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A Note on Biregular and Triregular Hyperenergetic Graphs

Smitha M. [1], Shashirekha S.[1], Ashawini M. P.[2], Honnegowda C. K.[3]

ABSTRACT: If G is a molecular graph with eigenvalues $\mu_1, \mu_2, \mu_3, ..., \mu_p$ for the adjacency matrix of G then, energy $\zeta(G)$ of graph G is the sum of the absolute values of μ_i's. The graphs having energy greater than the energy of complete graph are known as hyperenergetic graphs. Mathematically we say G is hyperenergetic if it satisfies the equation $\zeta(G) > (2p-2)$, where p is the count of vertices in G. In [1,8,6] the authors have studied various hyperenergetic graphs and established many results regarding hyperenergetic graphs. In this research paper we obtain constraints for a graph to be hyperenergetic which are biregular and triregular in terms of count of their vertices and edges.

Key Words: graph energy, biregular graphs, triregular graphs.

Contents

1. Introduction

We consider graphs which are finite, undirected and without loops. Basic terminologies and notations can be found in [10]. Let G represent a simple graph with p vertices. Then we have the adjacency matrix $Adj(G)$ of G as a matrix of order $p \times p$ whose entries are equal to one if vertices v_i and v_j are adjacent in G. If vertices v_i and v_j are not adjacent in G, then entry is zero. We also observe that all the entries in the principal diagonal elements of $Adj(G)$ are zero.
Let $\mu_1, \mu_2, \mu_3, ..., \mu_p$ be the eigenvalues of the graph G. The energy of G is defined as in [5], given by

$$\zeta(G) = \sum_{i=1}^{p} |\mu_i|. \tag{1.1}$$

The roots of the characteristic polynomial of adjacency matrix of the graph G is known as eigenvalues of G. The set of eigenvalues of the graph with their multiplicities is known as spectrum of the graph. Hence we have,

$$Spectrum(G) = \begin{pmatrix} \mu_1, \mu_2, \mu_3, \cdots, \mu_p \\ r_1, r_2, r_3, \cdots, r_p \end{pmatrix}. \tag{1.2}$$

Energy for a graph was introduced by I. Gutman [5], motivated by pi-electron energy in chemistry. A brief study in energy of graph and various significant results can be found in D. Cvetkovic et.al, [4] and X. Li et.al, [16]. For added information and history of graph energy one can study in detail in [3,5,7,8] and the references mentioned in the cited papers.

Two graphs which are non-isomorphic are concluded as co-spectral when they have identical spectra, otherwise we call them as non-co-spectral. Any two graphs are said to be equienergetic if they have same energy. Further equienergetic graphs have been studied in X. Li et.al [16]. A symmetric study

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has been done using computers for equienergetic trees in [2] [11]. In [9], we find open problems for equienergetic graphs. The big challenge and interesting task is to find out equienergetic graphs which are non-copspectral as well. In [5] I.Gutman during 1978 conjectured that graphs K_p, which are complete has maximum energy. The conjecture given by I.Gutman was disproved by H.B. Walikar et.al, in [18]. Further they introduced the concept of hyperenergetic graphs, where they proved that, there exists graphs whose energy exceeds energy of complete graphs. In [6] I.Gutman showed that hyperenergetic graphs exists for graphs with vertices $p \geq 8$. Further they proved that, there does not exist o hyperenergetic graphs whose vertices are less than eight. Subsequent computer studies revealed that, among graphs with large number of edges hyperenergetic graphs are encountered quite frequently.
There are many classes of graphs with p vertices satisfying the inequality,

$$\zeta(G) \geq p. \tag{1.3}$$

Here we mention two of them.

Theorem 1.1 *[7] Consider graph G which is not-singular, then*

$$\zeta(G) \geq p. \tag{1.4}$$

Theorem 1.2 *[7]Consider graph G , we have*

$$\zeta(G) \geq p. \tag{1.5}$$

while we also have certain graphs with

$$\zeta(G) < p. \tag{1.6}$$

These include acyclic graphs such as $K_{1,2}$ and $K_{1,3}$ whose energies are $2\sqrt{2}$ and $2\sqrt{3}$ respectively. Another example is $K_{2,3}$ whose energy is $2\sqrt{6}$ which is less than 5. The energy of the complete graph K_n is given by $\zeta(K_n) = (2p-2)$. A graph G of order n is called hyperenergetic if $\zeta(G) > (2p-2)$. If $\zeta(G) \leq (2p-2)$ then it is called non-hyperenergetic. The line graph $L(G)$ of graph G is constructed by taking edges of G as vertices of $L(G)$ and joining two vertices in $L(G)$, whenever the corresponding edges in G have a common vertex. It is well known that line graph of all k-regular graphs, for $k \geq 4$, are hyperenergetic. Two non-isomorphic graphs G_1 and G_2 of same order are said to be equienergetic if $\zeta(G_1) = \zeta(G_2)$. In [14,15] H.S.Ramane et.al, have proved that if G_1 and G_2 are regular graphs of sameorder then for $k \geq 2, L_k(G_1)$ and $L_k(G_2)$ are equienergetic. Here, $L_k(G)$ is called iterated line graph of G.
In [17] Samir K. Vaidya et.al, have studied hyperenergetic, equienergetic, and hypoenergetic graphs and have obtained significant results. Motivated by the above works in this research work we have obtained conditions under which Biregular and Triregular graphs satisfy the condition $\zeta(G) > p$.

2. Pre-Requisites

Let $\mu_1, \mu_2, \mu_3, ..., \mu_p$ be the eigen values of the graph G. It is well known that

$$M_2 = 2m. \tag{2.1}$$

$$M_4 = 2\sum_{i=1}^{p} d_i^2 - 2m + 8q. \tag{2.2}$$

where d_i is the degree of the i^{th} vertex. m is the number of edges in G, q is the number of quadrangles and M_K denote the k^{th} spectral moment of the graph G. In [13,12,19], we have lower bound expression for energy of graph given by,

$$\zeta(G) \geq \sqrt{\frac{M_2^3}{M_4}}. \tag{2.3}$$

Further we have,

$$\zeta(G) \geq 2m\sqrt{\frac{2m}{2\sum_{i=1}^{n} d_i^2 - 2m + 8q}}. \tag{2.4}$$

Now we shall recall the following definitions.

Definition 2.1 *A graph G is said to be w regular graph, if every vertex of G has degree w.*

Definition 2.2 *Let r, s be positive integers, $1 \leq r < s$. A graph G is said to be biregular graph of degree r and s, if atleast one vertex of G has degree r and atleast one vertex has degree r, and no vertex of G has degree other than r and s. The set of all p- vertex biregular graphs of degree r and s will be denoted by $\beta_p(r, s)$.*

Definition 2.3 *Let r, s, t be positive integers, $1 \leq r < s < t$. A graph G is said to be triregular graph of degree r, s and t, if atleast one vertex of G has degree r, atleast one vertex has degree s, and atleast one vertex of G has degree t no vertex of G has degree other than r, s and t. The set of all p- vertex triregular graphs of degree r, s and t will be denoted by $\Gamma_p(r, s, t)$.*

If $G \in \beta_p(r, s)$, then using (2.4) I. Gutman [8] deduced that

$$\zeta(G) \geq 2m\sqrt{\frac{2m}{(2r+2s-1)2m - 2rsp + 8q}}. \tag{2.5}$$

By employing the above inequality to obtain conditions for biregular and triregular graphs to be hyperenergetic.

3. Main Results: Biregular Hyperenergetic Graphs

In this section, we determine conditions under which biregular graphs satisfy the condition $\zeta(G) > 2p$.

Theorem 3.1 *Let $G \in \beta_p(r, s)$ and let the number of quadrangles q of G be less than or equal to $\frac{rsp}{4}$. If $17 \leq r < s \leq \frac{(r-2)^2-4r}{8}$, then G is hyperenergetic.*

Proof: If $q \leq \frac{rsp}{4}$ then $-2rsp + 8q \leq 0$, Using this in 2.5 we obtain

$$\zeta(G) \geq \sqrt{\frac{1}{(2r+2s-1)}}. \tag{3.1}$$

This implies

$$\frac{\zeta(G)}{p} \geq d\sqrt{\frac{1}{(2r+2s-1)}} > r\sqrt{\frac{1}{(2r+2s-1)}}, \tag{3.2}$$

where $d = \frac{2m}{p}$ is the average vertex degree of the graph G. But $\frac{r}{(2r+2s-1)} \geq 2$, if and only if, $s \leq \frac{(r-2)^2-4r}{8}$. Hence

$$\zeta(G) > 2p \quad \textit{if} \quad 17 \leq r < s \leq \frac{(r-2)^2 - 4r}{8}. \tag{3.3}$$

$\square$

That is G is hyperenergetic if $17 \leq r < s \leq \frac{(r-2)^2-4r}{8}$ and $q \leq \frac{rsp}{4}$.

Corollary 3.1 *Let $G \in \beta_p(r, s)$ be quadrangle free. If $17 \leq r < s \leq \frac{(r-2)^2-4r}{8}$, then G is hyperenergetic.*

Theorem 3.2 *Let $G \in \beta_p(r, s)$ be quadrangle free. If $9 \leq r < s < \frac{(r^2+4)}{8}$, then G is hyperenergetic.*

Proof: Since G is quadrangle free we have $q = 0$. Putting $q = 0$ in 2.5 and then dividing both sides by p we obtain,

$$\frac{\zeta(G)}{p} \geq d\sqrt{\frac{d}{(2r+2s-1)d - 2rs}} \geq r\sqrt{\frac{1}{(2s-1)}}, \tag{3.4}$$

where $d = \frac{2m}{p}$.
But

$$\frac{r}{\sqrt{(2s-1)}} > 2 \text{ if and only if } s < \frac{r^2+4}{8}. \tag{3.5}$$

Hence, if $9 \leq r < s < \frac{(r^2+4)}{8}$, then G is hyperenergetic. □

Now we consider biregular graphs G in which all quadrangles are mutually disjoint. That is no two of them have a common vertex. Then we have $q \leq \frac{p}{4}$. Using this fact and the function

$$g(x) = x\sqrt{\frac{x}{(2r+2s-1)x-(2rs-1)}}, \qquad x \in (r,s). \tag{3.6}$$

I. Gutman [8] proved that

$$\frac{\zeta(G)}{p} > \frac{r\sqrt{r}}{\sqrt{(2r^2-r+2)}} \text{ if } r < s \leq 2r-1+\frac{3}{r}. \tag{3.7}$$

Now, consider the function $f(x) = x^3 - 8x^2 + 4x - 8$. Since $f'(x) = 3x^2 - 16x + 4 \geq 0$ if $x \geq 6$, the function $f(x)$ increases monotonically in $[6, \infty)$. But $f(7) < 0$ and $f(8) > 0$. This implies $f(r) > 0$ if $r \geq 8$, i.e, $\frac{r\sqrt{r}}{\sqrt{2r^2-r+2}}$ if $r \geq 8$. Hence $\zeta(G) \geq 2p$ if $8 \leq r < s \leq 2s - 1 + \frac{3}{r}$. Thus we have proved the following theorem.

Theorem 3.3 *Let $G \in \beta_p(r,s)$ in which all quadrangles are mutually disjoint, and if $8 \leq r < s \leq 2r - 1 + \frac{3}{r}$ then G is hyperenergetic.*

4. Main Results: Triregular Hyperenergetic Graphs

In this section we consider triregular graphs and determine the conditions under which some classes of triregular graphs satisfy the condition $\zeta(G) > 2p$.

If $G \in \Gamma_n(1,r,s)$ and G has m edges, then the following inequality has been recently deduced in [1],

$$\zeta(G) \geq 2m\sqrt{\frac{2m}{(2r+2s-1)(2m-k)-2rs(n-k)+k+8q}}, \tag{4.1}$$

where k and q are respectively, the number of pendent vertices and quadrangles in G. If $G \in \Gamma_p(1,2,3)$ is a quadrangle free molecular graph with m edges and k pendent vertices, then from 4.1 we have

$$\zeta(G) \geq 2m\sqrt{\frac{2m}{9(2m-k)-12(p-k)+k}}.$$

i.e,

$$\frac{\zeta(G)}{p} \geq \frac{2m}{p}\sqrt{\frac{m}{9m-6p+2k}}.$$

But

$$\frac{2m}{p}\sqrt{\frac{m}{9m-6p+2k}} \qquad \geq 2.$$

if and only if $6p^3 - (9m+2k)p^2 + m^3 \geq 0$. Thus we have proved the following theorem :

Theorem 4.1 *Let G be a quadrangle free molecular graph with p vertices and m edges and k pendent vertices. If $6p^3 - (9m+2k)p^2 + m^3 \geq 0$, then G is hyperenergetic.*

Corollary 4.1 *Let G be a quadrangle free molecular graph with p vertices and m edges. If $4p^3 + m^3 \geq 9mp^2$, then G is hyperenergetic.*

Proof: If k is the number of pendent vertices in G then we have $-2kp^2 \geq -2p^3$. If

$$9mp^2 \leq 4p^3 + m^3$$

then

$$0 \leq 4p^3 - 9mp^2 + m^3 = 6p^3 - (9m + 2p)p^2 + m^3 \leq 6p^3 - (9m + 2k)p^2 + m^3.$$

□

Now, from 4.1 it follows that G is hyperenergetic.

Finally to conclude in this the paper, we discuss the conditions under which certain biregular and triregular graphs are hyperenergetic in terms of their number of vertices and number of edges. For a biregular graph we have related quadrangles of graph with degree of biregular graph and obtained the conditions under which biregular graphs are hyperenergetic. Also obtained conditions under which quadrangle free biregular graphs are hyperenergetic. Further we have obtained conditions under which triregular graphs are hyperenergetic, as a corallary we obtained conditions under which quadrangle free triregular moelecular graphs are hyperenergetic.

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[1] *Department of Mathematics, JSS Science and Technology University, Mysuru-57006, INDIA,*

[2] *Department of Mathematics, Vidya Vikas Institute of Engineering and Technology, Mysuru-570028, INDIA*

[3] *Department of Mathematics, Government First Grade College, Krishnarajapete, Mandya - 571426, INDIA*

E-mail address: [1] smitham@jssstuniv.in, shashirekha.s157@gmail.com,
[2] ashvviet22@gmail.com, [3] honnegowdack@gmail.com