kerdock, Preparata, Goethals and related codes*, IEEE Trans. Inform. Theory 40(2), 301-319, (1994).

15. H. Islam and O. Prakash, *Skew cyclic codes and skew* $(\alpha_1 + u\alpha_2 + v\alpha_3 + uv\alpha_4)$*-constacyclic codes over* $F_q + uF_q + uvF_q$, Int. J. Inf. Coding Theory 5(2), 101-116, (2018).

16. H. Islam and O. Prakash, *Structure of skew cyclic codes over* $\mathbb{F}_p[u]/\langle u^k \rangle$, J. Discrete Math. Sci. Cryptography 26(8), 1-13, (2022).

17. S. Jitman, S. Ling and P. Udomkavanich, *Skew constacyclic codes over finite chain rings*, Adv. Math. Commun. 6(1), 39-63, (2012).

18. S. Ling and P. Solé, *On the algebraic structure of quasi-cyclic codes I: Finite fields*, IEEE Trans. Inform. Theory 47(7), 2751-2760, (2001).

19. M. Özen, N.T. Özzaim and H. Ince, *Skew quasi-cyclic codes over* $\mathbb{F}_q + v\mathbb{F}_q$, J. Algebra Appl. 18(4), Paper No. 1950077, 16 p., (2019).

20. V. Pless, P. Solé and Z. Qian, *Cyclic Self-Dual* Z_4*-Codes*. Finite Fields Appl. 3(1), 48-69, (1997).

21. O. Prakash, H. Islam, S. Patel and P. Solé, *New quantum codes from skew constacyclic codes over a class of non-chain rings* $R_{e,q}$, Int. J. Theoret. Phys. 60(9), 3334-3352, (2021).

22. O. Prakash, A. Singh, R.K. Verma, P. Solé and W. Cheng, *DNA Code from Cyclic and Skew Cyclic Codes over* $F_4[v]/\langle v^3 \rangle$, Entropy 25(2), Paper No. 239, (2023).

23. G.E. Séguin, *A class of 1-generator quasi-cyclic codes*, IEEE Trans. Inform. Theory 50(8), 1745-1753, (2004).

24. P. Seneviratne and T. Abualrub, *New linear codes derived from skew generalized quasi-cyclic codes of any length*, Discrete Math. 345(11), Paper No. 113018, (2022).

25. I. Siap, T. Abualrub, N. Aydin and P. Seneviratne, *Skew cyclic codes of arbitrary length*, Int. J. Inf. Coding Theory 2(1), 10-20, (2011).

26. I. Siap and N. Kulhan, *The structure of generalized quasi-cyclic codes*, Appl. Math. E-Notes [electronic only] 5, 24-30, (2005).

27. S. Zhu, Y. Wang and M. Shi, *Some Results on Cyclic Codes Over* $F_2 + vF_2$, IEEE Trans. Inform. Theory 56(4), 1680-1684, (2010).

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