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A variational approach to a discrete fourth-order boundary value problem with three parameters

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ABSTRACT: In this article, we study discrete nonlinear fourth order boundary value problems with three parameters. We establish the existence of at least one solution under an asymptotical behaviour of the potential of the nonlinear term at zero by using variational methods. Some recent results are extended and improved. We provide sufficient conditions for the existence of positive solutions. An example is presented to demonstrate the applications of our main results.

Key Words: Discrete boundary value problem, fourth order boundary value problem, one solution, variational methods, critical point theory.

Contents

1. Introduction

Let $T > 2$ be a positive integer and $[2, T]_{\mathbb{Z}}$ be the discrete interval given by $\{2, 3, 4, \dots, T\}$. In this paper, we will examine a discrete nonlinear fourth order boundary value problems (BVP) with three parameters with intention of proving the existence of one solution. The problem to be studied can be viewed as a discrete version of the generalized beam equation. Consider the fourth BVP:

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda f(k, u(k)), & k \in [2, T]_{\mathbb{Z}}, \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0 \end{cases} \qquad (P_\lambda^f)$$

where Δ denotes the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$, $\Delta^{i+1}u(k) = \Delta(\Delta^i u(k))$, $\lambda > 0$, $f : [2, T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and α, β are real parameters and satisfy:

$$1 + (T-1)T\alpha_- + T(T-1)^3\beta_- > 0 \qquad (1.1)$$

where

$$\alpha_- = \min\{\alpha, 0\}$$

and

$$\beta_- = \min\{\beta, 0\}.$$

In recent years, a great deal of work has been done in the study of the existence of solutions for discrete boundary value problems (BVP), by which a number of physical, computer science, mechanical engineering, control systems, artificial or biological neural networks, phenomena are described. Recently, there is a trend to study difference equations by using fixed point theory, lower and upper solutions method, variational methods and critical point theory, Morse theory and the mountain-pass theorem. Many interesting results are obtained see for example, [6,8,9,18,20] and the references therein.

On the other hand, in the last few years, many researchers have used variational methods to study the existence of solutions for the discrete nonlinear fourth order boundary value problems. For related basic

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information, we refer the reader to [3,4,15,17,19,21,24,25] and the references therein. For example, Yang in [24] by using topological degree theory and fixed point index theory, provided sufficient conditions for the existence of sign-changing solutions, positive solutions, and negative solutions for the discrete fourth-order Neumann boundary value problem. In [15] by using a consequence of the local minimum theorem due Bonanno the existence of one solution under algebraic conditions on the nonlinear terms and two solutions the following discrete nonlinear fourth-order boundary value problem

$$\begin{cases} \Delta^4 u(t-2) + \delta\Delta^2 u(t-1) - \xi u(t) = \\ \lambda f(t,u(t)) + \mu g(t,u(t)) + h(u(t)), \quad t \in [a+1,b+1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0,\ u(b+2) = \Delta^2 u(b+1) = 0 \end{cases}$$

where a, b are two fixed integer numbers with $a < b$, $[a+1,b+1]_{\mathbb{Z}}$ is the discrete interval $\{a+1, a+2, \dots, b+1\}$, $f, g : [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions, $h : \mathbb{R} \to \mathbb{R}$ is a strictly monotone Lipschitz continuous function and δ, ξ, λ and μ are four real parameters, under algebraic conditions with the classical Ambrosetti-Rabinowitz (AR) condition on the nonlinear terms was ensured. Furthermore, by employing two critical point theorems, one due Averna and Bonanno, and another one due Bonanno the existence of two and three solutions for the above problem in the case $\mu = 0$ were discussed. Ousbika and El Allali in [21] based on the critical point theory, proved the existence of three solutions for the following discrete nonlinear fourth order boundary value problems with four parameters

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda f(k,u(k)) + \mu g(k,u(k)), \quad k \in [2,T]_{\mathbb{Z}}, \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0 \end{cases}$$

where $f, g : [2,T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions.

In [23] the authors have discussed the existence of three solutions and infinitely many solutions for discrete fourth-order boundary value problems with multiple parameters under the different suitable hypotheses, respectively applying the critical point theory. In [5] several sufficient conditions for the existence of at least three classical solutions for the problem (P_λ^f) using variational methods were presented.

We also refer the interested reader to the papers [11,12,13] in which fourth order discrete problems have been studied. For a through on the subject we refer to [2].

The objective of the present paper is to establish the existence of at least one solution for the problem (P_λ^f) and its parametric version by employing Ricceri's variational principle [22, Theorem 2.5]. Precisely, in Theorem 3.1 we establish the existence of least one solution for the problem (P_λ^f) requiring an algebraic condition on the nonlinear term f. Also in Corollary 3.1 a version of the result of Theorem 3.1 is successively discussed in which, for value of the parameter $\lambda = 1$ and requiring an additional asymptotical behaviour of the potential at zero if $f(k, 0) = 0$ for some $k \in [2,T]_{\mathbb{Z}}$, the existence of one non-trivial solution is established; see Remark 3.2. Moreover, we deduce the existence of solutions for small positive values of the parameter λ such that the corresponding solutions have smaller and smaller energies as the parameter goes to zero; see Theorem 3.2. As a consequence of Theorem 3.1, we obtain Theorem 3.4 for the autonomous case. We present Example 3.1 in which the hypotheses of Theorem 3.4 are fulfilled.

2. Preliminaries

In the present paper E denotes a finite dimensional real Banach space and $I_\lambda : E \to \mathbb{R}$ is a functional satisfying the following structure hypothesis:
$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for all $u \in E$ where $\Phi, \Psi : E \to \mathbb{R}$ are two functions of class C^1 on E with Φ coercive, i.e. $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$, and λ is a positive real parameter.
In this framework a finite dimensional variant of Ricceri's variational principle [22, Theorem 2.5] as given by Bonanno and Molica Bisci in [7].

Theorem 2.1 *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda = \Phi - \lambda\Psi$,*

$\lambda \in \mathbb{R}$, and for every $r > \inf_X \Phi$, let φ be the function defined as

$$\varphi(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

Then, for every $r > \inf_X \Phi$ and every $\lambda \in \left(0, \frac{1}{\varphi(r)}\right)$, the restriction of the functional I_λ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (precisely, a local minimum) of I_λ in X.

We refer the interested reader to the papers [1,10,14,16] in which Theorem 2.1 has been successfully employed to the existence of at least one non-trivial solution for boundary value problems.

We define the real vector space E

$$E = \left\{u : [0, T+2]_Z \to \mathbb{R},\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0\right\}$$

which is a $(T-2)$-dimentional Hilbert space, see [24] with the inner product

$$(u, v) = \sum_{k=2}^{T} u(k)v(k).$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=2}^{T} |u(k)|^2\right)^{\frac{1}{2}}.$$

Definition 2.1 *We say that $u \in E$ is a weak solution of problem (P_λ^f) if for any $v \in E$, we have*

$$\sum_{k=2}^{T} \Delta^4 u(k-2)v(k) - \alpha \sum_{k=2}^{T} \Delta^2 u(k-1)v(k) + \beta \sum_{k=2}^{T} u(k)v(k) = \lambda \sum_{k=2}^{T} f(k, u(k))v(k).$$

Lemma 2.1 *[21, Lemma 2.5] For any $u, v \in E$, we have*

$$\sum_{k=2}^{T} \Delta^4 u(k-2)v(k) = \sum_{k=2}^{T+1} \Delta^2 u(k-2)\Delta^2 v(k-2)$$

and

$$\sum_{k=2}^{T} \Delta u(k-1)\Delta v(k-1) = -\sum_{k=2}^{T} \Delta^2 u(k-1)v(k).$$

Put

$$F(k, t) = \int_0^t f(k, \xi)d\xi \quad \text{for all} \quad (k, t) \in [2, T]_{\mathbb{Z}} \times \mathbb{R}$$

and

$$\rho = \left(1 + (T-1)T\alpha_- + T(T-1)^3\beta_-\right) T^{-1}(T-1)^3.$$

We consider the functionals as follows:

$$\Phi(u) = \frac{1}{2}\left(\sum_{k=2}^{T+1} |\Delta^2 u(k-2)|^2 + \alpha \sum_{k=2}^{T} |\Delta u(k-1)|^2 + \beta \sum_{k=2}^{T} |u(k)|^2\right), \tag{2.1}$$

$$\Psi(u) = \sum_{k=1}^{T} F(k, u(k)) \tag{2.2}$$

and

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for every $u \in E$.

Lemma 2.2 *[21, Lemma 2.6] For any $u \in E$, we have*

$$\Phi(u) \geq 0$$

and

$$\Phi(u) \geq \frac{1}{2}\rho\|u\|^2.$$

Lemma 2.3 *[21, Lemma 2.7] If $u \in E$ is a critical point of the functional I_λ then u is a solution of BVP (P_λ^f).*

3. Main Result

We formulate our main result on the existence of one solution for the problem (P_λ^f) as follows:

Theorem 3.1 *For every λ small enough, i.e.*

$$\lambda \in \left(0, \frac{\rho}{2}\sup_{\gamma>0}\frac{\gamma^2}{\sum_{k=2}^{T}\max_{|t|\leq\gamma}F(k,t)}\right),$$

the problem (P_λ^f) admits at least one solution $u_\lambda \in E$.

Proof: Our goal is to apply Theorem 2.1 to the problem (P_λ^f). Take Φ and Ψ as given in (2.1) and (2.2), respectively. Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in E$ is

$$\Psi'(u)(v) = \sum_{k=2}^{T} f(k, u(k))v(k)$$

for every $v \in E$, as well as is sequentially weakly upper semicontinuous. By lemma 2.2, we prove that Φ is coercive. Moreover, Φ is continuously differentiable whose differential at the point $u \in E$ is

$$\Phi'(u)(v) = \sum_{k=2}^{T+1}\Delta^2u(k-2)\Delta^2v(k-2) + \alpha\sum_{k=2}^{T}\Delta u(k-1)\Delta v(k-1) + \beta\sum_{k=2}^{T}u(k)v(k)$$

for every $v \in E$. Furthermore, Φ is sequentially weakly lower semicontinuous. Therefore, we observe that the regularity assumptions on Φ and Ψ, as requested in Theorem 2.1 are verified. We now look on the existence of a critical point of the functional I in E. Let $\bar{\gamma} > 0$, choose

$$r_0 = \frac{\rho}{2}\bar{\gamma}^2.$$

For any $k \in [2, T]_{\mathbb{Z}}$, we have

$$|u(k)| \leq \|u\| \leq \sqrt{\frac{2\Phi(u)}{\rho}}.$$

From the definition of r_0, it follows that

$$\Phi^{-1}(-\infty, r_0) = \{u \in E : \ \Phi(u) < r_0\} \subseteq \{u \in E : \ |u| \leq \bar{\gamma}\}.$$

Therefore, we have that

$$\Psi(u) = \sum_{k=2}^{T} F(k, u(k)) \leq \sum_{k=2}^{T} \max_{|t|\leq\bar{\gamma}} F(k,t)$$

for every $u \in E$ such that $\Phi(u) < r_0$. Then

$$\sup_{\Phi(u)<r_0} \Psi(u) \leq \sum_{k=2}^{T} \max_{|t|\leq\bar{\gamma}} F(k,t).$$

Let us pick

$$0 < \lambda < \frac{\rho}{2} \sup_{\gamma>0} \frac{\gamma^2}{\sum_{k=2}^{T} \max_{|t|\leq\gamma} F(k,t)}.$$

Hence, there exists $\bar{\gamma} > 0$ such that

$$\lambda\frac{2}{\rho} < \frac{\bar{\gamma}^2}{\sum_{k=2}^{T} \max_{|t|\leq\bar{\gamma}} F(k,t)}.$$

One has

$$\varphi(r_0) \leq \frac{\sup_{v\in\Phi^{-1}(-\infty,r_0)} \Psi(v)}{r_0} \leq \frac{2}{\rho}\frac{\sum_{k=2}^{T} \max_{|t|\leq\bar{\gamma}} F(k,t)}{\bar{\gamma}^2} < \frac{1}{\lambda}.$$

Hence, since $\lambda \in \left(0, \frac{1}{\varphi(r_0)}\right)$, Theorem 2.1 ensures that the functional I_λ admits at least one critical point (local minima) $u_\lambda \in \Phi^{-1}(-\infty, r_0)$ and since the critical points of the functional I_λ are the solutions of the problem (P_λ^f) we have the conclusion. □

Now, we deduce the following straightforward consequence of Theorem 3.1.

Corollary 3.1 *Assume that*

$$\sup_{\gamma>0} \frac{\gamma^2}{\sum_{k=2}^{T} \max_{|t|\leq\gamma} F(k,t)} > \frac{2}{\rho}. \tag{D_F}$$

Then, the problem (P_λ^f) *in the case* $\lambda = 1$, *admits at least one solution in* E.

Proof: Our goal is to apply Theorem 2.1 to the problem (P_λ^f) in the case $\lambda = 1$. Take Φ and Ψ as given in the proof of Theorem 3.1. From the condition (D_F), there exists $\bar{\gamma} > 0$ such that

$$\frac{\bar{\gamma}^2}{\sum_{k=2}^{T} \max_{|t|\leq\bar{\gamma}} F(k,t)} > \frac{2}{\rho}. \tag{3.1}$$

Choose

$$r_0 = \frac{\rho}{2}\bar{\gamma}^2.$$

By a simple computation and from the definition of $\varphi(r_0)$, since $0 \in \Phi^{-1}(-\infty, r_0)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}\varphi(r_0) &= \inf_{u\in\Phi^{-1}(-\infty,r_0)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r_0)} \Psi(v)) - \Psi(u)}{r_0 - \Phi(u)} \leq \frac{\sup_{v\in\Phi^{-1}(-\infty,r_0)} \Psi(v)}{r_0} \\ &\leq \frac{2}{\rho} \frac{\sum_{k=2}^{T} \max_{|t|\leq\bar{\gamma}} F(k,t)}{\bar{\gamma}^2}.\end{aligned}$$

At this point, observe that

$$\varphi(r_0) \leq \frac{2}{\rho} \frac{\sum_{k=2}^{T} \max_{|t|\leq\bar{\gamma}} F(k,t)}{\bar{\gamma}^2}. \tag{3.2}$$

Consequently, by (3.1) and (3.2) one has $\varphi(r_0) < 1$. Hence, since $1 \in \left(0, \frac{1}{\varphi(r_0)}\right)$, applying Theorem 2.1 the functional I_λ admits at least one critical point (local minima) $\tilde{u} \in \Phi^{-1}(-\infty, r_0)$. The proof is complete. □

Now, we give consequences of our results.

Remark 3.1 *In Theorem 3.1 we looked for the critical points of the functional I_λ naturally associated with the problem (P_λ^f). We note that, in general, I_λ can be unbounded from the following in E. Indeed, for example, in the case when $f(\xi) = 1 + |\xi|^{\gamma-2}\xi$ for every $\xi \in \mathbb{R}$ with $\gamma > 2$, for any fixed $u \in E\backslash\{0\}$ and $\iota \in \mathbb{R}$, we obtain*

$$I_\lambda(\iota u) = \Phi(\iota u) - \lambda \sum_{k=2}^{T} F(k, \iota u(k)) \leq \iota^2\Phi(u) - \lambda\iota(T-1)\|u\| - \lambda\frac{\iota^\gamma}{\gamma}(T-1)\|u\|^\gamma \to -\infty$$

as $\iota \to +\infty$. Hence, we can not use direct minimization to find critical points of the functional I_λ.

Lemma 3.1 *If in Theorem 3.1 the function $f(k,\xi) \geq 0$ for every $k \in [2,T]_{\mathbb{Z}}$ and $\xi \in \mathbb{R}$, the condition (D_F) takes the following more simple and significative form*

$$\sup_{\gamma>0} \frac{\gamma^2}{\sum_{k=2}^{T} F(k,\gamma)} > \frac{2}{\rho}. \tag{D_F'}$$

Moreover, if the following assumption holds

$$\limsup_{\gamma\to+\infty} \frac{\gamma^2}{\sum_{k=2}^{T} F(k,\gamma)} > \frac{2}{\rho},$$

then the condition (D_F') is automatically verified.

Remark 3.2 *If in Theorem 3.1, $f(k,0) \neq 0$ for all $k \in [2,T]_{\mathbb{Z}}$, then the ensured solution is obviously non-trivial. On the other hand, the non-triviality of the solution can be achieved also in the case $f(k,0) = 0$ for some $k \in [2,T]_{\mathbb{Z}}$ requiring the extra condition at zero, that is there are discrete intervals $[2,T_1]_{\mathbb{Z}} \subseteq [2,T]_{\mathbb{Z}}$ and $[2,T_2]_{\mathbb{Z}} \subset [2,T_1]_{\mathbb{Z}}$ where $T > T_1, T_2 \geq 3$, such that*

$$\limsup_{\xi\to0^+} \frac{\sup_{k\in[2,T_2]_{\mathbb{Z}}} F(k,\xi)}{|\xi|^2} = +\infty \tag{3.3}$$

and

$$\liminf_{\xi\to 0^+}\frac{\sup_{k\in[2,T_1]_{\mathbb{Z}}}F(k,\xi)}{|\xi|^2}>-\infty. \tag{3.4}$$

Indeed, let $0<\bar\lambda<\lambda^*$ *where*

$$\lambda^*=\frac{\rho}{2}\sup_{\gamma>0}\frac{\gamma^2}{\sum_{k=2}^{T}\max_{|t|\le\gamma}F(k,t)}.$$

Then, there exists $\bar\gamma>0$ *such that*

$$\bar\lambda\frac{2}{\rho}<\frac{\bar\gamma^2}{\sum_{k=2}^{T}\max_{|t|\le\bar\gamma}F(k,t)}.$$

Let Φ *and* Ψ *be as given in* (2.1) *and* (2.2)*, respectively. Due to Theorem 2.1, for every* $\lambda\in(0,\bar\lambda)$ *there exists a critical point of* I_λ *such that* $u_\lambda\in\Phi^{-1}(-\infty,r_\lambda)$ *where* $r_\lambda=\frac{\rho}{2}\bar\gamma^2$*. In particular,* u_λ *is a global minimum of the restriction of* I_λ *to* $\Phi^{-1}(-\infty,r_\lambda)$*. We will prove that the function* u_λ *cannot be trivial. Let us show that*

$$\limsup_{\|u\|\to 0^+}\frac{\Psi(u)}{\Phi(u)}=+\infty. \tag{3.5}$$

Due to the assumptions (3.3) *and* (3.4)*, we can consider a sequence* $\{\xi_n\}\subset\mathbb{R}^+$ *converging to zero and two constants* ϵ,κ *(with* $\epsilon>0$*) such that*

$$\lim_{n\to+\infty}\frac{\sup_{k\in[2,T_2]_{\mathbb{Z}}}F(k,\xi_n)}{|\xi_n|^2}=+\infty$$

and

$$\sup_{k\in[2,T_1]_{\mathbb{Z}}}F(k,\xi)\ge\kappa|\xi|^2$$

for every $\xi\in[0,\epsilon]$*. We consider a discrete interval* $[2,T_3]_{\mathbb{Z}}\subset[2,T_2]_{\mathbb{Z}}$ *where* $T_3\ge 3$ *and a function* $v\in E$ *such that*

(k_1) $v(k)\in[0,1]$ *for every* $k\in[2,T]_{\mathbb{Z}}$,

(k_2) $v(k)=1$ *for every* $k\in[2,T_3]_{\mathbb{Z}}$,

(k_3) $v(k)=0$ *for every* $k\in[T_1+1,T]_{\mathbb{Z}}$.

Hence, fix $M>0$ *and consider a real positive number* η *with*

$$M<\frac{\eta\ (T_3-2)+\kappa\sum_{T_3+1}^{T_1}|v(k)|^2}{\Phi(v)}.$$

Then, there is $n_0\in\mathbb{N}$ *such that* $\xi_n<\epsilon$ *and*

$$\sup_{k\in[2,T_3]_{\mathbb{Z}}}F(k,\xi_n)\ge\eta|\xi_n|^2$$

for every $n > n_0$. Now, for every $n > n_0$, by considering the properties of the function v (that is $0 \le \xi_n v(k) < \epsilon$ for n large enough), we have

$$\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = \frac{\sum_{k=2}^{T_3} F(k,\xi_n) + \sum_{T_3+1}^{T_1} F(k,\xi_n v(k))}{\Phi(\xi_n v)}$$
$$> \frac{\eta\,(T_3-2) + \kappa \sum_{T_3+1}^{T_1} |v(k)|^2}{\Phi(v)} > M.$$

Since M could be taken arbitrarily large, it follows that

$$\lim_{n\to\infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty,$$

from which (3.5) clearly follows. Hence, there exists a sequence $\{w_n\} \subset E$ strongly converging to zero such that, for n large enough, $w_n \in \Phi^{-1}(-\infty, r_\lambda)$ and

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) < 0.$$

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r_\lambda)$, we obtain

$$I_\lambda(u_\lambda) < 0, \tag{3.6}$$

so that u_λ is not trivial.

Theorem 3.2 *From (3.6) we easily observe that the map*

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda) \tag{3.7}$$

is negative. Also, one has

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

Indeed, bearing in mind that Φ is coercive and for every $\lambda \in (0,\lambda^*)$ the solution $u_\lambda \in \Phi^{-1}(-\infty, r_\lambda)$, one has that there exists a positive constant L such that $\|u_\lambda\| \le L$ for every $\lambda \in (0,\lambda^*)$. After that, it is easy to see that there exists a positive constant N such that

$$\left| \sum_{k=2}^{T} f(k, u_\lambda(k)) u_\lambda(k) \right| \le N\|u_\lambda\| \le NL \tag{3.8}$$

for every $\lambda \in (0,\lambda^*)$. Since u_λ is a critical point of I_λ, we have $I'_\lambda(u_\lambda)(v) = 0$ for every $v \in E$ and every $\lambda \in (0,\lambda^*)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \sum_{k=2}^{T} f(k, u_\lambda(k)) u_\lambda(k) \tag{3.9}$$

for every $\lambda \in (0,\lambda^*)$. We have

$$0 \le \rho\|u_\lambda\|^2 \le \Phi'(u_\lambda)(u_\lambda),$$

from (3.9), we obtain

$$0 \le \rho\|u_\lambda\|^2 \le \lambda \sum_{k=2}^{T} f(k, u_\lambda(k)) u_\lambda(k) \tag{3.10}$$

for any $\lambda \in (0, \lambda^*)$. Letting $\lambda \to 0^+$, by (3.10) together with (3.8) we get

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.$$

Then, we have obviously the desired conclusion. Finally, we have to show that the map

$$\lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing in $(0, \lambda^*)$. For our goal we see that for any $u \in E$, one has

$$I_\lambda(u) = \lambda \left(\frac{\Phi(u)}{\lambda} - \Psi(u) \right). \tag{3.11}$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi(-\infty, r_\lambda)$ for $i = 1, 2$. Also, set

$$m_{\lambda_i} = \left(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) \right) = \inf_{v \in \Phi^{-1}(-\infty, r)} \left(\frac{\Phi(v)}{\lambda_i} - \Psi(v) \right)$$

for $i = 1, 2$.
Clearly, (3.7) together with (3.11) and the positivity of λ imply that

$$m_{\lambda_i} < 0 \quad \textit{for} \quad i = 1, 2. \tag{3.12}$$

Moreover

$$m_{\lambda_2} \le m_{\lambda_1}, \tag{3.13}$$

due to the fact that $0 < \lambda_1 < \lambda_2$. Then, by (3.11)-(3.13) and again by the fact that $0 < \lambda_1 < \lambda_2$, we get that

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \le \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^*)$. The arbitrariness of $\lambda < \lambda^*$ shows that $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$.

Theorem 3.3 *If f is non-negative then the solution ensured in Theorem 3.1 and Corollary 3.1 is non-negative.*

Proof: Let u_* be a non-trivial solution of the problem (P^f_λ), then u_* is non-negative. Arguing by a contradiction, assume that the discrete numbers $\mathcal{A} = \{k \in [2, T]_{\mathbb{Z}};\ u_*(k) < 0\}$. Put $\bar{v}(k) = \min\{u_*(k), 0\}$ for every $k \in [2, T]_{\mathbb{Z}}$. Clearly, $\bar{v} \in E$ and one has

$$\sum_{k=2}^{T+1} \Delta^2 u_*(k-2)\Delta^2 \bar{v}(k-2) + \alpha \sum_{k=2}^{T} \Delta u_*(k-1)\Delta \bar{v}(k-1) + \beta \sum_{k=2}^{T} u_*(k)\bar{v}(k)$$
$$-\lambda \sum_{k=2}^{T} f(k, u_*(k))\bar{v}(k) = 0$$

and by choosing $\bar{v} = u_*$ and since f is non-negative, we have

$$0 \le \rho \|u_*\|^2_{\mathcal{A}} \le \sum_{\mathcal{A}} |\Delta^2 u_*(k-2)|^2 + \alpha \sum_{\mathcal{A}} |\Delta u_*(k-1)|^2 + \beta \sum_{\mathcal{A}} |u_*(k)|^2$$
$$-\lambda \sum_{k=2}^{T} f(k, u_*(k))u_*(k) \le 0$$

where $\|u_*\|_{\mathcal{A}} = \left(\sum_{\mathcal{A}} |u_*(k)|^2 \right)^{\frac{1}{2}}$, i.e.,

$$\|u_*\|^2_{\mathcal{A}} \le 0$$

which contradicts with this fact that u_* is a non-trivial solution. Hence, u_* is positive. □

Remark 3.3 *We observe that Theorem 3.1 is a bifurcation result in the sense that the pair* $(0,0)$ *belongs to the closure of the set*

$$\left\{(u_\lambda,\lambda)\in E\times(0,+\infty): u_\lambda \text{ is a non-trivial solution of } (P_\lambda^f)\right\}$$

in $E\times\mathbb{R}$. *Indeed, by Theorem 3.2 we have that*

$$\|u_\lambda\|\to 0 \quad \text{as} \quad \lambda\to 0.$$

Hence, there exist two sequences $\{u_j\}$ *in* E *and* $\{\lambda_j\}$ *in* $\mathbb{R}^+$ *(here* $u_j=u_{\lambda_j}$*) such that*

$$\lambda_j\to 0^+ \quad \text{and} \quad \|u_j\|\to 0,$$

as $j\to+\infty$. *Moreover, we emphasis that due to the fact that the map*

$$(0,\lambda^*)\ni\lambda\mapsto I_\lambda(u_\lambda)$$

is strictly decreasing, for every $\lambda_1,\lambda_2\in(0,\lambda^*)$, *with* $\lambda_1\neq\lambda_2$, *the solutions* u_{λ_1} *and* u_{λ_2} *ensured by Theorem 3.1 are different.*

When f doesn't depend on k, we present the following consequence of Theorem 3.1 and results above.

Theorem 3.4 *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a non-negative continuous function and denote*

$$F(\xi)=\int_0^\xi f(t)dt \text{ for all } \xi\in\mathbb{R}.$$

Assume that

$$\lim_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}=+\infty.$$

Then, for each

$$\lambda\in\left(0,\frac{\rho}{2(T-2)}\sup_{\gamma>0}\frac{\gamma^2}{F(\gamma)}\right),$$

the problem

$$\begin{cases}\Delta^4u(k-2)-\alpha\Delta^2u(k-1)+\beta u(k)=\lambda f(u(k)), & k\in[2,T]_{\mathbb{Z}},\\ u(1)=\Delta u(0)=\Delta u(T)=\Delta^3u(0)=\Delta^3u(T-1)=0\end{cases}\tag{3.14}$$

admits at least one non-trivial solution $u_\lambda\in E$ *such that*

$$\lim_{\lambda\to 0^+}\|u_\lambda\|=0$$

and the real function

$$\lambda\to\frac{1}{2}\left(\sum_{k=2}^{T+1}|\Delta^2u(k-2)|^2+\alpha\sum_{k=2}^{T}|\Delta u(k-1)|^2+\beta\sum_{k=2}^{T}|u(k)|^2\right)-\lambda\sum_{k=2}^{T}F(u_\lambda(k))$$

is negative and strictly decreasing in $\left(0,\frac{\rho}{2(T-2)}\sup_{\gamma>0}\frac{\gamma^2}{F(\gamma)}\right)$.

Finally, we present the following example to illustrate Theorem 3.4.

Example 3.1 *Let* $T=4$. *We consider the following problem*

$$\begin{cases}\Delta^4u(k-2)-\Delta^2u(k-1)+u(k)=\lambda f(u(k)), & k\in[2,4]_{\mathbb{Z}},\\ u(1)=\Delta u(0)=\Delta u(4)=\Delta^3u(0)=\Delta^3u(3)=0\end{cases}\tag{3.15}$$

where

$$f(t) = \frac{1}{e}\left(2t + 2\sin(t)\cos(t) + e^t\right)$$

for every $t \in \mathbb{R}$. *By simple computations, we have*

$$F(t) = \frac{1}{e}\left(t^2 + \sin^2(t) + e^t - 1\right)$$

for every $t \in \mathbb{R}$. *Taking into account that* $\rho = \frac{27}{4}$, *all the assumptions of Theorem 3.4 are satisfied, and it implies that the problem (3.15) for each* $\lambda \in \left(0, \frac{27e}{16}\right)$, *admits at least one non-trivial solution* u_λ *in*

$$\left\{u : [0,6]_Z \to \mathbb{R},\ u(1) = \Delta u(0) = \Delta u(4) = \Delta^3 u(0) = \Delta^3 u(3) = 0\right\}$$

such that

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \to \frac{1}{2}\left(\sum_{k=2}^{5} |\Delta^2 u(k-2)|^2 + \sum_{k=2}^{4} |\Delta u(k-1)|^2 + \sum_{k=2}^{4} |u(k)|^2\right) - \lambda \sum_{k=2}^{4} F(u_\lambda(k))$$

is negative and strictly decreasing in $\left(0, \frac{27e}{16}\right)$.

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