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New Results of Characterization on Symmetric Ordered S-Banach Algebra

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ABSTRACT: This paper investigates the character of symmetric S-ordered Banach algebra with identity. First, we introduce the concepts of s-norm and quasi-s-norm and then define the symmetric ordered Banach algebra (SOBA) with Symmetric algebra cone C. We introduce the concept of S-character on (SOBA) and show that the set of all S-characters is compact, continuous, and has the s-norm, as well as being hermitian functional and proving new results.

Keywords: S-norm, quasi-S-norm, symmetric ordered Banach algebra, S-character.

Contents

1. Introduction

Banach algebra theory is an abstract mathematical theory that combines several specific situations from various branches of mathematics. It was a fundamental concept in the early twentieth century. Gelfand is the creator of the Banach algebra theory. On Banach space, he is also being examined for an additional multiplication. In 1936, Gelfand proposed the concept of abstract Banach algebra via linear metric, and in 1939, he introduced the foundations of commutative Banach algebra theory. "Normed Rings" by Naimark is the first book on Banach algebra, which was fundamental to its development, and Ambrose examined it in 1945. Scientists' interest in the new theory grew, and they presented a wealth of literature and notions that contributed to the development of Banach algebra theory. The new theory is a generic theory that bridges the gap between functional and classical analysis. It's a more sophisticated way of discussing algebra. If you're looking for a unique The theory is applied to physics, statistical mechanics, and quantum mechanics, among other fields. Sifis [20] investigated the commutative and universal Gelfand-Naimark, while de Jeu [4] provided an excellent introduction to functional analytic objects. Kulkarni and Sukumar [15] investigate the condition spectrum's relationship with a nearly multiplicative linear functional. On Symmetric Banach algebra, Bruce [2] built linkages. In complex Banach algebra, Kowalski and Slodkowski [14] investigated the characterization of multiplicative linear functionals. Thanyacharoen and Sintunavarat [21] demonstrated the composite functional equation's generalized Hyers-Ulam stability results in (β, p)- Banach space.

Hussein and Fahim introduced some results about characterization in ordered Banach algebra [7,8,10,9]. Hussein found equivalent measure by use the space of ordered Banach algebra [6]. de Jeu introduced thesis in ordered Banach algebra in 2010 [4]. Krishna and Kulkarni studied linear map which preserving pseudospectral [16]. Huda and Hussein studied characterization on symmetric Banach algebra [23,11,5]. Muzundu intrduced in commutative ordered Banach algebra [17]. Kulkarni and Sukumar found the properties of almost multiplicative functions [15]. Nielsen studies Hermitian and symmetric algebra [19].

Toure and Brits [22] found various character generating functions using Banach algebras. We begin by defining the concept of S-normed space in this paper. Many spaces do not act as normed spaces

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because one or more of the norm's prerequisites are not met. As a result, the S-norm is more useful in terms of application and understanding the relationship between it and the normed space. However, there are two crucial aspects to keep in mind: Symmetric algebra is the first, and ordered Banach algebra is the second. We might wonder if there is a link between the two concepts. Kadhim and Hussein [13,12] found proximit Banach algebra with some main results. In [18] Neamah and Hussein studied character in symmetric $t^w - \Delta$-Banach algebras. Is there any order in the symmetric Banach algebra? Isn't the contrary also true? When the Banach algebra is both symmetrical and ordered, what happens? To answer these questions, we'll use symmetric algebra cone C to explain the concept of Symmetric ordered Banach Algebra (SOBA). It has symmetrical and well-ordered qualities. Finally, we prove several properties while defining the S-character on a SOBA.

This paper's outline is divided into four pieces. This is the first part of the paper's introduction. Section 2 introduces ideas such as the S-norm, SOBA, and S-character. In section three, we go over several properties related to SOBA and S-character, such as if a SOBA has W*-topology, then s-char (Ω) is compact, if C is a valid symmetric algebra cone and is surjective, then is proper, and if in s-char (Ω), then is a bounded (or $\|\varphi\|_s \leq 1$) The major results are in Section 4: if is S-character, then is hermitian, and s-character is continuous. As a result, the functional: $\varphi : \Omega \longrightarrow \mathbb{C}$, $u \longmapsto \overline{h(u^*)}$ continuous, with h in s-ch (Ω) and other results.

2. Preliminaries

In this section, we will go over some fundamental ideas

Definition 2.1 *[3] (Linear algebra) Let Ω be a linear space with respect to the field $\mathcal{F} = \mathbb{C}$ or $\mathbb{R} \cdot \Omega$ is said to be a linear algebra, if satisfies the following conditions:*

i. $\lambda\,(uv) = (\lambda u)\,v = u(\lambda v)$.

ii. $(uv)\,z = u\,(vz)$.

iii. $(u+v)\,z = uz + vz$ *and* $u\,(v+z) = uv + uz$.

If any $u, v \in \Omega$ satisfies $uv = vu$, we say Ω is commutative linear algebra.

Let $\Omega_S \subseteq \Omega$ we say that Ω_S is sub algebra if it is satisfies the same conditions of linear algebra

Definition 2.2 *[24] A linear algebra Ω is called an algebra with identity if Ω contains an element e which satisfies the condition: $u \cdot e = eu = u$ for all $u \in \Omega$.*

Definition 2.3 *[15] Let Ω is algebra and $u \in \Omega$, u is invertible if there exists an element $v \in \Omega$ such that $uv = vu = e$. So we say that v inverse to u and denoted by u^{-1}.*

Definition 2.4 *(Banach Algebra) Let Ω be a linear algebra, Ω is called Normed Algebra if satisfy the conditions:*

1. *Ω is a Banach space*
2. *Ω is an algebra*
3. *For each $u, v \in \Omega$, $\|uv\| \leq \|u\|\|v\|$,*
4. *Ω with identity, then $\|e\| = 1$*

Proposition 2.5 *[18] Let Ω is a Banach algebra and $u \in \Omega$ is an invertible element. Suppose $v \in \Omega$ is element with $\|v-u\|_s < \frac{1}{\|u^{-1}\|_s}$, v is also invertible and*

$$\|v^{-1} - u^{-1}\|_s \leq \frac{\left|\|u^{-1}\|_s^2\right| \|v-u\|_s}{1 - \|u^{-1}\|_s . \|v-u\|_s}$$

Definition 2.6 *(W^*-Topology) Let Ω be a Banach space and Ω^* be the dual space of Ω and $f : \Omega \rightarrow \Omega^{**}$ be the natural inclusion of Ω on to the second dual. W^*-Topology is the weakest topology making all functional on Ω^* are continuous.*

Theorem 2.7 *[24] (Alaoglu's Theorem) Suppose A is a Banach space and A^* is dual space. Let $(A^*)_1$ be the closed unit ball in A^* (consisting of bounded linear functions on A whose s-norms are less than or equal to 1). Then $(A^*)_1$ is compact in the W^*-Topology.*

Theorem 2.8 *(Closed Graph Theorem) Let Ω_1, Ω_2 are Banach space. Then a function $\varphi : \Omega_1 \longrightarrow \Omega_2$ is a continuous if and only if the graph $\Lambda = (u, \varphi(u)) : u \in \Omega_1$ is linear subspace of $\Omega_1 \times \ \Omega_2$ and closed in the product topology of $\Omega_1 \times \ \Omega_2$.*

Definition 2.9 *[1] (Symmetric Algebra) We called Ω is a Symmetric algebra if it is satisfy the following conditions:*

1. *Ω is Algebra.*
2. *An operation is defined in Ω which each element $u \in \Omega$ the element $u^* \in \Omega$ satisfies the following conditions:*
   - *i.* $(\lambda u + \mu v)^* = \bar{\lambda} u^* + \bar{\mu} v^*$ *for all* $u, v \in \Omega$, $\lambda, \mu \in \mathcal{F}$
   - *ii.* $u^{**} = u$.
   - *iii.* $(uv)^* = v^* u^*$

$u \longrightarrow u^*$ we shall name this operation involution on an algebra Ω if satisfying (i), (ii), (iii) and the element u^* called the adjoin of u.

Definition 2.10 *[20] Let Ω be a Symmetric Banach algebra u is an element in Ω then:*

- *i. We called u is a self-adjoin or hermitian if $u^* = u$*
- *ii. We called u is a normal element if $u^* u = u u^*$*
- *iii. We called u is a projection element if $u^* = u$ and $u^2 = u$.*
- *iv. We called u is an unitary element if $u^* u = u u^* = 1$.*
- *v. We called u is anti-hermitian element if $u^* = -u$*

3. New Results Related to Symmetric Ordered Banach Algebra

The following definition will be used to begin this section:

Definition 3.1 *Let Ω a linear space over the field $\mathcal{F}$. S-norm on Ω is a function $\|\cdot\|_s \Omega \ \rightarrow \mathbb{R}$ such that for all u, v in Ω satisfies the following conditions:-*

- *i.* $\|u\|_s \geq 0$
- *ii.* $\|\lambda u\|_s \leq \|u\|_s |\lambda| \leq 1$
- *iii.* $\lim_{\lambda \rightarrow 0} \|\lambda u\|_s = 0$.
- *iv.* $\|\lambda u + \mu v\|_s \leq |\lambda + \mu| \left(\|u\|_s + \|v\|_s\right)$, *for all* $\lambda, \mu \geq 1$

The pair $(\Omega, \|\cdot\|_s)$ is called S-normed space.

Definition 3.2 *The S-normed space is called quasi-s-normed space if it is satisfies the following conditions:*

- *i.* $\|\lambda u\|_s = |\lambda| \, \|u\|_s, \forall u \in \Omega, \lambda \in \mathcal{F}$

ii. $\|u+v\|_s \leq \|u\|_s + \|v\|_s, \forall u, v \in \Omega$

Proposition 3.3 *A quasi-S-normed space is normed space.*

Proof: Let $(\Omega, \|\cdot\|_s)$ is quasi-s-normed space to prove it is normed space:
We define $\|\cdot\| : \Omega \to \mathbb{R}$ by $u = us$ for all $u \in \Omega$. It is clear. is well defined and:

i. $||u|| = \|u\|_s \geq 0, \forall u \in \Omega$

ii. If $u = 0$ then $||0|| = \|0\|_s = \|0.u\|_s = |0|\|u\|_s = 0$

iii. $||\lambda u|| = \|\lambda u\|_s = |\lambda|\|u\|_s$ then $||\lambda u|| = |\lambda|\, ||u||$

iv. $||u+v|| = \|u+v|_s \leq |\,|u|_s + |\,|v|_s = ||u|| + ||v||$

Hence $(\Omega, \|\cdot\|)$ is normed space. □

Definition 3.4 *A complete S-normed space is called S-Banach space.*

Definition 3.5 *Let Ω be a real or complex symmetric algebra with identity and C non-empty subset of Ω we called C a cone if it is satisfies the following conditions:*

i. $u + v \in C$ *for each* $u, v \in C$

ii. $\lambda u \in C$ *for each* $u \in C, \lambda \geq 0$

If in addition C satisfies $C \cap -C = \{0\}$ then C will be called proper cone. Any cone C on Ω induce a relation $(\preccurlyeq)$ on Ω called an ordering in the following way $u \preccurlyeq v$ if and only if $v^* - u^* \in C$ for all $u, v \in \Omega$.
We say that C is Symmetric algebra cone if it is satisfies the following conditions:

i. $u.v \in C$ *for each* $u, v \in C$

ii. $e \in C$

It can be show that for all $u, v \in \Omega$ this order satisfies:

i. Reflexive: For all $u \in (\Omega, C)$ then $u \preccurlyeq u$

ii. Transitive: For all $u, v, z \in (\Omega, C)$ if $u \preccurlyeq v$ and $v \preccurlyeq z$ then $u \preccurlyeq z$.

So Ω is ordered algebra with Symmetric algebra cone C by relation $(\preccurlyeq)$ such that for all u, v and $z \in \Omega$ and $\lambda \geq 0$ we have:

i. $0 \preccurlyeq u, 0 \preccurlyeq v$ *then* $0 \preccurlyeq u + v$

ii. $0 \preccurlyeq u, \lambda \geq 0$ *then* $0 \preccurlyeq \lambda u$

iii. $0 \preccurlyeq u, 0 \preccurlyeq v$ *then* $0 \preccurlyeq u.v$

iv. $e \succcurlyeq 0$

So if Ω is Banach algebra and ordered by a Symmetric algebra cone C we will called $(\Omega,\ C)$ is a Symmetric Ordered Banach Algebra (SOBA).

Definition 3.6 *Let (Ω, C) is Symmetric algebra cone C, and u, v and $z \in \Omega$:*

i. C is called normal if there exists a constant $\lambda \geq 1$ we have $0 \leq u \leq v$ then $|\,|u\|_s \leq \lambda\|\,v\|_s$

ii. C is called λ-normal if there exists a constant $\lambda \geq 1$ we have $u \leq v \leq z$ then $\|v\|_s \leq \lambda(\max\|u\|_s, \|z\|_s)$

If normality λ is equal to 1 we say that the C is 1-normal.

Definition 3.7 *(Symmetric inverse-closed) Let C be an algebra cone then C is a Symmetric inverse-closed if:*

i. *C is inverse-closed*

ii. *For all $u \in C$ we obtain $u^* \in C$.*

Definition 3.8 *(S-character) Let Ω is a SOBA for all $u, v \in \Omega$ and $\lambda \in \mathcal{F}$, we call that function φ from Ω into field $\mathcal{F}$ is s-character if it is satisfy the following conditions:*

i. $\varphi(\lambda u) = \lambda \varphi(u)$

ii. $\varphi(u^*) = \varphi(u)^*$

iii. $\|_{\varphi}(u+v) - \varphi(u) - \varphi(v)\|_s \leq \alpha(\|_u\|_{s+}^p\|_v\ \|_s^p), \alpha > 0$ *and* $0 < p < 1$

iv. $\|\varphi(uv) - \varphi(u)\varphi(v)\|_s \leq \alpha\|u\|_s \cdot \|v\|_s, \alpha > 0$

v. $\varphi(e) = 1$

If $\mathcal{F} = \mathbb{C}$ we say that φ is complex S-character and s- char (Ω) denotes the set of all complex character. We proved that the Symmetric ordered Banach Algebra has given various new results in this proposition.

Proposition 3.9 *Let (Ω, C) be a SOBA, and let $u, v \in \Omega$ be such that $uv^* \preccurlyeq u^*v$ then:*

i. *If $u^{-1}v^{-1} \in C$ then $u(u^{-1})^* \preccurlyeq v(v^{-1})^*$*

ii. *If $(u^*)^{-1}(v^*)^{-1} \in C$ then $v^{-1}v^* \preccurlyeq u^{-1}u^*$*

Proof:

i. since $uv^* \preccurlyeq u^*v$ then $(u^*v)^* - (uv^*)^* = v^*u - vu^* \in C$
Since $u^{-1}v^{-1} \in C$ then $u^{-1}v^{-1}(v^*u - vu^*) \in C$

$$\begin{aligned} u^{-1}v^{-1}(v^*u - vu^*) &= u^{-1}v^{-1}v^*u - u^{-1}v^{-1}vu^* = v^{-1}v^*\left(uu^{-1}\right) - u^{-1}\left(v^{-1}v\right)u^* \\ &= (v(v^{-1})^*)^* - (u(u^{-1})^*)^* \in C. \end{aligned}$$

Therefore

$$u(u^{-1})^* \preccurlyeq v(v^{-1})^*$$

ii. since $uv^* \preccurlyeq u^*v$ then $(u^*v)^* - (uv^*)^* = v^*u - vu^* \in C$
Since $(u^*)^{-1}(v^*)^{-1} \in C$ then

$$\begin{aligned} (v^*u - vu^*)(u^*)^{-1}(v^*)^{-1} &= v^*u(u^*)^{-1}(v^*)^{-1} - vu^*(u^*)^{-1}(v^*)^{-1} \\ &= [(v^*)^{-1}v^*]u(u^*) - 1 - v[u^*(u^*) - 1](v^*) - 1 = u(u^*)^{-1} - v(v^*)^{-1} \\ &= u(u^{-1})^* - v(v^{-1})^* = (u^{-1}u^*)^* - (v^{-1}v^*)^* \in C. \end{aligned}$$

Thus

$$v^{-1}v^* \preccurlyeq u^{-1}u^*$$

□

Proposition 3.10 *Let (Ω, C) be a SOBA and let $u, v \in \Omega$ be such that $u \preccurlyeq u^*, v \preccurlyeq v^*$ then:*

i. $u + v \preccurlyeq u^* + v^*$

ii. $vu^* + v^*u \preccurlyeq vu + v^*u^*$

Proof:

i. since $u \preccurlyeq u^*$ then $(u^*)^* - u^* \in C \implies u - u^* \in C$

So $v \preccurlyeq v^*, (v^*)^* - v^* \in C$ then $v - v^* \in C$ and $(u-u^*)+(v-v^*) \in C$, that is $(u+v)-(u^*+v^*) \in C$, therefore $(u^*+v^*)^* - (u+v)^* \in C$. Thus $u+v \preccurlyeq u^*+v^*$

ii. since $u - u^* \in C, v - v^* \in C$ $(by\ i)$ then $(u-u^*)(v-v^*) \in C$,

$$
\begin{aligned}
(u-u^*)(v-v^*) = uv - uv^* - u^*v + u^*v^* &= (uv + u^*v^*) - (uv^* + u^*v) \\
&= (v^*u^* + vu)^* - (vu^* + v^*u)^* \in C.
\end{aligned}
$$

Hence $vu^* + v^*u \preccurlyeq vu + v^*u^*$.

□

Theorem 3.11 *Let (Ω, C) be SOBA with a Symmetric algebra cone C, S is closed sub algebra and $e \in S \subseteq (\Omega, C)$. Then:*

i. S is SOBA with $C \cap S$ is a Symmetric algebra cone in S.

ii. If C is a proper symmetric algebra cone in (Ω, C) then $C \cap S$ is proper symmetric algebra cone in S.

iii. If C is a Symmetric inverse-closed in $(\Omega,\ C)$ and S is normal, then $C \cap S$ is a Symmetric inverse-closed in S.

Proof:

I. To prove $(S, C \cap S)$ is a SOBA with symmetric algebra cone $C \cap S$. We satisfies the following conditions:

(a) To prove $(S, C \cap S)$ is a Symmetric algebra: Let $u, v \in S$ and $\lambda, \mu \in \mathcal{F}$

i. since S sub algebra of Ω then $\lambda u + \mu v \in \Omega$, and since (Ω, C) is a SOBA, then $(\lambda u + \mu v)^* = \bar{\lambda} u^* + \bar{\mu} v^*$

ii. Let $u \in S$ then $u \in \Omega$ and since (Ω, C) SOBA then $u^{**} = u$.

iii. Let $u, v \in S$ then $uv \in S$, so $uv \in \Omega$ and (Ω, C) is a SOBA then $(uv)^* = v^*u^*$.

Hence $(S, C \cap S)$ Symmetric algebra.

(b) Let $u \in S$. Then $u \in \Omega$ and since (Ω, C) SOBA then $||u||_s = ||u^*||_s$

(c) To prove $(S, C \cap S)$ is an OBA. First show that $C \cap S$ is a symmetric algebra cone: Let $u, v \in C \cap S$ and $\lambda \in \mathcal{F}$ so $u, v \in C$ and $u, v \in S$. Since C is a symmetric algebra and S is closed sub algebra:

i. $u + v \in C$ and $u + v \in S$ then $u + v \in C \cap S$

ii. $\lambda u \in C$ and $\lambda u \in S$ then $\lambda u \in C \cap S$

iii. $uv \in C$ and $uv \in S$ then $uv \in C \cap S$

iv. $e \in C$ and $e \in S$ then $e \in C \cap S$

A cone $C \cap S$ on S induced an ordering relation $(\preccurlyeq)$ on S by the way for all $u, v \in C \cap S, u \preccurlyeq v$ if $v^* - u^* \in C \cap S$. This ordering satisfies:

i. Reflexive; For all $u \in S$ since $S \subseteq \Omega$ so $u \in \Omega$ then $u \preccurlyeq u$

ii. Transitive; For all u, v and $z \in S$, if $u \preccurlyeq v$ and $v \preccurlyeq z$ then

$v^* - u^* \in C \cap S$ and $z^* - v^* \in C \cap S$. Since C is a symmetric algebra cone and $S \subseteq \Omega$ is sub algebra then $z^* - u^* = (z^* - v^*) + (v^* - u^*) \in C \cap S$. That is $u \preccurlyeq z$.

Now to satisfy conditions of OBA for all $u, v \in S$ and $\lambda \in \mathcal{F}$ such that $u \preccurlyeq v$. Let $u, v \in S$ and since $S \subseteq (\Omega, C)$

i. $0 \preccurlyeq u, 0 \preccurlyeq v \implies (u+v)^* - 0^* \in C$ and S sub algebra that $(u+v)^* - 0^* \in S$ then $(u+v)^* - 0^* \in C \cap S$ therefore $0 \preccurlyeq u + v$

ii. $0 \preccurlyeq u, \lambda \geq 0, C$ is a symmetric algebra cone and S sub algebra, that is $(\lambda u)^* - 0^* \in C \cap S$ therefore $0 \preccurlyeq \lambda u$

iii. $0 \preccurlyeq u$,$0 \preccurlyeq v$, therefore $uv \in C$ and $uv \in S$, that is $(uv)^* - 0^* \in C \cap S$, then $0 \preccurlyeq uve \in C$ and S is sub algebra, then $e^* - 0^* \in C \cap S$ and $0 \preccurlyeq e$

Hence $(S, C \cap S)$ is OBA and Therefore $(S, C \cap S)$ is a SOBA.

II. Let $u \in (C \cap S) \cap (-(C \cap S)$ then $u \in (C \cap S) \cap (-C \cap -S)$ then $u \in C, u \in S$ and $u \in -C, u \in -S$ then $u \in (C \cap -C)$ and $u \in (S \cap -S)$, C is a proper, that is $C \cap -C = 0$, and $u \in (C \cap -C)$ then $u = 0$.

III. Let $u \in (\Omega, C)$ and u has inverse u^{-1}, To prove $u^{-1} \in C \cap S$. since C is a symmetric inverse-closed in (Ω, C) and u has an inverse so $u^{-1} \in C, S$ closed sub algebra and u has an inverse then $u^{-1} \in S$. Hence $u^{-1} \in C \cap S$, therefore $C \cap S$ is inverse-closed. So since C a symmetric inverse-closed and S symmetric **(by i)** and closed sub algebra then $C \cap S$ is symmetric inverse-closed.

□

Theorem 3.12 *[24] Let (Ω, C) be SOBA with a Symmetric algebra cone C and Γ is a Symmetric Banach algebra with identity and φ a homomorphism from Ω into Γ then:*

i. If Γ is a SOBA and C is a Symmetric algebra cone of (Ω, C) and φ is surjective, then φC is a Symmetric algebra cone of Γ.

ii. If C is a proper and φ is surjective, then φC is proper.

iii. If C is closed, φ is continuous and surjective, then φC is closed.

iv. If C is a Symmetric inverse-closed cone in (Ω, C) and φ is injective, then φC is a Symmetric inverse-closed in Γ.

Proof:

I. To prove $(\Gamma, \varphi C)$ is a SOBA with a symmetric algebra cone. we satisfies the following conditions:

1. To prove $(\Gamma, \varphi C)$ is a Symmetric algebra

   i. Let $u, y \in \Gamma$ then there exists $a, b \in \Omega$ such that $\varphi a = u, \varphi b = v$ and $\lambda u + \mu v = \lambda(\varphi a) + \mu(\varphi b) = \varphi(\lambda a + \mu b) \in \Gamma$ then $\lambda a + \mu b \in \Omega$. Since (Ω, C) is a symmetric algebra then:

$$(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^* then (\bar{\lambda} a^* + \bar{\mu} b^*) \in \Gamma.$$

   So

$$\bar{\lambda}(\varphi a^*) + \overline{\mu}(\varphi b^*) = \bar{\lambda}(\varphi a)^* + \bar{\mu}(\varphi b)^* = \bar{\lambda} u^* + \bar{\mu} v^*.$$

   ii. Let $u \in \Gamma$ then there exists $a \in \Omega$ such that $\varphi a = u$ and since (Ω, C) is a symmetric algebra then $a = a^{**}$ so $\varphi(a^{**}) \in \Gamma$ then $u^{**} \in \Gamma$ hence $u^{**} = u$

   iii. Let, $v \in \Gamma$. There exists $a, b \in \Omega$ such that $\varphi a = u$, $\varphi b = v$ and $uv = \varphi a . \varphi b = \varphi(ab) \in \Gamma$ then $ab \in \Omega$ since (Ω, C) symmetric algebra then $(ab)^* = b^* a^*$ so $\varphi(b^* a^*) \in \Gamma$ that is $\varphi b^* \varphi a^* \in \Gamma$. Therefore $(uv)^* = v^* u^*$

2. $\|u^*\|_s = \|\varphi a^*\|_s = \|\varphi a\|_s = \|u\|_s$

3. To prove $(\Gamma, \varphi C)$ is a OBA

   First show that φC is a symmetric algebra cone. Let $u, v \in \varphi C$ and $\lambda \geq 0$ then there exists $a, b \in C$ such that $\varphi a = u$, $\varphi b = v$ since C is cone then $a + b$, $\lambda a \in C$ so $u + v = \varphi a + \varphi b = \varphi(a + b) \in \varphi C$ and $\lambda u = \lambda \varphi a \in \varphi C, \lambda v = \lambda \varphi b \in \varphi C$, since φ is homomorphism and C is a symmetric algebra, So $a, b \in C$. then we have $u.v = \varphi a.\varphi b = \varphi(a.b) \in \varphi C$. Since φ is surjective. So $\varphi e = e'$ we obtain $e' \in \varphi C$ Therefore φC is a Symmetric algebra cone of Γ. A cone φC on Γ induced an ordering relation $(\preccurlyeq)$ on Γ by the way for all $u, v \in \varphi C$, $u \preccurlyeq v$ if $v^* - u^* \in \varphi C$. this ordering satisfies:

   i. Reflexive; For all $u \in \Gamma$ there exists $a \in \Omega$ such that $\varphi a = u$ and since (Ω, C) is a SOBA then $a \preccurlyeq a$ hence $a^* - a^* \in C$ then $\varphi(a^* - a^*) \in \varphi C$. therefore $\varphi a \preccurlyeq \varphi a$ so $u \preccurlyeq u$.

   ii. Transitive ; For all u, v and $z \in \Gamma$, if $u \preccurlyeq v$ and $v \preccurlyeq z$ then there exists a, b and $c \in \Omega$ such that $\varphi a = u$, $\varphi b = v$, $\varphi c = z$ then $\varphi a \preccurlyeq \varphi b$ and $\varphi b \preccurlyeq \varphi c$, hence $\varphi(b)^* - \varphi(a)^* \in \varphi C$ and $\varphi(c)^* - \varphi(b)^* \in \varphi C$. Since φ is homomorphism, then $\varphi(b - a)^* \in \varphi C, \varphi(b - c)^* \in \varphi C$, then $b^* - a^* \in C, c^* - b^* \in C$ and since (Ω, C) is a SOBA with a symmetric algebra cone. Then $c^* - a^* \in C$ hence $\varphi(c - a)^* \in \varphi C$ and so $\varphi a \preccurlyeq \varphi c$, therefore $u \preccurlyeq z$.

   Now to satisfy conditions of OBA for all $a, b \in C$ and $\lambda, \mu \in \mathscr{F}$ such that $a \preccurlyeq b$. Let $u, v \in \Gamma$:

   i. $0 \preccurlyeq u, 0 \preccurlyeq v \Longrightarrow 0 \preccurlyeq \varphi a$, $0 \preccurlyeq \varphi b$ then $\varphi(a)^* - \varphi(0)^* \in \varphi C$ and $\varphi(b)^* - \varphi(0)^* \in \varphi C$ So $a^* - 0^* \in C$ and $b^* - 0^* \in C$ then $a^* + b^* - 0^* \in C$ and $\varphi(a + b)^* - \varphi(0)^* \in \varphi C$ hence $0 \preccurlyeq u + v$

   ii. $0 \preccurlyeq u$, $\lambda \geq 0 \Longrightarrow 0 \preccurlyeq \varphi a$, then $\varphi(a)^* - \varphi(0)^* \in \varphi C$, So $a^* - 0^* \in C$ and since C a symmetric algebra cone then $(\lambda a)^* - 0^* \in C$ then $\varphi(\lambda a)^* - \varphi(0)^* \in \varphi C$ therefore $0 \preccurlyeq \lambda u$

   iii. $0 \preccurlyeq u, 0 \preccurlyeq v \Longrightarrow 0 \preccurlyeq \varphi a, 0 \preccurlyeq \varphi b$ then:
   $\varphi(a)^* - \varphi(0)^* \in \varphi C$ and $\varphi(b)^* - \varphi(0)^* \in \varphi C$ So $a^* - 0^* \in C$ and $b^* - 0^* \in C$ then $a^*.b^* - 0^* \in C$ and $\varphi(a.b)^* - \varphi(0)^* \in \varphi C$ hence $0 \preccurlyeq u.v$

   iv. since $\varphi(e) = e'$ and $0 \preccurlyeq e$ then $e^* - 0^* \in C$ then $\varphi(e^*) - \varphi(0^*) \in \varphi C$ and $0 \preccurlyeq e'$.

   Hence $(\Gamma, \varphi C)$ is OBA and Therefore $(\Gamma, \varphi C)$ is a SOBA.

II. Let $u \in (\varphi C \cap -\varphi C)$, since $u \in (\varphi C \cap -\varphi C) = \varphi(C \cap -C)$ then there exists $a \in C$ such that $\varphi a = u$, $\varphi a \in \varphi(C \cap -C)$ since C is proper then $a \in (C \cap -C) = 0$ so $a = 0$ since φ is homomorphism then $\varphi(a) = \varphi(0) = 0 = u$.

III. Let $\{a_j\}$ be a sequence in φC such that $\lim_{j\to\infty} a_j = a$ to prove that $a \in \varphi C$. Since $a_j \in \varphi C$ and since φ is surjective, then there exists $\{b_j\}$ a sequence in C such that $\varphi b_j = a_j$ since $\{b_j\}$ in C and C is closed, then there exists $n \in C$ such that $\lim_{j\to\infty} b_j = n \in C$. since φ is surjective, then there exists $m \in \varphi C$ such that $\varphi n = m$.

   Then $a = \varphi b_j$ convergent to a and $a_j = \varphi\, b_j$ convergent to $\varphi n = m$, therefore $\varphi n = m$ and $\varphi n = a$, that is $a = m \in \varphi C$ then φC is closed.

IV. φC is a Symmetric algebra cone **by (i)**. To prove that φC is a Symmetric inverse-closed. Let $u \in (\Omega, C)$ and u has inverse $u^{-1} \in \Omega$ to prove $u^{-1} \in \varphi C$. let $a \in C$ such that $\varphi a = u$ then $u^{-1} = \varphi(a)^{-1} = \varphi(a^{-1})$, since C is a symmetric inverse-closed and a has an inverse. So $a^{-1} \in C$. Hence $\varphi(a^{-1}) \in \varphi C$. Therefore u has an inverse in φC. Then φC is a symmetric inverse-closed.

□

4. Characterization on Symmetric Ordered S-Banach Space

In this section, we show that the s-character is continuous in the following way:

Theorem 4.1 *(Automatic continuity of S-character) Let (Ω, C) be a Symmetric ordered Banach algebra over $\mathbb{C}$ and $\varphi : \Omega \to \mathbb{C}$ be a s-character then φ is continuous and has the s-norm $|||\varphi\||_s \leq 1$*

Proof: show that φ is continuous and $\|\varphi\|_s \leq 1$ we need prove that $|\varphi(u)| \leq \|u\|_s$ for all $u \in \Omega$. We prove by contradiction. Assume there exists $u_0 \in \Omega$ with $|\varphi(u_0)| \geq \|u_0\|_s$, Let $\beta = \varphi(u_0)$, So that element $u_1 = \beta^{-1}u_0$ has s-norm $\|u_1\|_s < 1$ (by Proposion 2.5) applied to $u = 1$ and $m = 1 - u_1$, hence that the element $1 - u_1 = 1 - \beta^{-1}u_0$ is invertible. Since $\beta \neq 0$, it follows that element $v = \beta(1 - u_1) = \beta 1 - u_0$ is invertible. But $\varphi(v^*) \neq \varphi(v)^*$ since : □

$$\varphi(v^*) = \varphi((\beta 1 - u_0)^*) = \varphi\left(\overline{\beta 1} - u_0^*\right)$$

$$\varphi(v)^* = \varphi(\beta 1 - u_0)^* = \overline{\varphi(\beta 1 - u_0)}$$

and:

$$\|\varphi\left(\overline{\beta 1} - u_0^*\right) - \varphi\left(\overline{\beta 1}\right) - \varphi(-u_0^*)\|_s \leq \alpha(\|\overline{\beta 1}Vert_s^p + \|u_0\|_s^p)$$

$$\|\overline{\varphi(\beta 1 - u_0)} - \overline{\varphi(\beta 1)} - \overline{\varphi(-u_0)}\|_s \leq \alpha(\|\beta 1\|_s^p + \|u_0\|_s^p)$$

Hence $\varphi(v^*) \neq \varphi(v)^*$, which is impossible.

Corollary 4.2 *Let (Ω, C) be a SOBA when equipped with the W^*-topology then the space s-char (Ω) is compact.*

Proof: (by Theorem 2.7) the unit ball $(\Omega^*)_1$ is compact in the W^*-topology, we need to show that s-char (Ω) is W^*-closed in $(\Omega^*)_1$. Take a net $(\varphi_\lambda)_{\lambda\in\Lambda} \subset$ s-char (Ω) and $\alpha \in (\Omega^*)_1$, with $\alpha = W^* - \lim_{\lambda\to\Lambda}\varphi_\lambda$, then we show that $\alpha \in s$-char (Ω), by the definition of the W^*-topology $\alpha(u) = \lim_{\lambda\to\Lambda}\varphi_\Lambda(u)$ for all $u \in \Omega$. Hence that α is linear multiplicative, So α is not identically zero since $\alpha(1) = \lim_{\lambda\to\Lambda}\varphi_\Lambda(1) = 1$. □

Corollary 4.3 *Let (Ω, C) be a SOBA. If s-char $(\Omega) \neq \emptyset$, then it is locally compact with W^*-topology.*

Proof: Let $M = s$-char $(\Omega) \cup 0 \subset (\Omega^*)_1$ with is compact in the W^*-Topology (by Theorem7) then s-char $(\Omega) = M - 0$ is open in M and locally compact, when equipped with W^*-topology.

The following theorem establishes that every s-character is hermitian and vice versa: □

Theorem 4.4 *Let Ω be a SOBA then φ is a s-character if and only if φ is hermitian functional*

Proof: Let φ is s-character and for all $u, v \in \Omega$ such that:

$$u = u_1 + iu_2, v = v_1 + iv_2$$

$$\begin{aligned}
\|\varphi(u^* + v^*) - \varphi(u^*) - \varphi(v^*)\|_s &= \|\varphi((u_1 + iu_2)^* + (v_1 + iv_2)^*) - \varphi(u_1 + iu_2)^* - \varphi(v_1 + iv_2)^*\|_s \\
&= \|\varphi(u_1 + iu_2 + v_1 + iv_2)^* - \varphi(u_1 + iu_2)^* - \varphi(v_1 + iv_2)^*\|_s \\
&= \|\overline{\varphi((u_1 + iu_2) + (v_1 + iv_2)) - \varphi(u_1 + iu_2) - \varphi(v_1 + iv_2)}\|_s \\
&= |\varphi((u_1 + iu_2) + (v_1 + iv_2)) - \varphi(u_1 + iu_2) - \varphi(v_1 + iv_2\|_s \\
&\leq \alpha(\|u\|_s^p + \|v\|_s^p) \\
&\|\varphi((u_1 + iu_2)^* + (v_1 + iv_2)^*) - \varphi(u_1 + iu_2)^* - \varphi(v_1 + iv_2)^*\|_s \\
&\leq \alpha(\|u^*\|_s^p + \|v^*\|_s^p)
\end{aligned}$$

So

$$\begin{aligned}\|\varphi(u^*v^*) - \varphi(u^*)\varphi(v^*)\|_s &= \|\varphi((u_1+iu_2)^*(v_1+iv_2)^*) - \varphi(u_1+iu_2)^*.\varphi(v_1+iv_2)^*\|_s \\ &= \|\varphi(u_1v_1+iu_2v_1+iu_1v_2-u_2v_2)^* - \varphi(u_1+iu_2)^*.\varphi(v_1+iv_2)^*\|_s \\ &= \|\overline{\varphi(u_1v_1+iu_2v_1+iu_1v_2-u_2v_2)} - \overline{\varphi(u_1+iu_2)}\cdot\overline{\varphi(v_1+iv_2)}\|_s \\ &= \|\overline{\varphi((u_1+iu_2)(v_1+iv_2)) - \varphi(u_1+iu_2)\cdot\varphi(v_1+iv_2)}\|_s \\ &= \varphi((u_1+iu_2)(v_1+iv_2)) - \varphi(u_1+iu_2)\cdots\varphi(v_1+iv_2)\|_s \\ &\le \alpha\|u\|_s\cdot\|v\|_s \\ &\|\varphi((u_1+iu_2)^*(v_1+iv_2)^*) - \varphi(u_1+iu_2)^*\cdot\varphi(v_1+iv_2)^*\|_s \\ &\le \alpha\|u^*\|_s\cdot\|v^*\|_s\end{aligned}$$

Therefore φ is hermitian functional. On the other hand: Let $\varphi\colon \Omega \to \mathbb{C}$ such that $u \longmapsto \overline{\varphi(u^*)}$, to prove $\varphi \in s-ch(\Omega)$.

i. $\varphi(\lambda u) = \overline{\varphi((\lambda u^*))} = \lambda\overline{\varphi(u^*)} = \lambda\varphi(u)$.

ii. $\varphi(u^*) = \overline{\varphi(u)} = \varphi(u)^*$.

iii. $\|\varphi(u+v) - \varphi(u) - \varphi(v)\|_s = \|\overline{\varphi((u+v)^*)} - \overline{\varphi(u^*)} - \overline{\varphi(v^*)}\|_s \le \alpha(\|u\|_s^p + \|v\|_s^p)\,\alpha > 0, 0 < p < 1$.

iv. $\|\varphi(uv) - \varphi(u)\cdot\varphi(v)\|_s = \|\overline{\varphi((uv)^*)} - \overline{\varphi(u^*)}\cdot\overline{\varphi(v^*)}\|_s \le \alpha\|u\|_s\cdot\|v\|_s\alpha > 0$, So $\varphi(e) = 1$.

Therefore φ is a s-character functional

□

Corollary 4.5 *Let Ω be a SOBA and $h \in s-ch(\Omega)$. Then the functional $\varphi : \Omega \longrightarrow \mathbb{C}$ defined by,$\varphi(u) = \overline{h(u^*)}$ is continuous.*

Proposition 4.6 *Let Ω is a SOBA then the involution is continuous.*

Proof: We must prove if $u_{n n\in\mathbb{N}} \subset \Omega, u \in \Omega$ and $\lim_{n\to+\infty} u_n = u$ then $lim_{n\to+\infty} u_n^* = u^*$. (by Closed Graph theorem) if $\lim_{n\to+\infty} u_n = u$ and $\lim_{n\to+\infty} u_n^* = v$ Then $v = u^*$. then (by Corollary 4.5)

$$\overline{h(u^*)} = \varphi(u) = \lim_{n\to+\infty}\varphi(u_n) = \lim_{n\to+\infty}\overline{h(u_n^*)} = \overline{h(v)}$$

Therefore

$$h(u^*) = h(v) \iff u^* = v$$

□

Theorem 4.7 *Let Ω be a SOBA with quasi-s-norm such that for all $0 \neq u \in \Omega$ then $u^{-1} \in \Omega$ then $\varphi : \Omega \to s-ch(\Omega)$ is s-character.*

Proof: We define $\varphi(u) = \pi_u$ such that $\pi_u(h) = h_u$ for all $h \in \Omega^*, u \in \Omega$

i. $\varphi(\lambda u) = \pi_{\lambda u} \Longrightarrow \pi_{\lambda u}(h) = h_{\lambda u} = \lambda h_u = \lambda\varphi(u)$.

ii. $\varphi(u^*) = \pi_{u^*} \Longrightarrow \pi_{u^*}(h) = (h_u)^* = \varphi(u)^*$.

iii. $\|\varphi(u+v) - \varphi(u) - \varphi(v)\|_s = \|\pi_{u+v} - \pi_u - \pi_v\|_s\ \|\pi_{u+v}(h) - \pi_u(h) - \pi_v(h)\|_s = \|h_{u+v} - h_u - h_v\|_s \le \alpha(\|u\|_s^p + \|v\|_s^p)\,\alpha > 0, 0 < p < 1$.

iv. $\|\varphi(uv) - \varphi(u)\cdot\varphi(v)\|_s = \|\pi_{u+v} - \pi_u\cdot\pi_v\|_s\ \|\pi_{u+v}(h) - \pi_u(h).\pi_v(h)\|_s = \|h_{u+v} - h_u.h_v\|_s \le \alpha\|u\|_s\cdot\|v\|_s\alpha > 0$ (v)$\varphi(e) = \pi_e \Longrightarrow \pi_e(h) = h_e = 1$. Therefore φ is a s-character.

□

5. Conclusion

In this paper, we used the concept of S-normed space instead of Norm space to define Banach algebra. We introduced the concept of a Symmetric Ordered Banach Algebra (SOBA) with Symmetric Algebra Cone C, and if in C for all, where is Symmetric Banach algebra? As a result, we created an S-character and demonstrated that every S-character function is a hermitian, continuous, and bounded function. We proved that char($*$) is a locally compact and W*-topology if it is not empty.

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