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Multipliers and Reverse Generalized $(\alpha,*)$-n-Derivations on Prime Rings

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ABSTRACT: While α is an automorphism of R and $*$ denotes an involution of R, the goal of the current study is to define the notion of reverse generalized $(\alpha,*)$-derivations on ring R. Using the roles of α and $*$, we derive certain commutativity theorems in the case of prime rings. The proofs of the theorems in the situation of a non-commutative prime ring and the conditions under which a reverse generalized $(\alpha,*)$-derivation acts as an α-multiplier will also be covered. Appropriate examples are provided to support the proposed idea.

Keywords: Prime ring, automorphism, involution, reverse generalized derivation.

Contents

1. Introduction

Throughout the manuscript, the notation $\mathcal{Z}(R)$ stands for the center of an associative ring R. The symbol $[b,d]$ specifies the commutator of $b,d\in R$, which is represented by the mathematical formula $bd-db$. If $pr=0$ implies $r=0$ for every $r\in R$ and $p>1$ is a fixed integer, then a ring R is a p-torsion free ring. A ring R is a prime if $rRt=\{0\}$ gives $t=0$ or $r=0$. It is called semiprime if it fulfills the requirement that $cRc=\{0\}$ yields that $c=0$.

Herstein first proposed the conception of reverse derivation in one of his early papers [8], in which he examined Jordan derivations on prime associative rings. Some extensions of derivations are closely associated with the concept of reverse derivation. An additive mapping is said to be a reverse derivation $\mathfrak{D}$ from a ring R onto itself that fulfills the condition for every $x,y\in R$, $\mathfrak{D}(xy)=\mathfrak{D}(y)x+y\mathfrak{D}(x)$. Herstein established in [8] that R is a commutative integral domain and $\mathfrak{D}$ is a derivation if R is a prime ring and $\mathfrak{D}$ is a nonzero reverse derivation of R. Herstein's conclusion was further extended to semiprime rings by Samman and Alyamani in [13], demonstrating that if R is a semiprime ring, then a reverse derivation is actually a derivation from R to the $\mathcal{Z}(\mathcal{R})$.

Brešar [5] developed the generalized derivations. If there is a derivation $\mathfrak{D}: R\to R$ such that $\mathcal{G}(xy)=\mathcal{G}(x)y+x\mathfrak{D}(y)$ for all $x,y\in R$, then an additive mapping $\mathcal{G}: R\to R$ is referred to as a generalized derivation. Both the idea of derivation and the left multiplier (in the case $\mathfrak{D}=0$) are included in the concept of generalized derivation. Our investigation carries both the mappings, namely the extended generalized derivation and multipliers, together, and shows their relationship in specific conditions.

Let R be a ring whose automorphism is α. A map h on R satisfying $\mathrm{h}(dk)=\mathrm{h}(d)\alpha(k)+d\mathrm{h}(k)$ is recognized as the α-derivation (skew-derivation) if it holds for any pair d,k in R and h has additivity. The combination form $\mathrm{h}=\alpha-\mathcal{I}$ served as the α-derivation if we symbolize the identity map on R by $\mathcal{I}$.

We define certain key terms and concepts before going on to the main results of this section. Involution is an additive mapping that satisfies the following two conditions and is defined as $*$ from R to R, for every $p,q\in R$, $(pq)^*=q^*p^*$ and $(p^*)^*=p$. The most common forms of involution over the matrix ring are identity matrices and invertible matrices. An involution ring, or ring having an involution $*$, is another name for a $*$-ring.

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On the basis of [15], if $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all $x, y \in R$ and T is additive, then a mapping $T : R \to R$ is referred to as a left (right) multiplier. The expression for $*$-multiplier in the same line of study is as follows: The left $*$-multiplier and the right $*$-multiplier on R shall be categories for a mapping T on R that is additive and satisfies $T(xy) = T(x)y^*$ and $T(xy) = x^*T(y)$ for all $x, y \in R$. [1,3,4,7,8,9,10,11,12,14] presents an impressive analysis of the theory of multipliers, ϕ-multiplier and $*$-multipliers. The extended idea of $*$-derivation on standard operator algebra has been developed by the authors in [2].

Definition 1.1 [6] Let R be a ring. The additive maps $\mathfrak{D}, \mathcal{G} : R \to R$ are, respectively, called a reverse derivation and reverse generalized derivation if for all $\mu, \mu' \in R$, the following condition holds:

$$\mathfrak{D}(\mu\mu') = \mathfrak{D}(\mu')\mu + \mu'\mathfrak{D}(\mu)$$

$$\mathcal{G}(\mu\mu') = \mathcal{G}(\mu')\mu + \mu'\mathfrak{D}(\mu)$$

Example 1.1 *Consider the ring* $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in S \right\}$, *where* S *is a commutative ring such that* $S^2 \neq 0$. *Define* $\mathfrak{D} : R \to R$ *by*

$$\mathfrak{D}\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}.$$

Then it is easy to check that $\mathfrak{D}$ *is both a derivation and a reverse derivation.*

Example 1.2 *Let* S *be any ring and*

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a, b, c \in S \right\}$$

Define maps $\mathcal{G}, \mathfrak{D} : R \to R$ *by*

$$\mathcal{G}\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{D}\left(\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & a-c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then it is easy to check that $\mathcal{G}$ *is a generalized reverse derivation associated with reverse derivation* $\mathfrak{D}$.

Definition 1.2 Let $\mathfrak{D}, \mathcal{G} : R \longrightarrow R$ be additive maps. $\mathcal{G}$ is said to be reverse generalized $(\alpha, *)$-derivation associated with reverse $(\alpha, *)$-derivation $\mathfrak{D}$ on R if it satisfy the below condition

$$\mathcal{G}(\nu k) = \mathcal{G}(k)\alpha(\nu) + k^*\mathfrak{D}(\nu), \text{ for every } \nu, k \in R,$$

where $*$ is an involution on R and α is the automorphism on R. Notice that if we substitute $\mathfrak{D}$ for $\mathcal{G}$, then $\mathfrak{D}$ will be called a reverse $(\alpha, *)$-derivation on R.

Example 1.3 *Consider the ring*

$$R = \left\{ \begin{pmatrix} 0 & \mu & y & z \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, \mu, y, z, m, n \in \mathbb{Z} \right\}.$$

Define maps $\mathcal{G}, \mathfrak{D}, \alpha, * : R \to R$ *by*

$$\mathcal{G}\left(\begin{pmatrix} 0 & \mu & y & z \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 & y+n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{D}\left(\begin{pmatrix} 0 & \mu & y & z \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & y-m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\alpha\left(\begin{pmatrix} 0 & \mu & y & z \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & n & -y & -z \\ 0 & 0 & 0 & -m \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \mu & y & z \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{pmatrix}^{*} = \begin{pmatrix} 0 & \mu & m & z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then it is easy to check that $\mathcal{G}$ is a generalized reverse $(\alpha, *)$-derivation associated with reverse $(\alpha, *)$-derivation $\mathfrak{D}$.

Motivated by the above literature review and concepts, we put out the extension of the notion of reverse generalized $(\alpha, *)$-n-derivation on rings as follows:

Definition 1.3 A mapping $\mathcal{G} : R^n \to R$ is called a reverse generalized $(\alpha, *)$-n -derivation if there exists an $(\alpha, *)$-n-derivation $\mathfrak{D} : R^n \to R$ such that

$$\mathcal{G}(\varsigma_1, \ldots, \varsigma_k\varsigma'_k, \ldots, \varsigma_n) = \mathcal{G}(\varsigma_1, \ldots, \varsigma'_k, \ldots, \varsigma_n)\alpha(\varsigma_k) + \varsigma'^{*}_k\mathfrak{D}(\varsigma_1, \ldots, \varsigma_k, \ldots, \varsigma_n)$$

for all $\varsigma_1, \ldots, \varsigma_k, \varsigma'_k, \ldots, \varsigma_n \in R$. If we replace $\mathcal{G}$ by $\mathfrak{D}$, then $\mathfrak{D}$ will be read as the reverse $(\alpha, *)$-n-derivation.

In this work, we examine the behavior of certain outcomes using reverse generalized $(\alpha, *)$-n-derivations on prime rings. Previous studies have shown how important automorphism and involution are to the study of n-derivations and the maps that accompany them. We thus concentrate on comprehending certain related maps in the context of rings and their subsets. Analyzing how the study of multipliers affects the structure of reverse $(\alpha, *)$-n- derivations and reverse generalized $(\alpha, *)$-n-derivations is one of the main contributions of this paper.

2. Main Theorems

Lemma 2.1 *[4] Let R be a semiprime ring and let I be a nonzero left ideal. If R admits a non-zero derivation $\mathfrak{D}$ centralizing on I, then R has a nonzero central ideal.*

Theorem 2.1 *If a $*$-prime ring admits a non-zero $(\alpha, *)$-n-reverse derivation $\mathfrak{D}$, then R is commutative.*

Proof: Let $\mathfrak{D}$ be a nonzero $(\alpha, *)$-n-reverse derivation on a ring R. By definition, we have

$$\mathfrak{D}(\mu_1, \ldots, \mu_k\mu'_k, \ldots, \mu_n) = \mathfrak{D}(\mu_1, \ldots, \mu'_k, \ldots, \mu_n)\alpha(\mu_k) + \mu'^{*}_k\mathfrak{D}(\mu_1, \ldots, \mu_k, \ldots, \mu_n),$$

for every $\mu_1, \ldots, \mu_n \in R$. Now, substitute $\mu_k = \mu_k y$, for any $y \in R$

$$\mathfrak{D}(\mu_1, \ldots, (\mu_k y)\mu'_k, \ldots, \mu_n) = \mathfrak{D}(\mu_1, \ldots, \mu'_k, \ldots, \mu_n)\alpha(\mu_k y) + \mu'^{*}_k\mathfrak{D}(\mu_1, \ldots, \mu_k y, \ldots, \mu_n).$$

Expanding the terms, we obtain

$$\begin{aligned}\mathfrak{D}(\mu_1,\ldots,\mu_k y\mu_k',\ldots,\mu_n) = {} & \mathfrak{D}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)\alpha(\mu_k)\alpha(y) \\ & + \mu_k'^{*}\mathfrak{D}(\mu_1,\ldots,y,\ldots,\mu_n)\alpha(\mu_k) \\ & + \mu_k'^{*}y^{*}\mathfrak{D}(\mu_1,\ldots,\mu_k,\ldots,\mu_n),\end{aligned} \tag{2.1}$$

for every $\mu_1,\ldots,\mu_n, y \in R$.

Now, again use the reverse $(\alpha,*)$-n-derivation formula and reword the expression as

$$\begin{aligned}\mathfrak{D}(\mu_1,\ldots,\mu_k(y\mu_k'),\ldots,\mu_n) = {} & \mathfrak{D}(\mu_1,\ldots,y\mu_k',\ldots,\mu_n)\alpha(\mu_k) \\ & +(y\mu_k')^{*}\mathfrak{D}(\mu_1,\ldots,\mu_k,\ldots,\mu_n).\end{aligned}$$

Expansion of the last expression gives that

$$\begin{aligned}\mathfrak{D}(\mu_1,\ldots,\mu_k(y\mu_k'),\ldots,\mu_n) = {} & \mathfrak{D}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)\alpha(y)\alpha(\mu_k) \\ & + \mu_k'^{*}\mathfrak{D}(\mu_1,\ldots,y,\ldots,\mu_n)\alpha(\mu_k) \\ & + \mu_k'^{*}y^{*}\mathfrak{D}(\mu_1,\ldots,\mu_k,\ldots,\mu_n),\end{aligned} \tag{2.2}$$

for every $\mu_1,\ldots,\mu_n, y \in R$.

Subtract equation (2.1) from equation (2.2) to find

$$\mathfrak{D}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)(\alpha(y)\alpha(\mu_k) - \alpha(\mu_k)\alpha(y)) = 0, \quad \text{for every } \mu_1,\ldots,\mu_n, y \in R.$$

Which simplifies to the form

$$\mathfrak{D}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)[\alpha(y),\alpha(\mu_k)] = 0 \quad \text{for every } \mu_1,\ldots,\mu_n, y \in R.$$

Now, substitute $\alpha^{-1}(y)$ for y and $\alpha^{-1}(\mu_k)$ for μ_k to obtain

$$\mathfrak{D}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)[y,\mu_k] = 0 \quad \text{for every } \mu_1,\ldots,\mu_n, y \in R. \tag{2.3}$$

Put zy in place of y to obtain

$$\mathfrak{D}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)z[y,\mu_k] + \mathfrak{D}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)[z,\mu_k]y = 0. \tag{2.4}$$

Encounter equations (2.3) and (2.4) together to get

$$\mathfrak{D}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)z[y,\mu_k] = 0 \quad \text{for each } \mu_1,\ldots,\mu_n, y, z \in R.$$

Since $\mathfrak{D}(\mu_1,\ldots,\mu_k',\ldots,\mu_n) \neq 0$, we conclude by primeness of R

$$[y,\mu_k] = 0 \quad \text{for every} \quad y,\mu_k \in R.$$

Thus, R is commutative. □

Notice that in Example 1.3, it is obvious that R is not a prime ring and $\mathfrak{D}(Z(R)) = (0)$. It can be verified with ease that $\mathfrak{D}$ is $(\alpha,*)$-derivation on R, but R is not commutative. Hence, primeness is an essential condition in the hypothesis.

Theorem 2.2 *Consider a semi-prime $*$-ring R having a non-zero ideal J. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two nonzero reverse generalized $(\alpha,*)$-n-derivations on R associated with reverse $(\alpha,*)$-n-derivation $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively such that $\mathcal{G}_1 = \mathcal{G}_2$, then $\mathfrak{D}_1 = \mathfrak{D}_2$ on R.*

Proof: By the hypothesis, we are given that

$$\mathcal{G}_1(\mu_1,\mu_2,\ldots,\mu_n)=\mathcal{G}_2(\mu_1,\mu_2,\ldots,\mu_n),\quad \text{for every } \mu_1,\ldots,\mu_n\in J. \tag{2.5}$$

$$\mathcal{G}_1(\mu_1,\ldots,\mu_k\mu_k',\ldots,\mu_n)=\mathcal{G}_2(\mu_1,\ldots,\mu_k\mu_k',\ldots,\mu_n),\quad \text{for every } \mu_1,\ldots,\mu_n\in J. \tag{2.6}$$

By definition, the above equation can be rewritten as

$$\begin{aligned}&\mathcal{G}_1(\mu_1,\ldots,\mu_k',\ldots,\mu_n)\alpha(\mu_k)+\mu_k'^*\mathfrak{D}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\\&=\mathcal{G}_2(\mu_1,\ldots,\mu_k',\ldots,\mu_n)\alpha(\mu_k)+\mu_k'^*\mathfrak{D}_2(\mu_1,\ldots,\mu_k,\ldots,\mu_n).\end{aligned}$$

Applying equation (2.6) to the above expression, we get

$$\mu_k'^*\mathfrak{D}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)=\mu_k'^*\mathfrak{D}_2(\mu_1,\ldots,\mu_k,\ldots,\mu_n) \tag{2.7}$$

Semiprimeness of R yields that for each $\mu_1,..,\mu_k\in J$

$$\mathfrak{D}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)=\mathfrak{D}_2(\mu_1,\ldots,\mu_k,\ldots,\mu_n) \tag{2.8}$$

Put $r_k\mu_k$ for μ_k in (2.8) to obtain

$$\mathfrak{D}_1(\mu_1,\ldots,r_k\mu_k,\ldots,\mu_n)=\mathfrak{D}_2(\mu_1,\ldots,r_k\mu_k,\ldots,\mu_n),\quad \text{for each } \mu_1,\ldots,\mu_n\in J, r_k\in R.$$

This implies that

$$\begin{aligned}\mathfrak{D}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\alpha(r_k)+\mu_k^*\mathfrak{D}_1(\mu_1,\ldots,r_k,\ldots,\mu_n)=\mathfrak{D}_2(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\alpha(r_k)\\+\mu_k^*\mathfrak{D}_2(\mu_1,\ldots,r_k,\ldots,\mu_n),\quad \text{for each } \mu_1,\ldots,\mu_n\in J, r_k\in R.\end{aligned}$$

Making use of equation (2.8) and (2.7), we arrive at

$$\mathfrak{D}_1(\mu_1,\ldots,r_k,\ldots,\mu_n)=\mathfrak{D}_2(\mu_1,\ldots,r_k,\ldots,\mu_n),\quad \text{for each } \mu_1,\ldots,\mu_n\in J, r_k\in R.$$

Repeating the same process in all n-slots, we get

$$\mathfrak{D}_1(r_1,\ldots,r_k,\ldots,r_n)=\mathfrak{D}_2(r_1,\ldots,r_k,\ldots,r_n),\quad \text{for each } r_1,\ldots,r_n\in R.$$

Hence, $\mathfrak{D}_1=\mathfrak{D}_2$ on R. This completes the proof.

□

Theorem 2.3 *Let R be a prime $*$-ring. If R admits a nonzero generalized reverse $(\alpha,*)$-n-derivation $\mathcal{G}$ associated with a reverse$(\alpha,*)$-n-derivation $\mathfrak{D}$, then one of the conditions hold*
1. R is commutative.
2. $\mathcal{G}=0$

Proof: By definition, we have

$$\mathcal{G}(\mu_1,\ldots,\mu_k y\mu_k',\ldots,\mu_n)=\mathcal{G}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)\alpha(\mu_k y)+\mu_k'^*\mathfrak{D}(\mu_1,\ldots,\mu_k y,\ldots,\mu_n),$$

for each $\mu_1,\ldots,\mu_n,y,$ in R. On simplification, we get

$$\begin{aligned}&\mathcal{G}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)\alpha(\mu_k)\alpha(y)+\mu_k'^*(\mathfrak{D}(\mu_1,\ldots,y,\ldots,\mu_n)\alpha(\mu_k)+y^*\mathfrak{D}(\mu_1,\ldots,\mu_n))\\&=\mathcal{G}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)\alpha(\mu_k)\alpha(y)+\mu_k'^*\mathfrak{D}(\mu_1,\ldots,y,\ldots,\mu_n)\alpha(\mu_k)+\mu_k'^*y^*\mathfrak{D}(\mu_1,\ldots,\mu_n)\\&=\mathcal{G}(\mu_1,\ldots,\mu_k',\ldots,\mu_n)\alpha(\mu_k)\alpha(y)+\mu_k'^*\mathfrak{D}(\mu_1,\ldots,y,\ldots,\mu_n)\alpha(\mu_k)+(y\mu_k')^*\mathfrak{D}(\mu_1,\ldots,\mu_n)\end{aligned} \tag{2.9}$$

The left-hand side of (2.9) can be written as

$$\mathcal{G}(\mu_1,\ldots,\mu_k y\mu_k',\ldots,\mu_n)=\mathcal{G}(\mu_1,\ldots,y\mu_k',\ldots,\mu_n)\alpha(\mu_k)+(y\mu_k')^*\mathfrak{D}(\mu_1,\ldots,\mu_k,\ldots,\mu_n).$$

After simplifying the above equation, we arrive at

$$\begin{aligned}\mathcal{G}(\mu_1,\ldots,\mu_k y\mu'_k,\ldots,\mu_n) = \mathcal{G}(\mu_1,\ldots,\mu'_k,\ldots,\mu_n)\alpha(y)\alpha(\mu_k) + \mu'^*_k\mathfrak{D}(\mu_1,\ldots,y,\ldots,\mu_n)\alpha(\mu_k)\\ +(y\mu'_k)^*\mathfrak{D}(\mu_1,\ldots,\mu_n), \quad \text{for each } \mu_1,\ldots,\mu_n, y \text{ in } R.\end{aligned} \tag{2.10}$$

Evaluate (2.9) and (2.10) to find

$$\mathcal{G}(\mu_1,\ldots,\mu'_k,\ldots,\mu_n)(\alpha(y)\alpha(\mu_k) - \alpha(\mu_k)\alpha(y)) = 0, \quad \text{for each } \mu_1,\ldots,\mu_n, y, \text{in } R.$$

$$\mathcal{G}(\mu_1,\ldots,\mu'_k,\ldots,\mu_n)[\alpha(y),\alpha(\mu_k)] = 0, \quad \text{for each } \mu_1,\ldots,\mu_n, y \text{ in } R.$$

Since α is an automorphism, put $y = \alpha^{-1}(y)$ and $\mu_k = \alpha^{-1}(\mu_k)$ to obtain

$$\mathcal{G}(\mu_1,\ldots,\mu'_k,\ldots,\mu_n)[y,\mu_k] = 0.$$

Replace y by ry to get

$$\mathcal{G}(\mu_1,\ldots,\mu'_k,\ldots,\mu_n)r[y,\mu_k] = 0, \quad \text{for each } \mu_1,\ldots,\mu_n, y, r \text{ in } R.$$

Primeness gives that either $\mathcal{G} = 0$ or $[y,\mu_k] = 0$ for each $y,\mu_k \in R$. Consider the case $[y,\mu_k] = 0$ for each $y,\mu_k \in R$, then R is commutative by using Brauer's trick. □

Theorem 2.4 *let R be a prime $*$-ring having reverse generalized $(\alpha,*)$-n-derivations $\mathcal{G}_1$ and $\mathcal{G}_2$ associated with reverse $(\alpha,*)$-n-derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively such that $\mathfrak{D}_1 \neq \mathfrak{D}_2$ if*

$$\mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_2(y_1,\ldots,y_k,\ldots,y_n) - \mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_1(y_1,\ldots,y_k,\ldots,y_n) = 0,$$

or,

$$\mathcal{G}_2(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_2(y_1,\ldots,y_k,\ldots,y_n) - \mathcal{G}_2(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_1(y_1,\ldots,y_k,\ldots,y_n) = 0,$$

for each $\mu_1,...,\mu_n, y_1,...,y_n \in R$, then one of the following holds

1. *$\mathcal{G}_1 = 0$ or $\mathcal{G}_2$ acts as a left α-multiplier.*
2. *$\mathcal{G}_2 = 0$ or $\mathcal{G}_1$ acts as a left α-multiplier.*

Proof: By our hypothesis, we have for each $\mu_1,...,\mu_n, y_1,...,y_n \in R$,

$$\mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_2(y_1,\ldots,y_k,\ldots,y_n) - \mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_1(y_1,\ldots,y_k,\ldots,y_n) = 0. \tag{2.11}$$

Put $y_k z$ for y_k in above expression to obtain

$$\begin{aligned}&\mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_2(y_1,\ldots,y_k z,\ldots,y_n)\\ &-\mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_1(y_1,\ldots,y_k z,\ldots,y_n) = 0.\end{aligned}$$

Simplify the last equation to find

$$\begin{aligned}&\mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)[\mathfrak{D}_2(y_1,\ldots,z,\ldots,y_n)\alpha(y_k) + z^*\mathfrak{D}_2(y_1,\ldots,y_k,\ldots,y_n)]\\ &-\mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)[\mathfrak{D}_1(y_1,\ldots,z,\ldots,y_n)\alpha(y_k) + z^*\mathfrak{D}_1(y_1,\ldots,y_k,\ldots,y_n)] = 0\end{aligned}$$

for each $\mu_1,...,\mu_n, y_1,...,y_n \in R$. This yields that

$$\begin{aligned}&\mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)z^*\mathfrak{D}_2(y_1,\ldots,y_k,\ldots,y_n)\\ &-\mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)z^*\mathfrak{D}_1(y_1,\ldots,y_k,\ldots,y_n) = 0 \text{ for every } \mu_1,...,\mu_n, z, y_1,...,y_n \in R.\end{aligned} \tag{2.12}$$

Substitute $\mathfrak{D}_1(y'_1,\ldots,y'_k,\ldots,y'_n)z$ for z^* in (2.12) to obtain

$$\begin{aligned}&[\mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_1(y'_1,\ldots,y'_k,\ldots,y'_n)z\mathfrak{D}_2(y_1,\ldots,y_k,\ldots,y_n)\\ &-\mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_1(y'_1,\ldots,y'_k,\ldots,y'_n)z\mathfrak{D}_1(y_1,\ldots,y_k,\ldots,y_n)] = 0.\end{aligned} \tag{2.13}$$

After simplification and making use of (2.11), we find

$$[\mathcal{G}_1(\mu_1,\ldots,\mu_n)\mathfrak{D}_2(y'_1,\ldots,y'_k,\ldots,y'_n)]z\{\mathfrak{D}_2(y_1,\ldots,y_n)-\mathfrak{D}_1(y_1,\ldots,y_n)\}=0, \tag{2.14}$$

for every $\mu_1,...,\mu_n,y_1,...,y_n\in R$. Evaluate the above equation with the condition $\mathfrak{D}_1\neq\mathfrak{D}_2$, the following two cases arise:

Case 1: Using $*$-primeness of R, we obtain

$$\mathcal{G}_1(\mu_1,\ldots,\mu_k,\ldots,\mu_n)w\mathfrak{D}_2(y_1,\ldots,y_k,\ldots,y_n)=0.$$

Primeness argument again forces to conclude $\mathcal{G}_1=0$ or $\mathfrak{D}_2=0$. In case $\mathfrak{D}_2=0$, we observe that

$$\mathcal{G}_2(\mu_1,\ldots,\mu_k\mu'_k,\ldots,\mu_n)=\mathcal{G}_2(\mu_1,\ldots,\mu'_k,\ldots,\mu_n)\alpha(\mu_k),$$

for every $\mu_1,...,\mu_n\in R$. Which implies that $\mathcal{G}_2$ acts as an α-multiplier on R.

Case 2: Next, consider the condition
$\mathcal{G}_2(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_2(y_1,\ldots,y_k,\ldots,y_n)-\mathcal{G}_2(\mu_1,\ldots,\mu_k,\ldots,\mu_n)\mathfrak{D}_1(y_1,\ldots,y_k,\ldots,y_n)=0$, for each $\mu_1,...,\mu_n,y_1,...,y_n\in R$. Simplify on the parallel lines as in the first case to obtain

$$\mathcal{G}_2(\mu_1,\ldots,\mu_k,\ldots,\mu_n)z\mathfrak{D}_1(y_1,\ldots,y_n)=0 \text{ for each } \mu_1,...,\mu_n,y_1,...,y_n\in R.$$

Primeness condition forces us to conclude either $\mathcal{G}_2=0$ or $\mathfrak{D}_1=0$. For the part $\mathfrak{D}_1=0$, $\mathcal{G}_1$ acts as a left α-multiplier on R. □

Theorem 2.5 *Let R be a semi-prime ring with involution $*$.If $\mathcal{G}$ is a reverse generalized $(\alpha,*)$-n-derivation of R associated with a reverse $(\alpha,*)$-n-derivation $\mathfrak{D}$ such that*

$$\mathcal{G}(\mu_1,\ldots,\mu_k,\ldots,\mu_n)y_i=\mu_i\mathcal{G}(y_1,\ldots,y_k,\ldots,y_n),\quad \textit{for each } \mu_1,\ldots,\mu_n,y_1,\ldots,y_n\in R.$$

then $\mathcal{G}$ is a left α multiplier.

Proof: We are given that

$$\mathcal{G}(\mu_1,\ldots,\mu_k,\ldots,\mu_n)y_i=\mu_i\mathcal{G}(y_1,\ldots,y_k,\ldots,y_n),\quad \text{for each } \mu_1,\ldots,\mu_n,y_1,\ldots,y_n \text{ in } R.$$

Put y_kz in place of y_k to obtain

$$\mathcal{G}(\mu_1,\ldots,\mu_k,\ldots,\mu_n)y_i=\mu_i\mathcal{G}(y_1,\ldots,y_kz,\ldots,y_n).$$

$$\mathcal{G}(\mu_1,\ldots,\mu_k,\ldots,\mu_n)y_i=\mu_i(\mathcal{G}(y_1,\ldots,z,\ldots,y_n)\alpha(y_k)+z^*\mathfrak{D}(y_1,\ldots,y_k,\ldots,y_n)).$$

Replace y_k with $\alpha^{-1}(y_k)$

$$\mathcal{G}(\mu_1,\ldots,\mu_k,\ldots,\mu_n)y_i=\mu_i\mathcal{G}(y_1,\ldots,z,\ldots,y_n)y_k+\mu_iz^*\mathfrak{D}(y_1,\ldots,y_k,\ldots,y_n)), \text{ for each } \mu_1,\ldots,\mu_n,y_1,\ldots,y_n,z \text{ in } R. \tag{2.15}$$

$$\mathcal{G}(\mu_1,\ldots,\mu_k,\ldots,\mu_n)y_i-\mu_i\mathcal{G}(y_1,\ldots,z,\ldots,y_n)y_k=\mu_iz^*\mathfrak{D}(y_1,\ldots,y_k,\ldots,y_n)), \text{ for each } \mu_1,\ldots,\mu_n,y_1,\ldots,y_n,z \text{ in } R.$$

Substitute y_iy_k for y_i

$$\mathcal{G}(\mu_1,\ldots,\mu_k,\ldots,\mu_n)y_iy_k-\mu_i\mathcal{G}(y_1,\ldots,z,\ldots,y_n)y_k=\mu_iz^*\mathfrak{D}(y_1,\ldots,y_k,\ldots,y_n)).$$

$$(\mathcal{G}(\mu_1,\ldots,\mu_k,\ldots,\mu_n)y_i-\mu_i\mathcal{G}(y_1,\ldots,z,\ldots,y_n))y_k=\mu_iz^*\mathfrak{D}(y_1,\ldots,y_k,\ldots,y_n), \text{ for each } \mu_1,\ldots,\mu_n,y_1,\ldots,y_n,z \text{ in } R.$$

Then

$$\mu_i z^* \mathfrak{D}(y_1, \ldots, y_k, \ldots, y_n) = 0.$$
$$\mu_i z \mathfrak{D}(y_1, \ldots, y_k, \ldots, y_n) = 0. \tag{2.16}$$

Multiplying (2.16) by $D(y_1, \ldots, y_k, \ldots, y_n)$ from the left and μ_i from the right

$$\mathfrak{D}(y_1, \ldots, y_k, \ldots, y_n)\mu_i z \mathfrak{D}(y_1, \ldots, y_k, \ldots, y_n)\mu_i = 0.$$

Making use of the fact that R is semiprime to conclude

$$\mathfrak{D}(y_1, \ldots, y_k, \ldots, y_n)\mu_i = 0 \quad \text{for each } y_1, \ldots, y_n \text{ in } R..$$

Multiply above equation by $\mathfrak{D}(y_1, \ldots, y_k, \ldots, y_n)$ from right and semiprimeness of R yields that

$$\mathfrak{D}(y_1, \ldots, y_k, \ldots, y_n) = 0 \quad \text{for each } y_1, \ldots, y_n \text{ in } R.$$

By definition, $\mathcal{G}$ is a left α-multiplier.

□

3. Conclusion

In this study, a new class of maps called reverse generalized $(\alpha, *)$-n-derivations is thoroughly investigated, with a focus on prime (semiprime) rings. Introducing these concepts and seeing how these maps behave are the main goals. We discovered a number of connections between reverse generalized $(\alpha, *)$-n-derivations and α-multipliers throughout this extensive investigation. These results provide important new information about this field of study. We advance our knowledge of how these mappings interact with the underlying algebraic structures by exploring the characteristics of reverse generalized $(\alpha, *)$-n-derivations, the role of α, $*$, and multipliers. Our findings lay the groundwork for more future research into how these maps behave in various other settings.

Conflict of interest

All authors declare that they have no conflicts of interest in this document.

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