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Algebraic Properties of Generalized Inverse of Operators*

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ABSTRACT: This study explores the algebraic properties of Moore-Penrose inverse, Drazin-Moore-Penrose inverse, and the dual Drazin-Moore-Penrose inverse of operators defined on Hilbert spaces. It presents some sufficient conditions under which reverse order law, forward order law and absorption law hold, utilizing various types of generalized inverses.

Key Words: Absorption law, reverse order law, forward order law, Moore-Penrose inverse, DMP inverse, MPD inverse, operators on Hilbert space.

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1. Introduction and Preliminaries

The concept of generalized inverses has played a central role in the development of operator theory, especially when dealing with bounded linear operators on Hilbert spaces. In many cases, operators fail to be invertible in the classical sense, yet their behavior can still be meaningfully analyzed using tools like the Moore–Penrose inverse, Drazin inverse, group inverse, core inverse, Drazin Moore-Penrose inverse, and Dual Drazin Moore-Penrose inverse. Each of these provides a unique lens through which to study the structure and properties of operators, whether it's solving equations, understanding spectral characteristics, or decomposing spaces. These inverses have proven indispensable not only in finite-dimensional matrix theory but also in the more intricate setting of infinite-dimensional Hilbert spaces, where traditional techniques often fall short.

A classical result in algebra asserts that for invertible elements x and y in a semigroup with identity, the reverse order law $(xy)^{-1} = y^{-1}x^{-1}$ holds. However, this property does not generally extend to generalized inverses. For instance, the reverse order law fails in general for the Moore–Penrose inverse, Drazin inverse, and core inverse. Consequently, a substantial body of research has focused on identifying sufficient and necessary conditions under which reverse order laws are valid for these generalized inverses [6,4,5,8,1,7].

In particular, Djordjević and Dinčić [1] investigated the reverse order law for the Moore–Penrose inverse of bounded linear operators with closed range, extending finite-dimensional results to infinite-dimensional Hilbert spaces. Their approach utilized matrix representations induced by orthogonal decompositions. Building on foundational work by Greville [6], Bouldin [4], and Izumino [5], Tian [8] provided rank-based characterizations that further clarified the conditions under which reverse order laws hold.

More recently, Wang et al. [7] examined the reverse order law for the Drazin inverse, focusing on bounded linear operators $P, Q \in \mathcal{B}(H)$ and exploring conditions such as $[P, PQ] = 0$, $[Q, PQ] = 0$, $[P, PQQP] = 0$, and $[Q, PPPQ] = 0$. Their results offer equivalence conditions and matrix-based formulations that generalize known findings and support applications in areas such as Markov chains and Banach algebra theory.

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Parallel to the reverse order law, the absorption law for operators has also garnered significant attention. For two invertible operators $\phi, \psi \in \mathcal{B}(H)$, the identity

$$\phi^{-1} + \psi^{-1} = \phi^{-1}(\phi + \psi)\psi^{-1}$$

is known as the absorption law. While this law holds for invertible operators, it does not generally extend to all generalized inverses. Jin and Benítez [3] provided conditions under which the absorption law holds for ring elements, and Gao et al. [9] explored its validity for group and Drazin inverses in unitary rings, including counterexamples involving dual-core inverses.

Motivated by recent advances in the theory of generalized inverses, this paper explores the reverse order law, forward order law, and absorption law for the Drazin-Moore-Penrose and dual Drazin-Moore-Penrose inverses of bounded linear operators on Hilbert spaces. Our primary objective is to identify and characterize conditions under which the reverse and forward order laws coincide, thereby establishing equality in operator compositions involving generalized inverses.

We further investigate the mixed-type order laws, focusing on the structural requirements of operator products that ensure the validity of both reverse and forward order laws. This analysis reveals how specific operator configurations influence the preservation of these laws under multiplication. Additionally, we present sufficient conditions under which the absorption law holds for these generalized inverses.

Throughout this paper, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denotes the space of bounded linear operators from Hilbert space $\mathcal{H}$ to $\mathcal{K}$, and $\mathcal{B}(\mathcal{H})$ when $\mathcal{H} = \mathcal{K}$. The range and null space of an operator χ are denoted by $\mathcal{R}(\chi)$ and $\mathcal{N}(\chi)$, respectively. The ascent and descent of χ are defined as:

$$\mathrm{Asc}(\chi) = \inf\{p \in \mathbb{N} : \mathcal{N}(\chi^p) = \mathcal{N}(\chi^{p+1})\}, \quad \mathrm{Dsc}(\chi) = \inf\{p \in \mathbb{N} : \mathcal{R}(\chi^p) = \mathcal{R}(\chi^{p+1})\}.$$

If $\mathrm{Asc}(\chi) = \mathrm{Dsc}(\chi) < \infty$, the common value is called the index of χ, denoted $\mathrm{ind}(\chi)$.

An operator $S \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to have a Moore-Penrose inverse if there exists an operator $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying the following conditions:

$$\text{(i) } SXS = S \text{ (ii) } XSX = X \text{ (iii) } (SX)^* = SX \text{ (iv) } (XS)^* = XS.$$

The Moore-Penrose inverse of S exists if and only if S has a closed range. When it exists, the Moore-Penrose inverse is denoted by $S^\dagger$ and is unique.

In 2005, H.K. Du and C.Y. Deng [2] explored the existence and uniqueness of the Drazin inverse for bounded linear operators using Hilbert space decomposition. The operator $S \in \mathcal{B}(\mathcal{H})$ is said to have a Drazin inverse, denoted by S^D with index k, if there exists an operator $X \in \mathcal{B}(\mathcal{H})$ satisfying:

$$\text{(i) } XSX = X \text{ (ii) } SX = XS \text{ (iii) } S^{k+1}X = S^k.$$

In 2016, Yu and Deng [10] extended the concept of Drazin Moore-Penrose (DMP) inverse and Moore-Penrose Drazin (MPD) inverse from complex matrices to bounded linear operators on Hilbert spaces. If a closed range operator $S \in \mathcal{B}(\mathcal{H})$ with index k, the DMP inverse of S is unique and can be expressed as $S^{D,\dagger} = S^D S S^\dagger$. This inverse satisfies the following conditions:

$$\text{(i) } S^{D,\dagger} S S^{D,\dagger} = S^{D,\dagger} \text{ (ii) } S^{D,\dagger} S = S S^D \text{ (iii) } S^k S^{D,\dagger} = S^k S^\dagger.$$

Similarly, the MPD inverse, denoted by $S^{\dagger,D}$, is defined as $S^{\dagger,D} = S^\dagger S S^D$.

To prove the main results on reverse and forward order laws, as well as absorption laws for operators on Hilbert spaces, we begin by establishing a few key lemmas. These foundational results rely on Hilbert space decompositions and operator matrix representations, which help clarify the structural behavior of generalized inverses. The lemmas serve as essential tools for analyzing when these laws hold and under what conditions operator products preserve such equalities.

Lemma 1.1 *Let $\chi \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a closed range operator. Then the matrix representation of χ with respect to the orthogonal decompositions of Hilbert spaces $\mathcal{H} = \mathcal{R}(\chi^*) \oplus \mathcal{N}(\chi)$ and $\mathcal{K} = \mathcal{R}(\chi) \oplus \mathcal{N}(\chi^*)$: $\chi = \begin{bmatrix} \chi_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$ where χ_1 is invertible. Further, $\chi^\dagger = \begin{bmatrix} \chi_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix}$.*

Proof: The proof can be derived directly using the method outlined in the reference [1]. □

Lemma 1.2 *Let $\chi \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be a closed range operator. Then the matrix representation of χ with respect to the orthogonal decompositions of Hilbert spaces $\mathcal{H} = \mathcal{K} = \mathcal{R}(\chi) \oplus \mathcal{N}(\chi^*)$: $\chi = \begin{bmatrix} \chi_1 & \chi_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$ where χ_1 is invertible. Moreover,*

(i) $\chi^\dagger = \begin{bmatrix} \chi_1^* D^{-1} & 0 \\ \chi_2^* D^{-1} & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$ *where* $D = \chi_1\chi_1^* + \chi_2\chi_2^*$ *maps from* $\mathcal{R}(\chi)$ *into itself and* $D > 0$.

(ii) $\chi^D = \begin{bmatrix} \chi_1^{-1} & (\chi_1^{-1})^2\chi_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$ *where* $ind(\chi) = 1$.

(iii) $\chi^{D,\dagger} = \begin{bmatrix} \chi_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$.

Proof: The proof of (i) can be derived directly using the method outlined in the reference [1]. To prove (ii), one applies the definition of the Drazin inverse in the case where the operator has index 1.

To establish part (iii), we apply [Theorem 3.1, [10]]from Yu's work, which leads directly to the intended conclusion. □

Lemma 1.3 *[10] Let $\chi \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be a closed range operator. Then the matrix representation of χ with respect to the orthogonal decompositions of Hilbert spaces $\mathcal{H} = \mathcal{K} = \mathcal{R}(\chi^*) \oplus \mathcal{N}(\chi)$, then*

(i) $\chi = \begin{bmatrix} \chi_1 & 0 \\ \chi_3 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix} \to \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix}$ *where* χ_1 *is invertible.*

(ii) $\chi^\dagger = \begin{bmatrix} D^{-1}\chi_1^* & D^{-1}\chi_3^* \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix} \to \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix}$ *where* $D = \chi_1^*\chi_1 + \chi_3^*\chi_3$ *maps from* $\mathcal{R}(\chi^*)$ *into itself and* $D > 0$.

(iii) $\chi^D = \begin{bmatrix} \chi_1^{-1} & 0 \\ \chi_3(\chi_1^{-1})^2 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix} \to \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix}$ *where* $ind(\chi) = 1$.

(iv) $\chi^{\dagger,D} = \begin{bmatrix} \chi_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix} \to \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix}$.

Proof: The proof of this lemma closely follows the argument used in the previous one. For part (iii), we rely specifically on Theorem 3.2 from Yu's work [10]. □

In Section 2, the paper establishes conditions under which the reverse and forward order laws hold for various generalized inverses of bounded linear operators on Hilbert spaces. These results are derived using operator matrix representations and Hilbert space decompositions. Section 3 builds on this framework to explore absorption laws, presenting sufficient conditions for their validity when applied to combinations of Moore–Penrose, Drazin, and dual Drazin–Moore–Penrose inverses.

2. Reverse and forward order law

In the following lemmas, we present sufficient conditions under which various generalized inverses satisfy the reverse order law for bounded linear operators on Hilbert spaces.

Theorem 2.1 *Let $\chi \in \mathcal{B}(\mathcal{R}(\chi), \mathcal{N}(\chi^*))$ have a closed range. If χ_1 is invertible, then*

(i) $(\chi^\dagger\chi^{D,\dagger})^{\dagger,D} = (\chi^{D,\dagger})^\dagger(\chi^\dagger)^{\dagger,D}$

(ii) $(\chi^{D,\dagger}\chi^{\dagger})^{\dagger} = (\chi^{\dagger})^{\dagger,D}(\chi^{D,\dagger})^{\dagger}$

Proof: (i) From Lemma 1.2, we obtain that $\chi^{\dagger} = \begin{bmatrix} \chi_1^* D^{-1} & 0 \\ \chi_2^* D^{-1} & 0 \end{bmatrix}$ and $\chi^{D,\dagger} = \begin{bmatrix} \chi_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$.

Then multiplication of $\chi^{\dagger}\chi^{D,\dagger}$ yields $\begin{bmatrix} \chi_1^* D^{-1}\chi_1^{-1} & 0 \\ \chi_2^* D^{-1}\chi_1^{-1} & 0 \end{bmatrix}$, where $\chi_1^* D^{-1}\chi_1^{-1} : \mathcal{R}(\chi) \to \mathcal{R}(\chi)$.

Since $\chi_1^* D^{-1}\chi_1^{-1}$ is invertible, so using the property (i) and (iv) of Lemma 1.3 we can conclude that

$$(\chi^{\dagger}\chi^{D,\dagger})^{\dagger,D} = \begin{bmatrix} \chi_1 D(\chi_1^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}.$$

Similarly using Lemma 1.1, we get that $(\chi^{D,\dagger})^{\dagger} = \begin{bmatrix} \chi_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$, since χ_1 is invertible.

As χ_1 and D are invertible, so $(\chi_1^* D^{-1})^{-1} = D(\chi_1^*)^{-1}$. Thus the Moore-Penrose inverse of $(\chi^{\dagger})^{\dagger,D} = \begin{bmatrix} D(\chi_1^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$.

Hence, $(\chi^{D,\dagger})^{\dagger}(\chi^{\dagger})^{\dagger,D} = \begin{bmatrix} \chi_1 D(\chi_1^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = (\chi^{\dagger}\chi^{D,\dagger})^{\dagger,D}$.

(ii) This part can be proved using similar arguments as in part (i). □

Theorem 2.2 *Let $\chi \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be a closed range operator with matrix representation of χ with $\mathcal{H} = \mathcal{K} = \mathcal{R}(\chi^*) \oplus \mathcal{N}(\chi) : \chi = \begin{bmatrix} \chi_1 & 0 \\ \chi_3 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix} \to \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix}$ where χ_1 is invertible. Then*

(i) $(\chi^{\dagger,D}\chi^{\dagger})^{\dagger,D} = (\chi^{\dagger})^{\dagger,D}(\chi^{\dagger,D})^{\dagger}$

(ii) $(\chi^{\dagger}\chi^{\dagger,D})^{\dagger,D} = (\chi^{\dagger,D})^{\dagger}(\chi^{\dagger})^{\dagger,D}$

Proof: Using the values of $\chi^{\dagger,D}$ and $\chi^{\dagger}$ from Lemma 1.3, it can be easily computed that $\chi^{\dagger,D}\chi^{\dagger} = \begin{bmatrix} \chi_1^{-1}D^{-1}\chi_1^* & \chi_1^{-1}D^{-1}\chi_3^* \\ 0 & 0 \end{bmatrix}$ where $D = \chi_1^*\chi_1 + \chi_3^*\chi_3$ defines on $\mathcal{R}(\chi^*)$. Since χ_1 and D are invertible then $(D^{-1}\chi_1^*)^{-1} = (\chi_1^*)^{-1}D$ and $(\chi_1^{-1}D^{-1}\chi_1^*)^{-1} = (\chi_1^*)^{-1}D\chi_1$ maps from $\mathcal{R}(\chi^*)$ into itself. Hence, $(\chi^{\dagger,D}\chi^{\dagger})^{\dagger,D} = \begin{bmatrix} (\chi_1^*)^{-1}D\chi_1 & 0 \\ 0 & 0 \end{bmatrix}$. Using the Part (iii) of Lemma 1.2, One can easily calculate that $(\chi^{\dagger})^{\dagger,D} = \begin{bmatrix} (\chi_1^*)^{-1}D & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix} \to \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix}$. Thus, $(\chi^{\dagger})^{\dagger,D}(\chi^{\dagger,D})^{\dagger} = \begin{bmatrix} (\chi_1^*)^{-1}D\chi_1 & 0 \\ 0 & 0 \end{bmatrix} = (\chi^{\dagger,D}\chi^{\dagger})^{\dagger,D}$.

(ii) Using similar techniques, the part (ii) can be easily validated. □

In the following theorems, we establish conditions under which the reverse and forward order laws for generalized inverses coincide.

Theorem 2.3 *Let $\chi \in \mathcal{B}(\mathcal{R}(\chi) \oplus \mathcal{N}(\chi^*))$ be a closed range operator. If χ_1 is invertible, then*

(i) $(\chi^D\chi^{D,\dagger})^{\dagger} = (\chi^{D,\dagger})^{\dagger}(\chi^D)^{D,\dagger} = (\chi^D)^{D,\dagger}(\chi^{D,\dagger})^{\dagger}$

(ii) $(\chi\chi^{D,\dagger})^{\dagger} = (\chi^{D,\dagger})^{\dagger}(\chi^{D,\dagger}) = (\chi^{D,\dagger})(\chi^{D,\dagger})^{\dagger}$.

Proof: (i) From Lemma 1.2, $\chi^D\chi^{D,\dagger} = \begin{bmatrix} (\chi_1^2)^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$.

Now using the Lemma 1.1, on the result we get $(\chi^D\chi^{D,\dagger})^{\dagger} = \begin{bmatrix} \chi_1^2 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$.

Further calculating

$$\begin{aligned}(\chi^{D,\dagger})^{\dagger}(\chi^{D})^{D,\dagger} &= (\chi^{D})^{D,\dagger}(\chi^{D,\dagger})^{\dagger} \\ &= \begin{bmatrix} \chi_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \chi_1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= (\chi^{D}\chi^{D,\dagger})^{\dagger}.\end{aligned}$$

(ii) The computation of $\chi\chi^{D,\dagger}$ results in $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ which is an idempotent. Now the value of $(\chi^{D,\dagger})^{\dagger} = \begin{bmatrix} \chi_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$ can be easily calculated. The remainder of the proof proceeds via direct computation.

□

Theorem 2.4 *Let $\chi \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be a closed range operator with matrix representation of χ with $\mathcal{H} = \mathcal{K} = \mathcal{R}(\chi^*) \oplus \mathcal{N}(\chi) : \chi = \begin{bmatrix} \chi_1 & 0 \\ \chi_3 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix} \to \begin{bmatrix} R(\chi^*) \\ N(\chi) \end{bmatrix}$ where χ_1 is invertible. Then*

(i) $(\chi^{\dagger,D}\chi^{D})^{\dagger} = (\chi^{D})^{\dagger,D}(\chi^{\dagger,D})^{\dagger} = (\chi^{\dagger,D})^{\dagger}(\chi^{D})^{\dagger,D}$

(ii) $(\chi\chi^{\dagger,D})^{\dagger,D} = (\chi^{\dagger,D})^{\dagger}(\chi^{\dagger,D}) = \chi^{\dagger,D}(\chi^{\dagger,D})^{\dagger}$

Proof: (i) Taking the expressions for $\chi^{\dagger,D}$ and χ^{D} from Lemma 1.3, we obtain

$$\begin{aligned}\chi^{\dagger,D}\chi^{D} &= \begin{bmatrix} \chi_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \chi_1^{-1} & 0 \\ \chi_3(\chi_1^{-1})^2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (\chi_1^{-1})^2 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Since χ_1 is invertible, then $(\chi^{\dagger,D}\chi^{D})^{\dagger} = \begin{bmatrix} \chi_1^2 & 0 \\ 0 & 0 \end{bmatrix}$.

Furthermore, the following expression can be readily computed $(\chi^{D})^{\dagger,D} = \begin{bmatrix} \chi_1 & 0 \\ 0 & 0 \end{bmatrix}$.

Thus,

$$(\chi^{\dagger,D}\chi^{D})^{\dagger} = (\chi^{D})^{\dagger,D}(\chi^{\dagger,D})^{\dagger} = (\chi^{\dagger,D})^{\dagger}(\chi^{D})^{\dagger,D}.$$

(ii) Analogously, it can be readily confirmed that $(\chi\chi^{\dagger,D})^{\dagger,D} = (\chi^{\dagger,D})^{\dagger}\chi^{\dagger,D} = \chi^{\dagger,D}(\chi^{\dagger,D})^{\dagger}$. □

Since the reverse order law does not hold universally for all generalized inverse operators, the following theorems establish mixed-type order laws, which provide sufficient conditions under which the reverse and forward order laws coincide.

Theorem 2.5 *Let $\chi \in \mathcal{B}(\mathcal{R}(\chi) \oplus \mathcal{N}(\chi^*))$ have a closed range. If χ_1 is invertible, then*

(i) $(\chi^{D,\dagger}\chi^{D})^{D}\chi = (\chi^{D})^{D}(\chi^{D,\dagger})^{\dagger}\chi = (\chi^{D,\dagger})^{\dagger}(\chi^{D})^{D}\chi$

(ii) $(\chi\chi^{D,\dagger})^{\dagger}\chi^{D} = (\chi^{D,\dagger})^{\dagger}\chi^{D}\chi^{D} = \chi^{D}(\chi^{D,\dagger})^{\dagger}\chi^{D}$.

Proof: (i) Computing $\chi^{D,\dagger}\chi^{D} = \begin{bmatrix} (\chi_1^{-1})^2 & (\chi^{-1})^3\chi_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$.

Since χ_1 is invertible, the following expression can be computed directly: $(\chi^{D,\dagger}\chi^{D})^{D} = \begin{bmatrix} \chi_1^2 & \chi_1\chi_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$.

Now, $(\chi^{D,\dagger}\chi^{D})^{D}\chi = \begin{bmatrix} \chi_1^3 & \chi_1^2\chi_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$.
Using the Lemma 1.2, we can calculate that $(\chi^{D})^{D} = \chi$, as χ_1 is invertible.
Hence

$$\begin{aligned}(\chi^{D})^{D}(\chi^{D,\dagger})^{\dagger}\chi &= (\chi^{D,\dagger})^{\dagger}(\chi^{D})^{D}\chi \\ &= \begin{bmatrix} \chi_1^3 & \chi_1^2\chi_2 \\ 0 & 0 \end{bmatrix} \\ &= (\chi^{D,\dagger}\chi^{D})^{D}\chi.\end{aligned}$$

(ii) This portion of the proof follows from a direct computation. □

Theorem 2.6 *Let $\chi \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be a closed range operator. Then the matrix representation of χ with respect to the orthogonal decompositions of Hilbert spaces $\mathcal{H} = \mathcal{K} = \mathcal{R}(\chi) \oplus \mathcal{N}(\chi^*)$: $\chi = \begin{bmatrix} \chi_1 & \chi_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$ where χ_1 is invertible. Further,*

(i) $(\chi^{D,\dagger}\chi^{D})^{D}\chi = (\chi^{D})^{D}(\chi^{D,\dagger})^{\dagger}\chi = (\chi^{D,\dagger})^{\dagger}(\chi^{D})^{D}\chi$.

(ii) $(\chi\chi^{D,\dagger})^{\dagger}\chi^{D} = (\chi^{D,\dagger})^{\dagger}\chi^{D}\chi^{D} = \chi^{D}(\chi^{D,\dagger})^{\dagger}\chi^{D}$.

Proof: Computation of $\chi^{D,\dagger}\chi^{D}$ yields $\begin{bmatrix} (\chi_1^{-1})^2 & (\chi_1^{-1})^3\chi_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$.

Since χ_1 is invertible, we can easily calculate that $(\chi^{D,\dagger}\chi^{D})^{D} = \begin{bmatrix} (\chi_1)^2 & \chi_1\chi_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}$.

Multiplying χ with the previous result, we get

$$(\chi^{D,\dagger}\chi^{D})^{D}\chi = \begin{bmatrix} (\chi_1)^3 & (\chi_1)^2\chi_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix} \to \begin{bmatrix} R(\chi) \\ N(\chi^*) \end{bmatrix}.$$

Further using the Lemma 1.2, we can compute that $(\chi^{D})^{D} = \chi$, as χ_1 is invertible.

Hence,

$$\begin{aligned}(\chi^{D})^{D}(\chi^{D,\dagger})^{\dagger}\chi &= (\chi^{D,\dagger})^{\dagger}(\chi^{D})^{D}\chi \\ &= \begin{bmatrix} (\chi_1)^3 & (\chi_1)^2\chi_2 \\ 0 & 0 \end{bmatrix} \\ &= (\chi^{D,\dagger}\chi^{D})^{D}\chi.\end{aligned}$$

(ii) The proof of this part involved straightforward calculations.

□

3. Absorption Law

In this section, we have provided sufficient conditions under which the absorption law holds for generalized invertible operators.

Theorem 3.1 *Let the operater $\chi \in \mathcal{B}(\mathcal{R}(\chi^*) \oplus \mathcal{N}(\chi))$ have closed range. If χ_1 is invertible, then*

(i) $(\chi^{\dagger,D})^{\dagger} + (\chi^{\dagger})^{D,\dagger} = (\chi^{\dagger,D})^{\dagger}(\chi^{\dagger,D} + \chi^{\dagger})(\chi^{\dagger})^{D,\dagger} = (\chi^{\dagger})^{D,\dagger}(\chi^{\dagger,D} + \chi^{\dagger})(\chi^{\dagger,D})^{\dagger}$.

(ii) $\chi^{\dagger,D} + (\chi^D)^{\dagger,D} = \chi^{\dagger,D}(\chi + \chi^D)(\chi^D)^{\dagger,D} = (\chi^D)^{\dagger,D}(\chi + \chi^D)\chi^{\dagger,D}$.

(iii) $(\chi^D)^{\dagger,D} + (\chi^{\dagger,D})^{\dagger} = (\chi^D)^{\dagger,D}(\chi^D + \chi^{\dagger,D})(\chi^{\dagger,D})^{\dagger} = (\chi^{\dagger,D})^{\dagger}(\chi^D + \chi^{\dagger,D})(\chi^D)^{\dagger,D}$.

Proof: (i) Since χ_1 and D are both invertible, using (iv) of Lemma 1.2, we obtain the invertibility $(\chi^{\dagger})^{D,\dagger} = \begin{bmatrix} (\chi_1^*)^{-1}D & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(\chi^*) \\ \mathcal{N}(\chi) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(\chi^*) \\ \mathcal{N}(\chi) \end{bmatrix}$, where $D = \chi_1^*\chi_1 + \chi_3^*\chi$ maps $\mathcal{R}(\chi^*)$ into itself.

Similarly, $(\chi^{\dagger,D})^{\dagger} = \begin{bmatrix} \chi_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(\chi^*) \\ \mathcal{N}(\chi) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(\chi^*) \\ \mathcal{N}(\chi) \end{bmatrix}$, as χ_1 is invertible.

Now addition of $(\chi^{\dagger,D})^{\dagger} + (\chi^{\dagger})^{D,\dagger}$, shows that

$$(\chi^{\dagger,D})^{\dagger} + (\chi^{\dagger})^{D,\dagger} = \begin{bmatrix} \chi_1 + (\chi_1^*)^{-1}D & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(\chi^*) \\ \mathcal{N}(\chi) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(\chi^*) \\ \mathcal{N}(\chi) \end{bmatrix}.$$

Further solving

$$\begin{aligned} (\chi^{\dagger,D})^{\dagger}(\chi^{\dagger,D} + \chi^{\dagger})(\chi^{\dagger})^{D,\dagger} &= \begin{bmatrix} \chi_1 & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} \chi_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D^{-1}\chi_1^* & D^{-1}\chi_3^* \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} (\chi_1^*)^{-1}D & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \chi_1 + (\chi_1^*)^{-1}D & 0 \\ 0 & 0 \end{bmatrix} \\ &= (\chi^{\dagger,D})^{\dagger} + (\chi^{\dagger})^{D,\dagger}. \end{aligned}$$

Similarly,

$$\begin{aligned} (\chi^{\dagger})^{D,\dagger}(\chi^{\dagger,D} + \chi^{\dagger})(\chi^{\dagger,D})^{\dagger} &= \begin{bmatrix} (\chi_1^*)^{-1}D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\chi_1^*)^{-1}D\chi_1^{-1} + I & (\chi_1^*)^{-1}\chi_3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \chi_1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= (\chi^{\dagger,D})^{\dagger} + (\chi^{\dagger})^{D,\dagger}. \end{aligned}$$

The arguments for the subsequent parts proceed analogously to the proof of part (i). □

Theorem 3.2 *Let $\chi \in (\mathcal{R}(\chi) \oplus \mathcal{N}(\chi^*))$ have closed range. If χ_1 is invertible, then*

(i) $(\chi^{D,\dagger})^{\dagger} + (\chi^{\dagger})^{\dagger,D} = (\chi^{D,\dagger})^{\dagger}(\chi^{D,\dagger} + \chi^{\dagger})(\chi^{\dagger})^{\dagger,D} = (\chi^{\dagger})^{\dagger,D}(\chi^{D,\dagger} + \chi^{\dagger})(\chi^{D,\dagger})^{\dagger}$.

(ii) $(\chi^{D,\dagger})^{\dagger} + (\chi^D)^{D,\dagger} = (\chi^{D,\dagger})^{\dagger}(\chi^{D,\dagger} + \chi^D)(\chi^D)^{\dagger,D} = (\chi^D)^{\dagger,D}(\chi^{D,\dagger} + \chi^{\dagger})(\chi^{D,\dagger})^{\dagger}$.

Proof:

(i) Using the values of all the required expressions from the preceding theorems and lemmas, we find that

$$\begin{aligned} (\chi^{D,\dagger})^{\dagger}(\chi^{D,\dagger} + \chi^{\dagger})(\chi^{\dagger})^{\dagger,D} &= \begin{bmatrix} \chi_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \chi_1^{-1} + \chi_1^* D^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D(\chi_1^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} D(\chi_1^*)^{-1} + \chi_1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Similarly easy multiplication of $(\chi^{\dagger})^{\dagger,D}(\chi^{D,\dagger} + \chi^{\dagger})(\chi^{D,\dagger})^{\dagger}$ show that,

$$\begin{bmatrix} D(\chi_1^*)^{-1} + \chi_1 & 0 \\ 0 & 0 \end{bmatrix} = (\chi^{\dagger})^{\dagger,D}(\chi^{D,\dagger} + \chi^{\dagger})(\chi^{D,\dagger})^{\dagger}.$$

Now it is easy to verify that,

$$(\chi^{D,\dagger})^{\dagger} + (\chi^{\dagger})^{\dagger,D} = (\chi^{D,\dagger})^{\dagger}(\chi^{D,\dagger} + \chi^{\dagger})(\chi^{\dagger})^{\dagger,D} = (\chi^{\dagger})^{\dagger,D}(\chi^{D,\dagger} + \chi^{\dagger})(\chi^{D,\dagger})^{\dagger}.$$

(ii) As $(\chi^{D,\dagger})^{\dagger} = (\chi^D)^{D,\dagger}$ lies in $\mathcal{R}(\chi) \oplus \mathcal{N}(\chi^*)$, part (ii) follows immediately.

□

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