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Higher Order Orthogonal Spline Collocation Method for Periodic Boundary Value Problems

Nitisha Pandey* and Reena Jain

ABSTRACT: This work thoroughly investigates various high-accuracy collocation methods for solving periodic boundary value problems (PBVPs), highlighting their efficiency and rapid convergence. The study demonstrates that collocation techniques can achieve high-order accuracy while reducing the computational resources required, making them a powerful alternative to traditional methods. Numerical experiments are conducted to validate the effectiveness of our proposed approach and confirmed sixth-order accuracy in both, the solution and its derivative. This finding is supported by error analysis and convergence rates for periodic boundary conditions in second-order differential equations. During the comparison of our method to the finite difference method, we found that our approach is superior. Specifically, increasing the number of grid points significantly reduces the error while maintaining a consistent order of convergence. Overall, the results underscore the effectiveness of collocation methods in addressing PBVPs for differential equations and complex boundary conditions.

Key Words: Linear ordinary differential equation, orthogonal spline collocation, periodic boundary conditions, superconvergence.

Contents

1. Introduction

Periodic boundary value problems arise frequently in scientific and engineering applications, especially in modeling multiple periodic events like wave propagation and multiple oscillatory systems. PBVPs are an important class of problems that are dealt with in the study of differential equations. Their popularity owes almost exclusively to their applicability in the modeling of phenomena exhibiting intrinsic periodicity. Interestingly, scientists Max Born and Theodore von Karman greatly developed the notion of periodic boundary conditions in the early part of the 20th century. They proposed what has come to be called the Born–von Karman boundary condition, which prescribes that a wave function must be periodic on a given Bravais lattice. This boundary condition has played a crucial role in solid-state physics, especially in the theory of ideal crystals.

The objective of this study is to investigate the higher-order convergence of periodic boundary value problems. Earlier research has played a significant role in developing the theory surrounding these problems for differential equations and their systems. Recent researches on PBVPs have expanded into various complex systems. For instance, Benner et al. [1] investigated weakly nonlinear PBVPs in ordinary differential equation systems with switching behaviors under nonlinear perturbations. They proposed an iterative algorithm to find solutions in critical cases and successfully applied their methodology to a mathematical model of nonisothermal chemical reactions.

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Wang et al. [2] studied Caputo-type fractional semilinear nonautonomous differential equations with periodic boundary value problems involving non-instantaneous impulses. Using semigroup theory in combination with the measure of non-compactness and fixed-point theorems, they proved the existence of PC-mild solutions. They also presented an example for demonstration.

The study of PBVPs has also been extended to fractional differential equations. Xue et al. [3] examined the existence of solutions for fractional PBVPs that involve the p(t)-Laplacian operator. They established a continuation theorem and included illustrative examples to support their results. The study explores the existence of positive solutions for systems governed by second-order, two-point PBVPs [4]. The analysis has also addressed the existence and multiplicity of positive solutions for systems of PBVPs of second-order, three-point [5]. Liu et. al. [6], have presented sufficient conditions for nontrivial periodic solutions to second-order, two-point periodic boundary value problems. Additionally,Yao [7] demonstrated the existence and multiplicity of positive solutions, which were further investigated for second-order two-point PBVPs. Solvability and an iterative scheme for systems of first-order PBVPs have been explored by Smadi et. al. [8] through the Reproducing Kernel Hilbert Space Method (RKHSM). Recently, Arqub [9] introduced analytical-numerical solutions for systems of second-order two-point singular PBVPs using the RKHSM.

Since the invention of digital computing, collocation techniques have been essential to the numerical solution of differential equations. The basic principle of collocation is to fit a function that satisfies the differential equation at specific discrete points, referred to as collocation points, to approximate the solution. High convergence and low errors can be obtained by applying a collocation point. When working with complicated boundary conditions, smoothness is essential, and this method successfully manages it. Additionally, increased precision reduces cumulative errors in time-dependent situations.

There are several approaches to solving differential equations with PBVPs, but collocation methods [10] have proven to be valuable tools. They provide high-order accuracy using fewer grid points and enjoy super convergence [11], which enhances solutions without additional computational expense. These methods ensure the accuracy of both the solution and its derivatives. This approach yields a smooth and highly precise solution while minimizing errors and remaining sensitive to stability conditions. This results in a balanced and consistent outcome. In this analysis, we will compare the finite difference method with our approach to determine the conditions under which our method performs optimally.

Table 1: Summary of Collocation Methods

Ref.	**ODE**	**PBVP**	**Collocation Types**	**Superconvergence**
[12]	✓	×	Chebyshev	×
[13]	✓	✓	Uniform	×
[8]	✓	✓	RKHS	×
[9]	×	✓	Kernel	✓
[14]	✓	×	Chebyshev Gauss Lobatto	×
[10]	×	×	Spline collocation	✓
[2]	×	✓	Analytic	×
[1]	✓	✓	Iterative	×
[3]	×	✓	Analytic	×
This Paper	✓	✓	Gauss quadrature points	✓

2. Collocation Method

Let n order linear differential equation with periodic boundary conditions. And r be the order of the approximate solution

$$b_n(x)\frac{d^n u}{dx^n} + b_{n-1}(x)\frac{d^{n-1}u}{dx^{n-1}} + \cdots + b_1(x)\frac{du}{dx} + b_0(x)u = g(x), \tag{2.1}$$

where $b_n(x), b_{n-1}(x), \ldots, b_0(x)$ are given functions, and $g(x)$ is the non-homogeneous term.
With interval

$$D : 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = N,$$

let's denote a partition of the interval $[0, N]$ such that $x \in D$, and set,

$$I_i = [x_i, x_{i+1}], \quad h_i = x_{i+1} - x_i, \quad i = 0, 1, \ldots, N-1.$$

in $[0, N]$, the two-point Gauss quadrature points

$$\xi_{2i-1} = x_i + \frac{1}{2}\left(1 - \sqrt{\frac{1}{3}}\right) h_i,$$

$$\xi_{2i} = x_i + \frac{1}{2}\left(1 + \sqrt{\frac{1}{3}}\right) h_i.$$

where, i = 0,1, 2, . . . , N - 1.
The solution of equation (2.1) satisfies the periodic boundary conditions

$$u(x_0) = u(x_N), \quad u'(x_0) = u'(x_N).$$

On each subinterval I_i, $i = 0, 1, \ldots, N-1$, We express the collocation point using an approximate solution in the following form

$$U_h(x) = y_{i1} + (x - x_i)y_{i2} + (x - x_i)^2 z_{i1} + (x - x_i)^3 z_{i2}. \tag{2.2}$$

We take $r = 3$, so the superconvergence of order $2r - 2 = 4$ is observed, as well as in its derivative.
The equation involving the four unknowns y_{i1}, y_{i2}, z_{i1}, and z_{i2}.
By differentiating (2.2) with respect to x, we find the following result,

$$U_h'(x) = y_{i2} + 2(x - x_i)z_{i1} + 3(x - x_i)^2 z_{i2}. \tag{2.3}$$

On differentiating (2.3) a second time, it follows that,

$$U_h''(x) = 2z_{i1} + 6(x - x_i)z_{i2}. \tag{2.4}$$

Using (2.2),(2.3), and (2.4) in (2.1), the collocation points on the interval I_i at $x = \xi_{2i-1}$ and $x = \xi_{2i}$ are for $n = 2$,

$$\begin{aligned} &b_2(x)\Big[2z_{i1} + 6(\xi_{2i-1} - x_i)z_{i2}\Big] + b_1(x)\Big[(\xi_{2i-1} - x_i) + 2(\xi_{2i-1} - x_i)z_{i1} + 3(\xi_{2i-1} - x_i)^2 z_{i2}\Big] \\ &+ b_0(x)\Big[y_{i1} + (\xi_{2i-1} - x_i)y_{i2} + (\xi_{2i-1} - x_i)^2 z_{i1} + (\xi_{2i-1} - x_i)^3 z_{i2}\Big] = g(\xi_{2i-1} - x_i), \end{aligned} \tag{2.5}$$

$$\begin{aligned} &b_2(x)\Big[2z_{i1} + 6(\xi_{2i} - x_i)z_{i2}\Big] + b_1(x)\Big[(\xi_{2i} - x_i) + 2(\xi_{2i} - x_i)z_{i1} + 3(\xi_{2i} - x_i)^2 z_{i2}\Big] \\ &+ b_0(x)\Big[y_{i1} + (\xi_{2i} - x_i)y_{i2} + (\xi_{2i} - x_i)^2 z_{i1} + (\xi_{2i} - x_i)^3 z_{i2}\Big] = g(\xi_{2i} - x_i), \end{aligned} \tag{2.6}$$

We can express this linear equation in matrix form. There are $2N$ collocation points and two boundary conditions; therefore, we derive a linear system of order $4N+2$ that is almost block diagonal (ABD).

$$\begin{bmatrix} D_a & & & & & & & \\ A_1 & B_1 & & & & & & \\ C_1 & D_1 & I_2 & & & & & \\ & & \ddots & & & & & \\ & & A_i & B_i & & & & \\ & & C_i & D_i & I_2 & & & \\ & & & & \ddots & \ddots & & \\ & & & & & A_N & B_N & \\ & & & & & C_N & D_N & I_2 \\ & & & & & & & D_b \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \\ y_1 \\ \vdots \\ y_{N-2} \\ z_{N-2} \\ \vdots \\ y_{N-1} \\ z_{N-1} \\ y_N \end{bmatrix} = \begin{bmatrix} g_0 \\ f_1 \\ 0 \\ \vdots \\ g_{n-1} \\ f_{n-1} \\ \vdots \\ f_n \\ 0 \\ g_1 \end{bmatrix} \tag{2.7}$$

where $D_a = (u(x_0), ..., u(x_N), .)$, $D_b = (., u'(x_0), ..., u'(x_N))$ boundary conditions respectively, and the matrices A_i, B_i, C_i,D_i and I_2 are given by:

$$A_i = \begin{bmatrix} b_0 & b_0(\xi_{2i-1} - x_i) + b_1(x) \\ b_0 & b_0(\xi_{2i} - x_i) + b_1(x) \end{bmatrix},$$

$$B_i = \begin{bmatrix} 2b_1(x)(\xi_{2i-1} - x_i) + b_0(x)(\xi_{2i-1} - x_i)^2 & b_0(x)(\xi_{2i-1} - x_i)^3 + 3b_1(x)(\xi_{2i-1} - x_i)^2 \\ 2b_1(x)(\xi_{2i} - x_i) + b_0(x)(\xi_{2i} - x_i)^2 & b_0(x)(\xi_{2i} - x_i)^3 + 3b_1(x)(\xi_{2i} - x_i)^2 \end{bmatrix},$$

$$C_i = \begin{bmatrix} -1 & -h \\ 0 & -1 \end{bmatrix}, D_i = \begin{bmatrix} -h^2 & -h^3 \\ -2h & -3h^2 \end{bmatrix}, I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now, extended this for $r = 4$, and for three-point Gauss quadrature points

The three-point Gauss quadrature points

$$\xi_{3i} = x_i + \frac{1}{2}\left(1 - \sqrt{\frac{3}{5}}\right) h_i,$$

$$\xi_{3i+1} = x_i + \frac{1}{2} h_i,$$

$$\xi_{3i+2} = x_i + \frac{1}{2}\left(1 + \sqrt{\frac{3}{5}}\right) h_i.$$

where, i = 0,1, 2, . . . , N - 1.
On each subinterval I_i, $i = 0, 1, \ldots, N-1$, We express the collocation point using an approximate solution in the following form

$$U_h(x) = y_{i1} + (x - x_i)y_{i2} + (x - x_i)^2 y_{i3} + (x - x_i)^3 z_{i1} + (x - x_i)^4 z_{i2}. \tag{2.8}$$

We take $r = 4$, so the superconvergence of order $2r - 2 = 6$ is observed, as well as in its derivative. The equation involving the five unknowns y_{i1}, y_{i2},y_{i3}, z_{i1}, and z_{i2}.
By differentiating (2.8) with respect to x, we find the following result,

$$U_h'(x) = y_{i2} + 2(x - x_i)y_{i3} + 3(x - x_i)^2 z_{i1} + 4(x - x_i)^3 z_{i2}. \tag{2.9}$$

On differentiating (2.9) a second time, it follows that,

$$U_h''(x) = 2y_{i3} + 6(x - x_i)z_{i1} + 12(x - x_i)^2 z_{i2}. \tag{2.10}$$

Using (2.8),(2.9), and (2.10) in (2.1) for n=2, the collocation points on the interval I_i at $x = \xi_{3i}$, $x = \xi_{3i+1}$ and $x = \xi_{3i+2}$ are

$$\begin{aligned} &b_2(x)\Big[2y_{i3} + 6(\xi_{3i} - x_i)z_{i1} + 12(\xi_{3i} - x_i)^2 z_{i2}\Big] \\ &+ b_1(x)\Big[y_{i2} + 2(\xi_{3i} - x_i)y_{i3} + 3(\xi_{3i} - x_i)^2 z_{i1} + 4(\xi_{3i} - x_i)^3 z_{i2}\Big] \\ &+ b_0(x)\Big[y_{i1} + (\xi_{3i} - x_i)y_{i2} + (\xi_{3i} - x_i)^2 y_{i3} + (\xi_{3i} - x_i)^3 z_{i1} + (\xi_{3i} - x_i)^4 z_{i2}\Big] \\ &= g(\xi_{3i} - x_i), \end{aligned} \tag{2.11}$$

$$\begin{aligned} &b_2(x)\Big[2y_{i3} + 6(\xi_{3i+1} - x_i)z_{i1} + 12(\xi_{3i+1} - x_i)^2 z_{i2}\Big] \\ &+ b_1(x)\Big[y_{i2} + 2(\xi_{3i+1} - x_i)y_{i3} + 3(\xi_{3i+1} - x_i)^2 z_{i1} + 4(\xi_{3i+1} - x_i)^3 z_{i2}\Big] \\ &+ b_0(x)\Big[y_{i1} + (\xi_{3i+1} - x_i)y_{i2} + (\xi_{3i+1} - x_i)^2 y_{i3} + (\xi_{3i+1} - x_i)^3 z_{i1} + (\xi_{3i+1} - x_i)^4 z_{i2}\Big] \\ &= g(\xi_{3i+1} - x_i), \end{aligned} \tag{2.12}$$

$$\begin{aligned} &b_2(x)\Big[2y_{i3} + 6(\xi_{3i+2} - x_i)z_{i1} + 12(\xi_{3i+2} - x_i)^2 z_{i2}\Big] \\ &+ b_1(x)\Big[y_{i2} + 2(\xi_{3i+2} - x_i)y_{i3} + 3(\xi_{3i+2} - x_i)^2 z_{i1} + 4(\xi_{3i+2} - x_i)^3 z_{i2}\Big] \\ &+ b_0(x)\Big[y_{i1} + (\xi_{3i+2} - x_i)y_{i2} + (\xi_{3i+2} - x_i)^2 y_{i3} + (\xi_{3i+2} - x_i)^3 z_{i1} + (\xi_{3i+2} - x_i)^4 z_{i2}\Big] \\ &= g(\xi_{3i+2} - x_i). \end{aligned} \tag{2.13}$$

There are $3N$ collocation points and two boundary conditions therefore, We derive a linear system of order $5N + 2$ that is almost block diagonal.

$$\begin{bmatrix} D_a & & & & & & \\ A_1 & B_1 & & & & & \\ C_1 & D_1 & I_2 & & & & \\ & & \ddots & & & & \\ & & A_i & B_i & & & \\ & & C_i & D_i & I_2 & & \\ & & & & \ddots & \ddots & \\ & & & & & A_N & B_N & \\ & & & & & C_N & D_N & I_2 \\ & & & & & & & D_b \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \\ y_1 \\ \vdots \\ \vdots \\ \vdots \\ y_{N-1} \\ z_{N-1} \\ y_N \end{bmatrix} = \begin{bmatrix} g_0 \\ f_0 \\ \vdots \\ \vdots \\ g_{n-1} \\ f_{n-1} \\ \vdots \\ g_n \\ f_n \\ g_1 \end{bmatrix} \tag{2.14}$$

where $D_a = (u(x_0), ..., u(x_N), .)$, $D_b = (., u'(x_0), ..., u'(x_N))$ boundary conditions respectively, and the matrices A_i, B_i, C_i and D_i are given by:

$$A_i = \begin{bmatrix} b_0 & b_0(\xi_{3i} - x_i) + b_1(x) & 2b_2(x) + 2b_1(x)(\xi_{3i} - x_i) + b_0(x)(\xi_{3i} - x_i)^2 \\ b_0 & b_0(\xi_{3i+1} - x_i) + b_1(x) & 2b_2(x) + 2b_1(x)(\xi_{3i+1} - x_i) + b_0(x)(\xi_{3i+1} - x_i)^2 \\ b_0 & b_0(\xi_{3i+2} - x_i) + b_1(x) & 2b_2(x) + 2b_1(x)(\xi_{3i+2} - x_i) + b_0(x)(\xi_{3i+2} - x_i)^2 \end{bmatrix},$$

$$B_i = \begin{bmatrix} b_2(x)6(\xi_{3i} - x_i) + b_1(x)3(\xi_{3i} - x_i)^2 + b_0(x)(\xi_{3i} - x_i)^3 & b_2(x)12(\xi_{3i} - x_i)^2 + b_1(x)4(\xi_{3i} - x_i)^3 + b_0(x)(\xi_{3i} - x_i)^4 \\ b_2(x)6(\xi_{3i+1} - x_i) + b_1(x)3(\xi_{3i+1} - x_i)^2 + b_0(x)(\xi_{3i+1} - x_i)^3 & b_2(x)12(\xi_{3i+1} - x_i)^2 + b_1(x)4(\xi_{3i+1} - x_i)^3 + b_0(x)(\xi_{3i+1} - x_i)^4 \\ b_2(x)6(\xi_{3i+2} - x_i) + b_1(x)3(\xi_{3i+2} - x_i)^2 + b_0(x)(\xi_{3i+2} - x_i)^3 & b_2(x)12(\xi_{3i+2} - x_i)^2 + b_1(x)4(\xi_{3i+2} - x_i)^3 + b_0(x)(\xi_{3i+2} - x_i)^4 \end{bmatrix}$$

$$C_i = \begin{bmatrix} -1 & -h & -h^2 \\ 0 & -1 & -2h \end{bmatrix}, D_i = \begin{bmatrix} -h^3 & -h^4 \\ -3h^2 & -4h^3 \end{bmatrix}.$$

In particular, a fourth-order accurate scheme is obtained by using cubic basis functions with two collocation points per element. Similarly, the accuracy level is raised to six by employing quartic basis functions. For the degree of the basis r, the method achieves an accuracy of the order $2r - 2$.

3. Numerical Illustrations

We present results from various numerical experiments that involve periodic boundary conditions. In each case, we calculate the experimental convergence rate of the error using

$$\text{Rate} = \frac{\log(E_{N_1}/E_{N_2})}{\log(N_2/N_1)},$$

where E_N denotes the norm of the error using the subintervals N .

In the numerical analysis of differential equations, particularly in collocation methods, superconvergence refers to a phenomenon where the numerical solution and its derivatives demonstrate a higher level of accuracy. Increasing the number of subintervals results in a more refined discretization of the domain, which enhances the accuracy of approximations of function values and derivatives. At these particular locations, both the numerical solution and its first derivative achieve an accuracy order of $2r-2$, as discussed in the previous section. This is significantly higher than the typical order of convergence.

We tested it on standard PBVPs for cubic and quartic approximate solutions, i.e., the Helmholtz equation, the Poisson equation, and the numerical approach. These results confirm that the high-accuracy collocation method is effective and efficient for solving periodic boundary value problems. Its ability to enforce periodicity precisely, achieve superconvergence, and maintain high accuracy makes it well-suited for applications in physics, engineering, and computational mathematics. Algorithm 1 is provided at the end of the paper as an appendix.

3.1. Example 1

Evaluate the effectiveness of the one-dimensional Helmholtz equation with periodic boundary conditions.

$$\frac{d^2y}{dx^2} - \alpha y = g(x). \tag{3.1}$$

where α is the real number. The domain is defined as

$$\mathrm{D} = \{x \mid x_0 \le x \le x_f\}, \tag{3.2}$$

x_0 and x_f are are known boundary points, and $g(x)$ is a defined source function.
We apply periodic boundary conditions to the boundaries of the domain $x = x_0$ and $x = x_f$.

$$y(x_0) = y(x_f), \quad y'(x_0) = y'(x_f).$$

The problem parameters are assigned the following values

$$x_0 = 0, \quad x_f = 4, \quad h = L = x_f - x_0 = 4.$$

The Helmholtz equation (3.1) has an analytic solution

$$y(x) = \sin\left[3\pi(x+0.05)\right]\cos\left[2\pi(x+0.05)\right] + 2.$$

This function is periodic with a period of $L = 4$ and satisfies the boundary conditions. After solving for $y(x)$, we get

$$\begin{aligned} g(x) = {} & (-13\pi^2 - \alpha)\sin\left(3\pi(x+0.05)\right)\cos\left(2\pi(x+0.05)\right) \\ & - 12\pi^2\cos\left(3\pi(x+0.05)\right)\sin\left(2\pi(x+0.05)\right) - 2\alpha. \end{aligned} \tag{3.3}$$

We express the approximate solution for $r = 4$ using three collocation points in the following form

$$U_h(x) = y_{i1} + (x - x_i)y_{i2} + (x - x_i)^2 y_{i3} + (x - x_i)^3 z_{i1} + (x - x_i)^4 z_{i2}. \tag{3.4}$$

Differentiating (3.4) two times, it follows that

$$U_h''(x) = 2y_{i3} + 6(x - x_i)z_{i1} + 12(x - x_i)^2 z_{i2}. \tag{3.5}$$

The collocation points on the interval I_i at $a = \xi_{3i}$, $b = \xi_{3i+1}$ and $c = \xi_{3i+2}$ are we have,

$$a = \frac{1}{2}h_i(1 - \sqrt{\frac{3}{5}}) \tag{3.6}$$

$$b = \frac{1}{2}h_i \tag{3.7}$$

$$c = \frac{1}{2}h_i(1 + \sqrt{\frac{3}{5}}) \tag{3.8}$$

Where $h(i) = \frac{x_f - x_0}{N_i}$, then using (3.4) and (3.5) in (3.1)

$$2y_{i3} + 6az_{i1} + 12a^2 z_{i2} - \alpha(y_{i1} + ay_{i2} + a^2 y_{i3} + a^3 z_{i1} + a^4 z_{i2}) = g(a), \tag{3.9}$$

$$2y_{i3} + 6bz_{i1} + 12b^2 z_{i2} - \alpha(y_{i1} + by_{i2} + b^2 y_{i3} + b^3 z_{i1} + b^4 z_{i2}) = g(b), \tag{3.10}$$

$$2y_{i3} + 6cz_{i1} + 12c^2 z_{i2} - \alpha(y_{i1} + cy_{i2} + c^2 y_{i3} + c^3 z_{i1} + c^4 z_{i2}) = g(c). \tag{3.11}$$

We can express the linear equation in the matrix for $i = 1$ where, the matrices A_1, B_1, C_1 and D_1 are given by:

$$A_1 = \begin{bmatrix} -\alpha & -\alpha a & 2 - \alpha a^2 \\ -\alpha & -\alpha b & 2 - \alpha b^2 \\ -\alpha & -\alpha c & 2 - \alpha c^2 \end{bmatrix}, B_1 \begin{bmatrix} 6a - \alpha a^3 & 12a^2 - \alpha a^4 \\ 6b - \alpha a^3 & 12b^2 - \alpha b^4 \\ 6c - \alpha a^3 & 12c^2 - \alpha c^4 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -1 & -h & -h^2 \\ 0 & -1 & -2h \end{bmatrix}, D_1 = \begin{bmatrix} -h^3 & -h^4 \\ -3h^2 & -4h^3 \end{bmatrix}.$$

Similarly, for $i = 1, 2 \ldots N - 1$ we get the almost block diagonal of $5N + 2$ order. After solving this by MatlabR2024a 2 reports a comprehensive analysis of errors and convergence rates for the Finite Difference Method (FDM) and the Collocation Method (CM) with subintervals (N). Clearly, the outcomes show that for different values of α, the CM achieves sixth-order accuracy both in the approximated solution and the derivative.

The CM method demonstrates a higher-order and more stable rate of convergence, as shown in table 2. Despite some initial oscillations, most values converge to approximately sixth order for both $\|u - u_h\|_\infty$ and $\|u' - u_h'\|_\infty$.

CM offers several advantages, including improved accuracy at higher orders, consistent convergence rates, and effective error reduction across a range of α values. It is particularly beneficial for problems that require precise numerical solutions, as its accuracy continuously improves with additional subintervals. In contrast, the finite difference method performs well but faces challenges on subintervals, making it less suitable for those scenarios.

Table 2: Error and Rate Comparisons for FDM and CM

α	METHOD	N	Error $\|y - y_h\|_\infty$	Rate	Error $\|y' - y'_h\|_\infty$	Rate
10	FDM	10	1.07×10^{1}	-	-	-
	CM		2.0994×10^{-1}	2.6537	1.5286×10^{0}	3.7364
	FDM	20	2.05×10^{0}	2.38	-	-
	CM		8.8854×10^{-3}	4.5624	7.7975×10^{-3}	7.615
	FDM	40	2.00×10^{0}	0.04	-	-
	CM		8.0491×10^{-5}	6.7865	1.4519×10^{-4}	5.747
	FDM	80	2.00×10^{0}	0	-	-
	CM		1.1109×10^{-6}	6.1791	2.2121×10^{-6}	6.0364
	FDM	160	2.00×10^{0}	0	-	-
	CM		1.6840×10^{-8}	6.0436	3.4292×10^{-8}	6.0114
-10	FDM	10	6.73×10^{0}	0	-	-
	CM		1.4872×10^{1}	4.8139	4.3433×10^{1}	4.8836
	FDM	20	2.10×10^{0}	1.68	-	-
	CM		1.4303×10^{-2}	10.022	5.1530×10^{-2}	9.7191
	FDM	40	2.10×10^{0}	0.06	-	-
	CM		9.2245×10^{-5}	7.2766	3.6471×10^{-4}	7.1425
	FDM	80	2.00×10^{0}	0	-	-
	CM		1.2231×10^{-6}	6.2368	4.9341×10^{-6}	6.2078
	FDM	160	2.00×10^{0}	0	-	-
	CM		1.8383×10^{-8}	6.056	7.4518×10^{-8}	6.049
16	FDM	10	7.45×10^{0}	-	-	-
	CM		2.1289×10^{-1}	2.3889	1.2604×10^{0}	3.6945
	FDM	20	2.05×10^{0}	1.86	-	-
	CM		7.6597×10^{-3}	4.7967	1.7495×10^{-3}	9.4928
	FDM	40	2.00×10^{0}	0.03	-	-
	CM		7.7233×10^{-5}	6.6319	7.8358×10^{-5}	4.4807
	FDM	80	2.00×10^{0}	0	-	-
	CM		1.0811×10^{-6}	6.1587	1.3540×10^{-6}	5.8548
	FDM	160	2.00×10^{0}	0	-	-
	CM		1.6439×10^{-8}	6.0392	2.1519×10^{-8}	5.9755
100	FDM	10	2.87×10^{0}	-	-	-
	CM		1.3081×10^{-1}	2.1746	5.4364×10^{-1}	3.1449
	FDM	20	2.02×10^{0}	0.51	-	-
	CM		1.4533×10^{-3}	6.492	6.4166×10^{-2}	3.0828
	FDM	40	2.00×10^{0}	0.01	-	-
	CM		3.8186×10^{-5}	5.2502	8.6369×10^{-4}	6.2152
	FDM	80	2.00×10^{0}	0	-	-
	CM		7.4974×10^{-7}	5.6705	1.1852×10^{-5}	6.1873
	FDM	160	2.00×10^{0}	0	-	-
	CM		1.2195×10^{-8}	5.942	1.7837×10^{-7}	6.0542

Table 3 presents the results obtained using the Collocation Method (CM) with both cubic and quartic basis functions for $\alpha = -10$.

Table 3: Error and Rate for cubic and Quartic

N	**Error** $\|y-y_h\|_\infty$	**Rate**	**Error** $\|y'-y'_h\|_\infty$	**Rate**
Cubic		$\alpha=-10$		
10	1.07×10^{2}	–	5.17×10^{1}	–
20	1.86×10^{2}	0.80	5.66×10^{2}	3.46
40	2.51×10^{-1}	9.53	1.13×10^{0}	8.97
80	5.97×10^{-3}	5.40	3.56×10^{-2}	4.09
160	3.52×10^{-4}	4.09	1.97×10^{-3}	4.07
Quartic				
10	1.4872×10^{1}	4.8139	4.3433×10^{1}	4.8836
20	1.4303×10^{-2}	10.022	5.1530×10^{-2}	9.7191
40	9.2245×10^{-5}	7.2766	3.6471×10^{-4}	7.1425
80	1.2231×10^{-6}	6.2368	4.9341×10^{-6}	6.2078
160	1.8383×10^{-8}	6.056	7.4518×10^{-8}	6.049

By examining the figures, the differences in error and rate between the methods are evident. Figure 1 shows the comparison of the error and rate for $\alpha=10$, while figure 2 displays the errors and rate for $\alpha=100$.

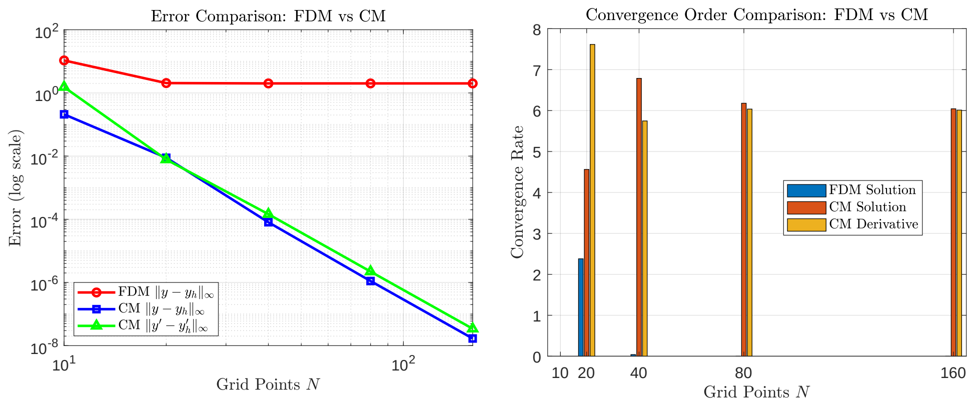

Figure 1: Comparison of Errors & Rate for $\alpha=10$

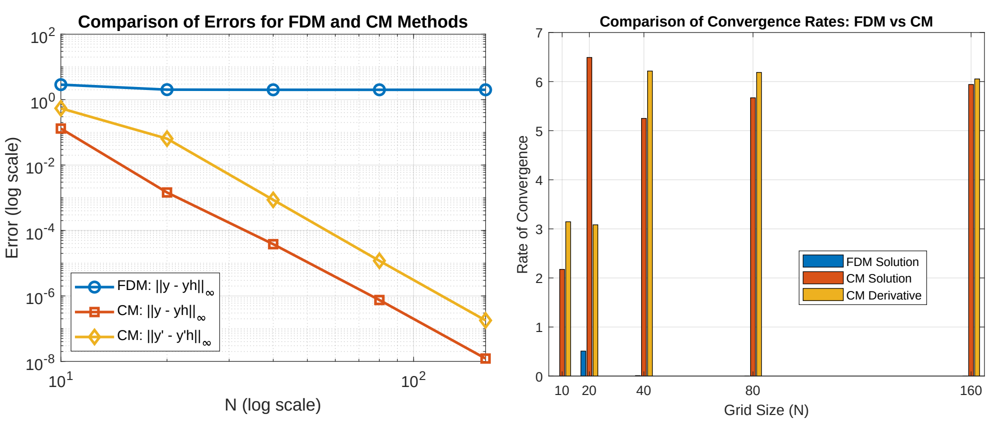

Figure 2: Comparison of Errors & Rate for $\alpha=100$

Now, figure 3 shows the convergent order and exact solution graph for $\alpha = 10$, and figure 4 displays the prime convergent order and numerical solution graph for $\alpha = 10$.

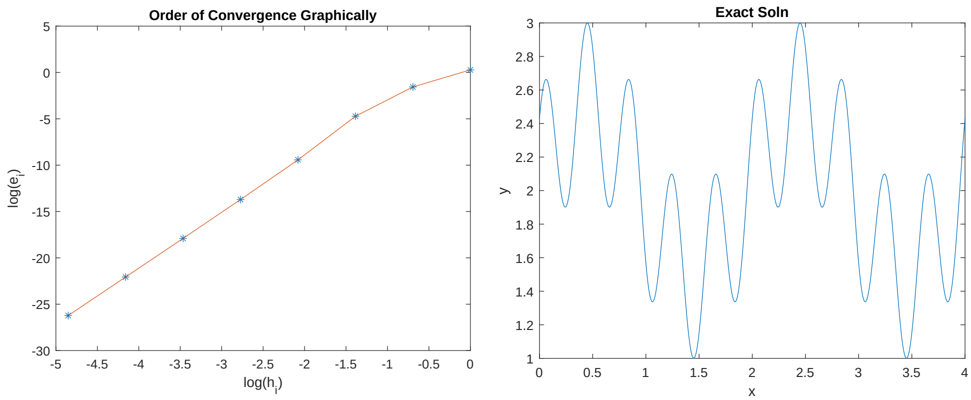

Figure 3: Convergent Order & Exact Solution Graph for $\alpha = 10$

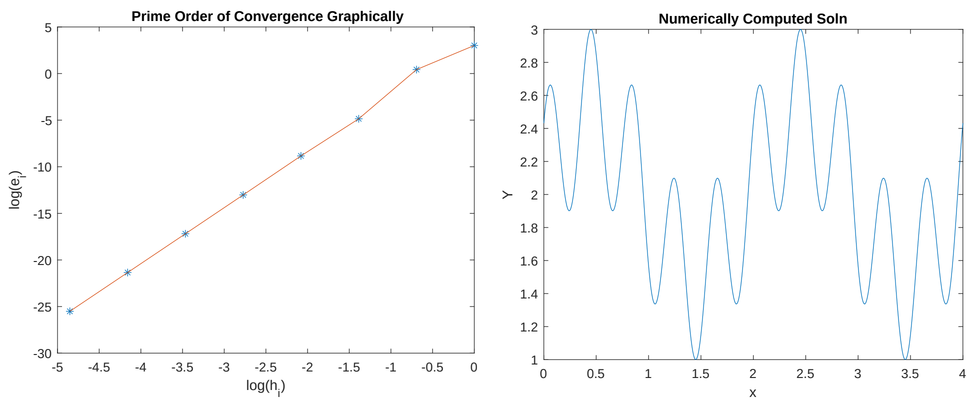

Figure 4: Prime Convergent Order & Numerical Solution Graph for $\alpha = 10$

3.2. Example 2

We analyze the Poisson equation in one dimension.

$$\frac{d^2y}{dx^2} = f(x), \quad x \in [0, 2\pi] \tag{3.12}$$

With periodic boundary conditions

$$y(0) = y(2\pi), \quad y'(0) = y'(2\pi). \tag{3.13}$$

Table 4: Periodic Solutions for Different Functions $f(x)$.

$f(x)$	Exact Solution $y(x)$
$\sin(x)$	$-\sin(x)$
$\sin(2x)$	$\frac{-1}{4}\sin(2x)$

We present the errors, convergence rates for both methods in table 5. This also indicates that both the approximate solution and its derivative achieve sixth-order accuracy using CM.

Table 5: Error and Rate Comparisons for for f(x) =sin(x)

METHOD	N	Error $\|y-y_h\|_\infty$	Rate	Error $\|y'-y'_h\|_\infty$	Rate
FDM	10	3.58×10^{-1}	-	-	-
CM		1.4790×10^{-7}	6.0832	3.0935×10^{-8}	6.0593
FDM	20	2.28×10^{-1}	0.65	-	-
CM		2.3956×10^{-9}	5.9481	4.7849×10^{-10}	6.0146
FDM	40	2.48×10^{-2}	3.2	-	-
CM		3.7304×10^{-11}	6.0049	7.4580×10^{-12}	6.0035
FDM	80	4.94×10^{0}	-7.64	-	-
CM		5.7976×10^{-13}	6.0077	1.2002×10^{-13}	5.9575

The data from the CM method shows higher accuracy, stability, and reliability compared to the FDM method. FDM is less reliable for solving the problem because it exhibits signs of numerical instability and implementation flaws. These issues suggest that this method is not dependable. In contrast, CM demonstrates a consistent and predictable convergence rate close to 6, indicating that the numerical technique used regularly achieves sixth-order accuracy. Therefore, our approach is clearly superior, as higher convergence rates signify a more precise and effective method.

The difference in errors and rate can be observed clearly in Figure 5.

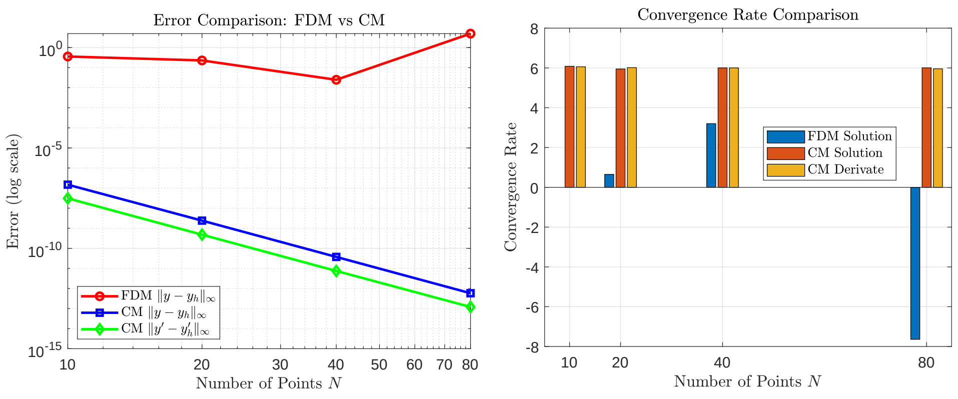

Figure 5: Comparison of Errors & Rate for f(x) = sin(x)

Table 6 and table 7 reports the performance of the Collocation Method using cubic and quartic basis functions, highlighting the order of solution accuracy.

Table 6: Error and Rate for cubic and Quartic for f(x) = sin(x)

N	Error $\|y-y_h\|_\infty$	Rate	Error $\|y'-y'_h\|_\infty$	Rate
Cubic				
10	4.83×10^{-3}	–	1.53×10^{-3}	–
20	2.73×10^{-4}	4.15	8.98×10^{-5}	4.09
40	1.67×10^{-5}	4.04	5.53×10^{-6}	4.02
80	1.03×10^{-6}	4.01	3.44×10^{-7}	4.01
160	6.45×10^{-8}	4.00	2.15×10^{-8}	4.00
Quartic				
10	1.4790×10^{-7}	6.0832	3.0935×10^{-8}	6.0593
20	2.3956×10^{-9}	5.9481	4.7849×10^{-10}	6.0146
40	3.7304×10^{-11}	6.0049	7.4580×10^{-12}	6.0035
80	5.7976×10^{-13}	6.0077	1.2002×10^{-13}	5.9575

Table 7: Error and Rate for cubic and Quartic basis for f(x) = sin(2x)

N	**Error** $\|y - y_h\|_\infty$	**Rate**	**Error** $\|y' - y'_h\|_\infty$	**Rate**
Cubic				
10	1.6045×10^{-2}	–	2.5857×10^{-7}	–
20	7.6369×10^{-4}	4.3930	1.2087×10^{-3}	4.256
40	4.4917×10^{-5}	4.0877	6.8266×10^{-5}	4.1462
80	2.7660×10^{-6}	4.0214	4.1626×10^{-6}	4.0356
160	1.7224×10^{-7}	4.0053	2.5857×10^{-7}	4.0088
Quartic				
10	1.4790×10^{-7}	6.0832	3.0935×10^{-8}	6.0593
20	2.3956×10^{-9}	5.9481	4.7849×10^{-10}	6.0146
40	3.7304×10^{-11}	6.0049	7.4580×10^{-12}	6.0035
80	5.7976×10^{-13}	6.0077	1.2002×10^{-13}	5.9575

Table 8 presents the errors and convergence order for both methods when applied to $f(x) = \sin(2x)$.

Table 8: Error and Rate Comparisons for f(x) =sin(2x)

METHOD	N	**Error** $\|y - y_h\|_\infty$	**Rate**	**Error** $\|y' - y'_h\|_\infty$	**Rate**
FDM	10	5.56×10^{-2}	-	-	-
CM		2.5069×10^{-6}	6.3578	1.0315×10^{-6}	6.2534
FDM	20	1.48×10^{-1}	-1.41	-	-
CM		3.6975×10^{-8}	6.0832	1.5467×10^{-8}	6.0593
FDM	40	8.80×10^{-2}	0.75	-	-
CM		5.9891×10^{-10}	5.9481	2.3924×10^{-10}	6.0146
FDM	80	2.97×10^{-2}	1.57	-	-
CM		9.3271×10^{-12}	6.0048	3.7287×10^{-12}	6.0037

The graph in Figure 6 shows the variation in errors and the rate that highlights their differences.

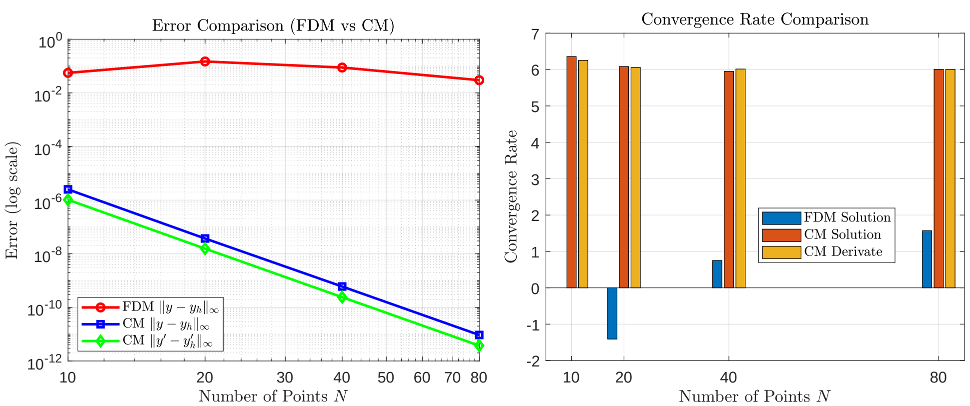

Figure 6: Comparison of Errors & Rate for f(x) = sin(2x)

Figure 7 and figure 9 illustrate the convergence order and exact solution graphs for $f(x) = \sin(x)$ and $f(x) = \sin(2x)$, respectively. Meanwhile, figure 8 and figure 10 present the prime convergence order and numerical solution graphs for $f(x) = \sin(x)$ and $f(x) = \sin(2x)$, respectively.

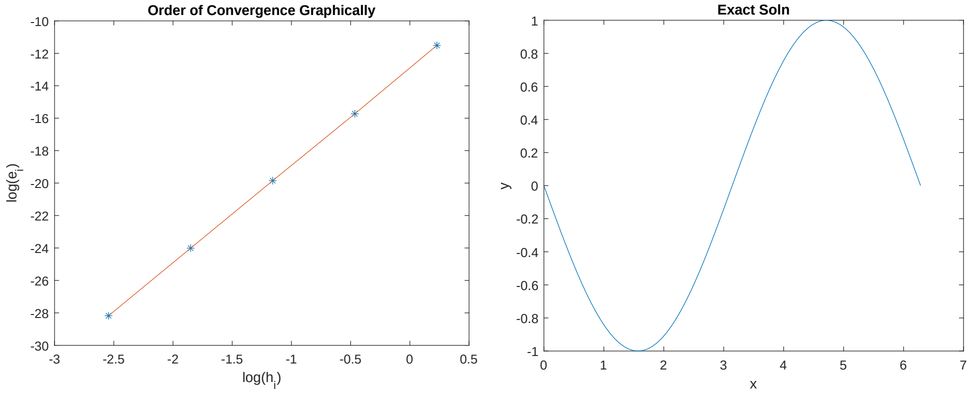

Figure 7: Convergent Order & Exact Solution Graph f(x) =sin(x)

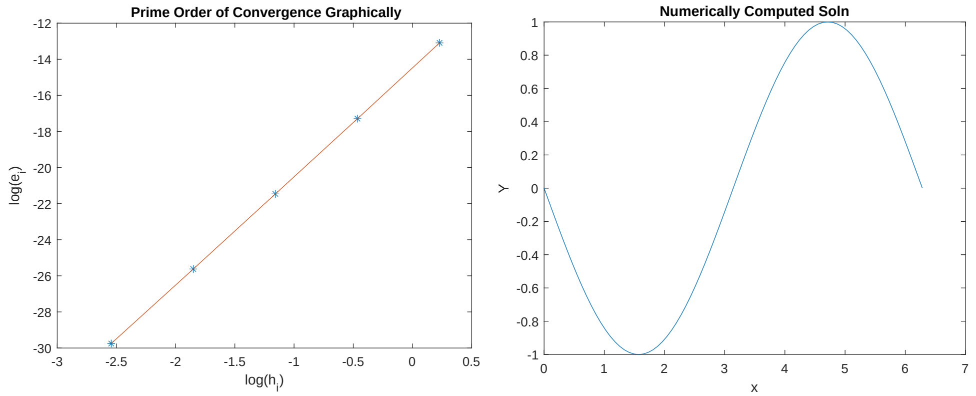

Figure 8: Prime Convergent Order & Numerical Solution Graph f(x)=sin(x)

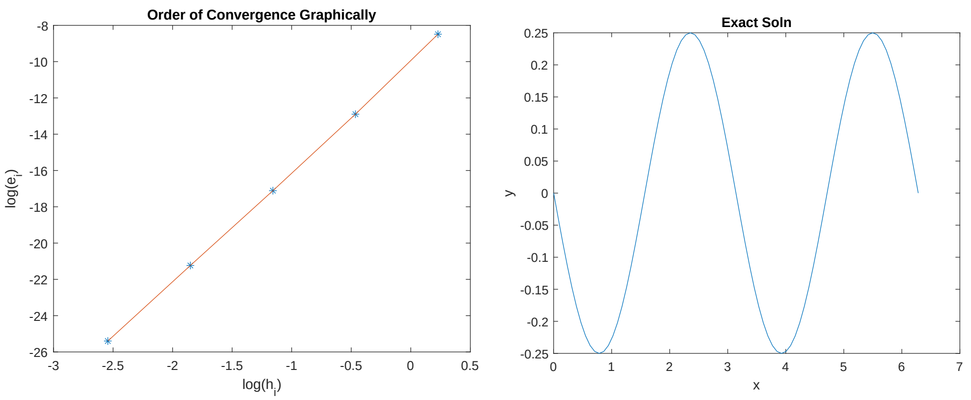

Figure 9: Convergent Order & Exact Solution Graph f(x)=sin(2x)

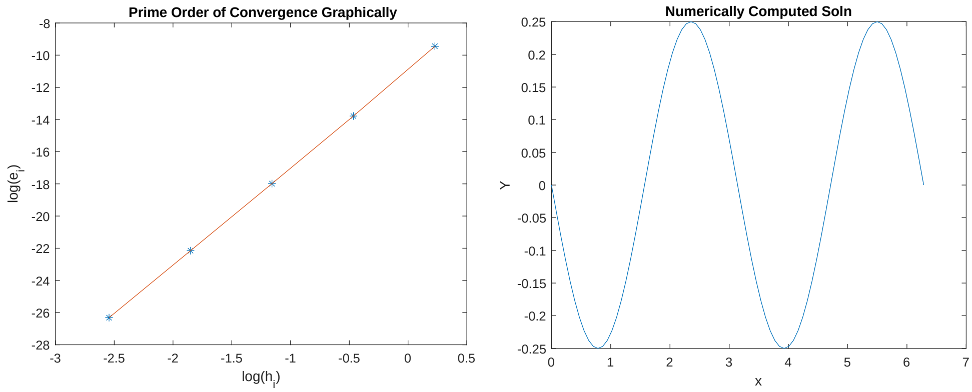

Figure 10: Prime Convergent Order & Numerical Solution Graph f(x) = sin(2x)

3.3. Example 3

Examine the following second-order linear non-homogeneous ODE

$$-0.4y'' + 2y' + 0.5y = f(x). \tag{3.14}$$

with boundary conditions.

$$y(0) = y(1) = 1, \quad y'(0) = y'(1) = 3. \tag{3.15}$$

The function $f(x)$ is given by

$$f(x) = 2.9e^{3x} + (1 - e^3)(-2.4x + 6x^2 + 0.5x^3). \tag{3.16}$$

The exact solution is

$$y(x) = e^{3x} + (1 - e^3)x^3. \tag{3.17}$$

Table 9 displays the errors, convergence rates for both methods.

Table 9: Error and Rate Comparison for FDM and CM Methods

METHOD	**N**	**Error** $\|y - y_h\|_\infty$	**Rate**	**Error** $\|y' - y'_h\|_\infty$	**Rate**
FDM	10	2.0378×10^{-2}	-	-	-
CM		1.6765×10^{-7}	6.0202	7.7206×10^{-7}	6.0202
FDM	20	4.7052×10^{-3}	2.1147	-	-
CM		2.6162×10^{-9}	6.0018	1.2054×10^{-8}	6.0018
FDM	40	1.1316×10^{-3}	2.0559	-	-
CM		4.0986×10^{-11}	5.9962	1.8877×10^{-10}	5.9962
FDM	80	2.7754×10^{-4}	2.0275	-	-
CM		6.5281×10^{-13}	5.9962	2.9439×10^{-12}	5.9723

FDM demonstrates a second-order convergence rate, leading to a predictable pattern of error reduction. In contrast, CM exhibits a sixth-order convergence rate, which means that the error decreases significantly more rapidly as the grid size increases. For larger values of N, the error levels in CM are noticeably lower, indicating that it produces much more precise solutions.
When evaluating effectiveness at larger grid sizes, FDM does reduce error but at a slower rate. In comparison, CM is both more accurate and efficient, as it minimizes error much more quickly. Due to its higher-order convergence, CM achieves faster error reduction, making it the superior choice. This improved accuracy makes CM ideal for high-precision numerical solutions, especially when working with

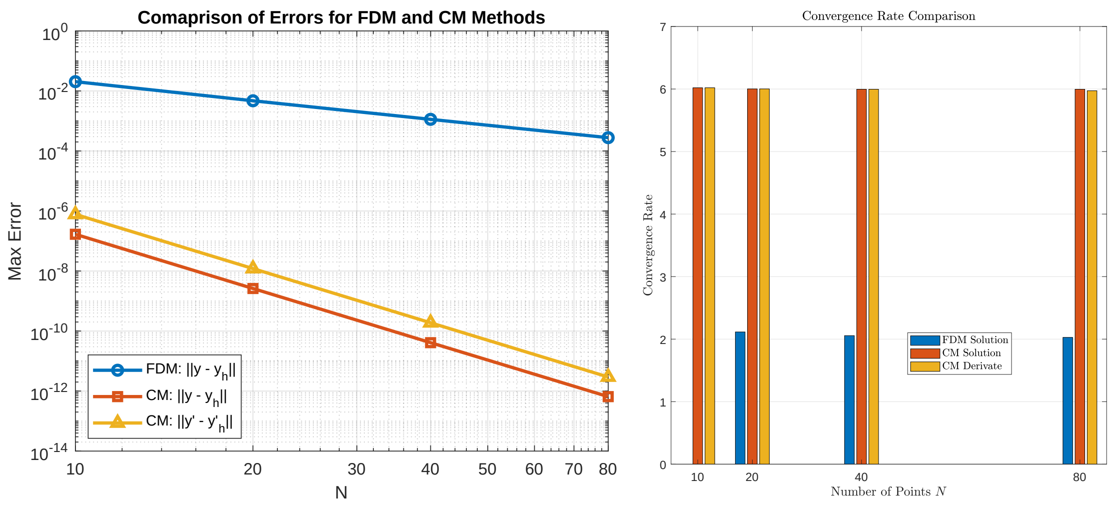

Figure 11: Comparison of Errors & Rate

larger grid sizes.
Figure 11 graphically represents the errors and rate for both methods.

Table 10 reports the performance of the Collocation Method using cubic and quartic basis functions, highlighting the effect of basis order on solution accuracy.

Table 10: Error and Rate for cubic and Quartic basis

N	**Error** $\|y - y_h\|_\infty$	**Rate**	**Error** $\|y' - y'_h\|_\infty$	**Rate**
Cubic				
10	2.45×10^{-3}	–	2.93×10^{-2}	–
20	1.38×10^{-4}	4.16	1.75×10^{-3}	4.07
40	8.71×10^{-6}	3.98	1.08×10^{-4}	4.02
80	5.46×10^{-7}	4.00	6.77×10^{-6}	4.00
160	3.42×10^{-8}	4.00	4.23×10^{-7}	4.00
Quartic				
10	1.6765×10^{-7}	6.0202	7.7206×10^{-7}	6.0202
20	2.6162×10^{-9}	6.0018	1.2054×10^{-8}	6.0018
40	4.0986×10^{-11}	5.9962	1.8877×10^{-10}	5.9962
80	6.5281×10^{-13}	5.9962	2.9439×10^{-12}	5.9723

figure 12 illustrates the convergence order alongside the exact solution graph. Furthermore, figure 13 shows the prime convergence order and the numerical solution graph.

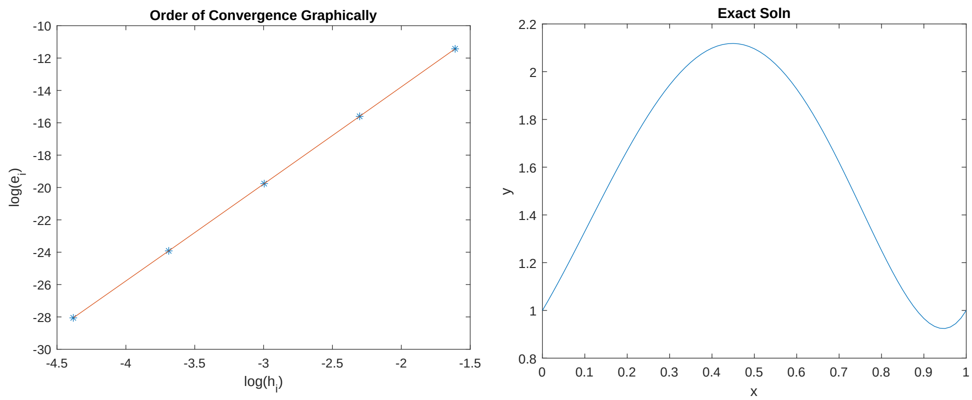

Figure 12: Convergent Order & Exact Solution Graph

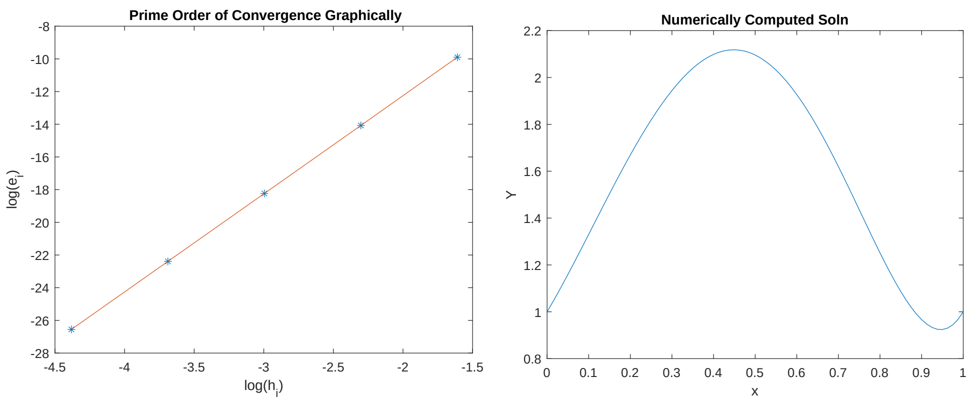

Figure 13: Prime Convergent Order & Numerical Solution Graph

4. Concluding Remarks

In this work, we examined high-accuracy collocation techniques to solve periodic boundary value problems. Our numerical experiments demonstrated that these techniques converge very quickly with high-order accuracy, highlighting their effectiveness in solving periodic differential equations. The examples show that with more node points, the collocation method significantly reduces the error while maintaining a stable convergence order. These results reinforce the notion that collocation methods are particularly effective for problems where periodicity is a crucial factor.

Furthermore, our numerical results indicate that our approach provides highly accurate approximations for both the numerical solutions and their derivatives, requiring fewer computational resources compared to conventional methods. Overall, the findings presented here collectively showcase the effectiveness of Collocation-based methods used to address periodic differential equations in engineering, physics, computational chemistry, and applied mathematics.

Our analysis demonstrates that the strategy is effective in managing discontinuities even when an interface is present at a node position. Future work will focus on a thorough convergence investigation and error analysis in suitable function spaces.

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Appendix

Algorithm 1 High-Accuracy Orthogonal Spline Collocation for PBVP

Step 1: Initialize Parameters
Define problem parameters: x_0, x_f
Set number of sub-intervals N for refinement
Step 2: Discretization
Define step size $h = \frac{(x_f - x_0)}{N}$
for $j = 1$ to $N + 1$ **do**
  Generate grid points: $x[j] = x_0 + (j - 1) \cdot h$
end for
Compute collocation points using Gaussian quadrature nodes
Step 3: Construct System of Equations
Initialize matrix A (system coefficients) and vector d (right-hand side)
Apply boundary conditions
for $j = 1$ to N **do**
  Populate matrix A using the differential equation coefficients
  Evaluate $f(x)$ and store in vector d
end for
Step 4: Solve the Linear System
Solve for the solution vector: $\text{sol} = A^{-1}d$
Step 5: Compute Numerical and Exact Solutions
for $j = 1$ to $N + 1$ **do**
  Numerical solution: $A_{\text{sol}}[j] = \text{sol}[5j - 4]$
  Compute exact solution: $exact[j] =$ (insert exact formula)
end for
for $j = 1$ to $N + 1$ **do**
  Compute numerical derivative: $A_{\text{sol1}}[j]$
  Compute exact derivative: $exact1[j]$
end for
Step 6: Error and Convergence Analysis
Compute maximum error: $error = \max |exact - A_{\text{sol}}|$
Compute derivative error: $error1 = \max |exact1 - A_{\text{sol1}}|$
for $j = 1$ to $p - 1$ **do**
  $order[j] = \frac{\log(error[j]/error[j+1])}{\log(h[j]/h[j+1])}$
  $order1[j] = \frac{\log(error1[j]/error1[j+1])}{\log(h[j]/h[j+1])}$
end for
Step 7: Visualization of graphs

Nitisha Pandey,
Mathematics Division,
School of Advanced Sciences and Languages,
VIT Bhopal University, Bhopal-Indore Highway,
Kothrikalan, Sehore, Madhya Pradesh, 466114, India.
E-mail address: nitisha.24phd10018@vitbhopal.ac.in, npandeyjkd@gmail.com

and

Reena Jain,
Mathematics Division,
School of Advanced Sciences and Languages,

VIT Bhopal University, Bhopal-Indore Highway,
Kothrikalan, Sehore, Madhya Pradesh, 466114, India
E-mail address: reena.jain@vitbhopal.ac.in