Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 2** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.78826

New Classes of Meir-Keeler Type Contractive Mappings

Deepak Khantwal[1] and Rajendra Pant[1,2*]

ABSTRACT: In this paper, we answer some open questions raised in [Filomat 34 (2020), no. 11, 3855–3860] by introducing some new classes of contractive mappings, including several classes as a special case. We establish some existence results for these mappings and present illustrative examples in their support. Moreover, we demonstrate that several well-known results follow as direct consequences of our findings.

Key Words: Fixed point, continuity, Banach contraction, Meir-Keeler contraction.

Contents

1. Introduction

We start this paper by listing some Meir-Keeler-type conditions. We consider (W, ϱ) a metric space and $P : W \to W$ a self-mapping and Φ denotes the class of mappings $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\varphi(\ell) < \ell$ for $\ell > 0$ throughout the paper. Let us consider the following conditions for $i = 1, 2, 3, 4$.

(a_i) For any $\varepsilon > 0 \ \exists \ \delta > 0$ such that for $\omega, \nu \in W$,

$$\varepsilon \le m_i(\omega, \nu) < \varepsilon + \delta \Rightarrow \varrho(P\omega, P\nu) < \varepsilon,$$

(b_i) for any $\varepsilon > 0 \ \exists \ \delta > 0$ such that for $\omega, \nu \in W$,

$$\varepsilon < m_i(\omega, \nu) < \varepsilon + \delta \Rightarrow \varrho(P\omega, P\nu) \le \varepsilon,$$

(c_i) $\varrho(P\omega, P\nu) < m_i(\omega, \nu)$ for $\omega, \nu \in W$ with $m_i(\omega, \nu) > 0$,

(d_i) $\varrho(P\omega, P\nu) \le \varphi(m_i(\omega, \nu))$ for $\omega, \nu \in W$ and $\varphi \in \Phi$,

where

$$\begin{aligned}
m_1(\omega, \nu) &= \varrho(\omega, \nu),\\
m_2(\omega, \nu) &= \max\{\varrho(\omega, P\omega), \varrho(\nu, P\nu)\}\\
m_3(\omega, \nu) &= \max\{\varrho(\omega, \nu), \varrho(\omega, P\omega), \varrho(\nu, P\nu)\},\\
m_4(\omega, \nu) &= \max\left\{\varrho(\omega, \nu), \varrho(\omega, P\omega), \varrho(\nu, P\nu), \frac{\varrho(\omega, P\nu) + \varrho(\nu, P\omega)}{2}\right\}.
\end{aligned}$$

* Corresponding author.
2010 *Mathematics Subject Classification:* 47H10, 54H25.
Submitted September 07, 2025. Published December 19, 2025

It is evident that (d_i) implies to (c_i). In [11] Jackymski studied various (ε,δ)-contractive conditions and have the following observations:

(i) $(a_i) \implies (b_i)$ and (c_i) but not conversely

(ii) $(a_1) \implies (a_3) \implies (a_4)$

(iii) $(b_1) \implies (b_3) \implies (b_4)$.

The condition (a_1) is an original Meir-Keeler contraction (MKC, in short). These mappings not only contain the classical contractions but also some nonlinear contractions previously explored by Matkowski [17], Browder [6], and Boyd and Wong [5]. Matkowski and Wegrzyk (see [18], also [8]) slightly generalized the MKC by proposing the condition (b_1). Note that the condition (b_1) is not sufficient enough to ensure the fixed point of the underlying mapping, so it requires an additional condition of (c_1). Several generalizations of these mappings can be found in [4,11,16,23,27,28].

In 1999, Pant [24] established an existence result for mappings satisfying conditions (b_2) and (d_2). This result does not force the underlying mapping to be continuous at the fixed point and also provides an answer to the Rhoades problem posed in [30]. In 2017, Bisht and Pant [3] provided another answer to the Rhoades problem, considering (ε,δ)-contractive mappings satisfying conditions (b_4) and (d_4) along with an additional assumption of continuity of P^2. In 2020, Joshi et al. [13] further studied (ε,δ)-contractive mappings under an alternative set of conditions. They considered (ε,δ)-contractive mappings satisfying conditions (b_i) and (c_i) for $i = 2$ or 3 under the hypothesis of weak-orbital continuity (see [22]). They established some existence results for these mappings and raised two open questions regarding the solutions of the Rhoades problem:

Q1 Does there exist a solution of Rhoades problem which satisfies (b_i) and (c_i) but not (d_i)?

Q2 Does there exist a Meir-Keeler type (a_i) solution of Rhoades problem which also satisfies (d_i)?

In this paper, we provide affirmative answers to the open questions discussed above. We do this by introducing some new classes of contractive mappings, including MKCs, Proinov contractive mappings, F-contractions, weak contractions and many more. We establish some existence results for these mappings and show that the assumption of continuity of P^2 remains redundant in the main result of [2]. We present some supportive examples and deduce several results as corollaries of our findings. Our findings generalize several existing results in the literature, including some recent results presented in [2,4,13,26,29] and many others.

2. Preliminaries

We denote by $\mathbb{N}$ and $\mathbb{R}^+$, the set of natural numbers and positive real numbers, respectively. We represent the orbit of P at a point $\omega \in W$ by

$$\mathcal{O}(P,\omega) = \{\omega, P\omega, P^2\omega, \dots, P^n\omega, \dots\}.$$

A fixed point of P is called *contractive* (see [28,15]) if all Picard iterates $\{P^n\omega\}$ converge to this fixed point. A mapping $P : W \to W$ is an *orbitally continuous* (see [7]) at $\omega^* \in W$ if $\{\omega_n\} \subseteq \mathcal{O}(P,\omega)$ for some $\omega \in W$ such that $\omega_n \to \omega^*$ implies $P\omega_n \to P\omega^*$ as $n \to \infty$. A mapping $P : W \to W$ is a *k-continuous* (see [21]), $k \in \mathbb{N}$, at $\omega^* \in W$ if for every sequence $\{\omega_n\} \subseteq W$, the condition $P^{k-1}\omega_n \to \omega^*$ implies $P^k\omega_n \to P\omega^*$ as $n \to \infty$. In 2019, authors [22] proposed a weaker assumption than orbital continuity and k-continuity, referred to as *weak orbital continuity.* A mapping $P : W \to W$ is *weakly orbitally continuous* if the set $\left\{\nu \in W : \lim_i P^{n_i}\nu = \omega^* \Rightarrow \lim_i PP^{n_i}\nu = P\omega^*\right\} \neq$ whenever the set $\left\{\omega \in W : \lim_i P^{n_i}\omega = \omega^*\right\} \neq \emptyset$. In 2021, Nguyen [20] proposed the notion of *P-orbitally lower semicontinuous* (P-OLS, in short) which states if $P\colon W \to W$ and $\mathcal{G}\colon W \to \mathbb{R}$ be mappings then $\mathcal{G}$ is called P-OLS at a point $\omega^* \in W$ if, for any sequence $\{\omega_n\} \subseteq \mathcal{O}(P,\omega)$ for some $\omega \in W$, $\lim_{n\to\infty} \omega_n = \omega^* \implies \mathcal{G}(\omega^*) \leq \liminf_{n\to\infty} \mathcal{G}(\omega_n)$. It is evident that these mappings are independent of weakly orbitally continuous mappings (see [20,1]).

In 1999, Pant [24] proved the following result, which answers the Rhoades problem.

Theorem 2.1 *If P is a selfmapping in a* complete metric space *(c.m.s., in short) W that satisfies conditions (b_2) and (d_2), then P has a unique fixed point $\omega^* \in W$. Moreover, P is continuous at ω^* iff* $\lim_{\omega\to\omega^*} m_2(\omega,\omega^*) = 0$.

In 2017, Bisht and Pant [3] provided another answer to the Rhoades problem.

Theorem 2.2 *Let P be a selfmapping in a c.m.s. W such that P^2 is continuous and satisfying conditions (b_4) and (d_4). Then P has a contractive fixed point $\omega^* \in W$. Moreover, P is discontinuous at ω^* iff $\lim_{\omega \to \omega^*} m_4(\omega, \omega^*) \neq 0$.*

3. Generalized (ε, δ)-Contractive Mappings

In this section, we propose the following conditions for $i = 1, 2, 3, 4$.

For each $w \in W$ and given $\varepsilon > 0$, there exist $\delta > 0$ and $n_\varepsilon \in \mathbb{N}$ such that

$$m_i(P^{n_\varepsilon}\upsilon, P^{n_\varepsilon}\nu) < \varepsilon + \delta \text{ implies } \varrho(P^{n_\varepsilon+1}\upsilon, P^{n_\varepsilon+1}\nu) \leq \varepsilon \tag{D_i}$$

for any $\upsilon, \nu \in \mathcal{O}(P, \omega)$.

It is easy to see that a mapping satisfying condition (D_i) also satisfies $(a_i), (b_i), (c_i)$ or (d_i) for each $i \in \{1, 2, 3, 4\}$ and $D_1 \implies D_2 \implies D_3 \implies D_4$.

Definition 3.1 A mapping $P : W \to W$ is said to be a generalized (ε, δ)-contractive mapping if P satisfies condition (D_4).

The following example demonstrates that generalized (ε, δ)-contractive mappings is different from those satisfying either $(a_i), (b_i), (c_i)$ or (d_i).

Example 3.1 *Let $W = \mathbb{R}$ and $P : W \to W$ defined by*

$$P\omega = \begin{cases} -1, & \text{if } \omega < 0, \\ 0, & \text{if } \omega = 0, \\ 1, & \text{if } \omega > 0. \end{cases}$$

Then P is a generalized (ε, δ)-contractive mappings with $\delta(\varepsilon) = \varepsilon$ and $n_\varepsilon = 2$. However P does not satisfy any one of the conditions $(a_i), (b_i), (c_i), (d_i)$ for $i = 1, 2, 3, 4$.

Example 3.2 *Let (W, ϱ) be a usual metric space, where $W = [0, 1]$ and $P : W \to W$ such that*

$$P\omega = \begin{cases} 1 & \text{if } \omega = 1, \\ 0 & \text{if } \omega \neq 1. \end{cases}$$

Then, P satisfies condition (D_4) for each $\omega \in W$.

Example 3.3 *Let (W, ϱ) be a usual metric space, where $W = [-1, 1]$ and $P : W \to W$ such that*

$$P\omega = \begin{cases} \omega, & \text{if } \omega \leq 0, \\ \omega/2, & \text{if } \omega > 0. \end{cases}$$

Then, P is a generalized (ε, δ)-contractive mapping with $\delta = \varepsilon/2$ and $n_\varepsilon = 1$.

4. Main Results

The following lemma is essential for our findings.

Lemma 4.1 *If $P : W \to W$ is a generalized (ε, δ)-contractive mapping satisfies the following condition*

$$\varrho(P\omega, P^2\omega) < \varrho(\omega, P\omega) \text{ for each } \omega \in W \text{ with } \omega \neq P\omega. \tag{4.1}$$

Then the sequence of iterations $\{P^n\omega\}$ for each $\omega \in W$ is Cauchy in W.

Proof: Take $\omega \in W$ and construct a sequence of iterations $\{P^n\omega\}$ for $n \in \mathbb{N}$. We assume that $P^n\omega \neq P^{n+1}\omega$ and so $\varrho(P^n\omega, P^{n+1}\omega) > 0$ for $n \in \mathbb{N}$. Applying condition (4.1), we get

$$\varrho(P^{n+1}\omega, P^{n+2}\omega) < \varrho(P^n\omega, P^{n+1}\omega) \quad \text{for all } n \in \mathbb{N}. \tag{4.2}$$

Let $\varrho_n := \varrho(P^n\omega, P^{n+1}\omega)$. Then from (4.2), the sequence $\{\varrho_n\}$ is strictly decreasing with n. If $\lim\limits_{n\to\infty} \varrho_n = \varepsilon > 0$ and since $m_4(P^n\omega, P^{n+1}\omega) = \varrho_n$, then by hypothesis (D_4), there exist $\delta > 0$ and $n_\varepsilon \in \mathbb{N}$ such that

$$\varrho_n < \varepsilon + \delta \implies \varrho_{n+1} \leq \varepsilon \qquad \text{for } n \geq n_\varepsilon. \tag{4.3}$$

Moreover, $\varrho_n \downarrow \varepsilon$ so $\delta > 0$, there exists $\ell \in \mathbb{N}$ (without any ambiguity we may assume $\ell \geq n_\varepsilon$) such that

$$\varepsilon < \varrho_n < \varepsilon + \delta, \quad \text{whenever } n \geq \ell. \tag{4.4}$$

Then the hypothesis (4.3) implies $\varrho_{n+1} \leq \varepsilon$ for $n \geq \ell$ which fails to (4.4). Hence, $\varrho_n \downarrow 0$ and for $\delta > 0$, there exists $\ell \in \mathbb{N}$ (we may choose $\ell \geq n_\varepsilon$) such that $\varrho_n < \delta/2$ for $n \geq \ell$.

We first prove that

$$\varrho(P^\ell\omega, P^{\ell+m}\omega) < \varepsilon + \delta/2 \tag{4.5}$$

is true for all $m \in \mathbb{N}$. We use the induction method to validate (4.5). For $m = 1$, (4.5) is true as $\varrho_n \downarrow 0$. Suppose that (4.5) is true for some $m \in \mathbb{N}$, we will prove it for $m+1$. By the triangle inequality, we have

$$\varrho(P^\ell\omega, P^{\ell+m+1}\omega) \leq \varrho(P^\ell\omega, P^{\ell+1}\omega) + \varrho(P^{\ell+1}\omega, P^{\ell+m+1}\omega).$$

It would be suffice to prove that $\varrho(P^{\ell+1}\omega, P^{\ell+m+1}\omega) \leq \varepsilon$. For which, we first show that $m_4(P^\ell\omega, P^{\ell+m}\omega) < \varepsilon + \delta$ and then apply the hypothesis (D_4) to achieve it. Since, $\varrho_n < \delta/2$ for $n \geq \ell$ and by assumption $\varrho(P^\ell\omega, P^{\ell+m}\omega) \leq \varepsilon + \delta/2$, we have

$$\begin{aligned}
&\frac{1}{2}\left[\varrho(P^\ell\omega, P^{\ell+m+1}\omega) + \varrho(P^{\ell+1}\omega, P^{\ell+m}\omega)\right] \\
&\leq \frac{1}{2}\left\{ \begin{array}{c} \varrho(P^\ell\omega, P^{\ell+m}\omega) + \varrho(P^{\ell+m}\omega, P^{\ell+m+1}\omega) \\ +\varrho(P^{\ell+1}\omega, P^\ell\omega) + \varrho(P^\ell\omega, P^{\ell+m}\omega) \end{array} \right\} \\
&< \varepsilon + \delta.
\end{aligned}$$

Thus, $m_4(P^\ell\omega, P^{\ell+m}\omega) < \varepsilon + \delta$ and then apply the hypothesis (D_4), we have $\varrho(P^{\ell+1}\omega, P^{\ell+m+1}\omega) \leq \varepsilon$. Thus, by the induction method, the condition (4.5) holds for every $m \in \mathbb{N}$. Hence $\{P^n\omega\}$ is a Cauchy sequence by hypothesis (D_4). □

We present the following result which ensures the existence of fixed points for generalized (ε, δ)-contractive mappings.

Theorem 4.1 *Let P be a generalized (ε, δ)-contractive mapping satisfying condition (c_4) in a c.m.s. W. If P satisfies any one of the following assumptions:*

(1) *P is weakly orbitally continuous;*

(2) *the mapping $\omega \mapsto \varrho(\omega, P\omega)$ is P-OLS;*

then P has a contractive fixed point.

Proof: Taking $\nu = P\omega$ in (c_4) reduces to (4.1). Hence from Lemma 4.1, we have the sequence of iterates $\{P^n\omega\}$ for each $\omega \in W$ is Cauchy in W. Since W is complete, there exists $\omega^* \in W$ such that $P^n\omega \to \omega^*$ and the set $\{\omega \in W : \lim\limits_i P^{n_i}\omega = \omega^*\} \neq \emptyset$. Now, if hypothesis (1) is true, then by assumption of weak orbital continuity, there exists $\nu \in W$ for which $P^n\nu \to \omega^*$ and $P^{n+1}\nu \to P\omega^*$. However, it follows from the uniqueness of the limit that $\omega^* = P\omega^*$. Next, suppose that hypothesis (2) is true. Since $\{P^n\omega\} \subseteq \mathcal{O}(P, \omega)$ and $P^n\omega \to \omega^*$ with $\varrho(P^n\omega, PP^n\omega) = \varrho(P^n\omega, P^{n+1}\omega) \to 0$ as $n \to \infty$, by the P-OLS of $\omega \mapsto \varrho(\omega, P\omega)$, we get $\varrho(\omega^*, P\omega^*) \leq \liminf_{n\to\infty} \varrho(P^n\omega, P^{n+1}\omega) = 0$, which implies that $\varrho(\omega^*, P\omega^*) = 0$, i.e., $P\omega^* = \omega^*$. Hence, ω^* is a contractive fixed point of P. □

Theorem 4.2 *Theorem 4.1 remains true by replacing the hypothesis of weak orbital continuity with the assumption of either continuity, orbital continuity, or k-continuity.*

Theorem 4.3 (Meir-Keller Contraction Principle) *Let P be a mapping satisfying condition (a_1) in a c.m.s. W. Then P has a contractive fixed point.*

Remark 4.1 *Theorem 4.1 provides an answer to Question 1 raised in [13] by Joshi et al.*

Our next result provides an answer to Question 2 posed in [13]

Theorem 4.4 *Let P be a generalized (ε, δ)-contractive mapping satisfying (d_4) in a c.m.s. W. Then P has a contractive fixed point $\omega^* \in W$. Moreover, ω^* is a point of continuity for P iff $\lim\limits_{\omega \to \omega^*} m_4(\omega, \omega^*) = 0$.*

Proof: Taking $\nu = P\omega$ in (d_4), it reduces to $\varrho(P\omega, P^2\omega) \leq \varphi(\varrho(\omega, P\omega)) < \varrho(\omega, P\omega)$, hence P satisfies the condition (4.1). By Lemma 4.1, it follows that for each $\omega \in W$, $\{P^n\omega\} \subseteq W$ is a Cauchy sequence. Then there has $\omega^* \in W$, by completeness of W, such that $P^n\omega \to \omega^*$. Suppose that $P\omega^* \neq \omega^*$, and let $\varrho(\omega^*, P\omega^*) = \ell > 0$. Since $P^n\omega \to \omega^*$ $\exists\, n_\ell \in \mathbb{N}$ such that $\varrho(P^n\omega, \omega^*) < \ell/2$ for $n \geq n_\ell$. Then for $n \geq n_\ell$,

$$\begin{aligned}\varrho(P^n\omega, P^{n+1}\omega) &\leq \varrho(P^n\omega, \omega^*) + \varrho(P^{n+1}\omega, \omega^*)\\ &< \ell/2 + \ell/2 = \ell,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}[\varrho(P^{n+1}\omega, \omega^*) + \varrho(P^n\omega, P\omega^*)] &\leq \frac{1}{2}[\varrho(P^{n+1}\omega, \omega^*) + \varrho(P^n\omega, \omega^*) + \varrho(\omega^*, P\omega^*)]\\ &< \frac{\ell/2 + \ell/2 + \ell}{2}\\ &= \ell.\end{aligned}$$

Thus,

$$\begin{aligned}m_4(P^n\omega, \omega^*) &= \max\left\{\begin{array}{c}\varrho(P^n\omega, \omega^*), \varrho(\omega^*, P\omega^*), \varrho(P^n\omega, P^{n+1}\omega),\\ \dfrac{\varrho(P^{n+1}\omega, \omega^*) + \varrho(P^n\omega, P\omega^*)}{2}\end{array}\right\}\\ &= \varrho(\omega^*, P\omega^*) \quad \text{for } n \geq n_\ell.\end{aligned}$$

From the hypothesis (d_4), we have

$$\varrho(P^{n+1}\omega, P\omega^*) \leq \varphi(\varrho(\omega^*, P\omega^*)) \quad \text{for all} \quad n \geq n_\ell.$$

Making $n \to \infty$, we get

$$\begin{aligned}\varrho(\omega^*, P\omega^*) &\leq \varphi(\varrho(\omega^*, P\omega^*))\\ &< \varrho(\omega^*, P\omega^*)\end{aligned}$$

a contradiction. Hence, ω^* is a fixed point of P. From (d_4), we can easily prove the uniqueness of ω^*. Thus, ω^* is a contractive fixed point of P.

Suppose that P is continuous at ω^* and $P^n\omega \to \omega^*$. Then, $P^{n+1}\omega \to P\omega^* = \omega^*$, and so $\lim\limits_{n\to\infty} m_4(P^n\omega, \omega^*) = 0$. Conversely, let $\lim\limits_{n\to\infty} m_4(P^n\omega, \omega^*) = 0$ then $\varrho(P^n\omega, \omega^*) \to 0$. Hence $P^n\omega \to \omega^*$ implies $P^{n+1}\omega \to \omega^* = P\omega^*$. Thus P is continuous at ω^*. □

The following result generalizes Theorem 2.1 and Theorem 2.2, and shows that the assumption of continuity of P^2 is redundant in Theorem 2.2.

Theorem 4.5 *Let P be a selfmapping on a c.m.s. W. If P meets the conditions (b_4) and (d_4) then P has a contractive fixed point. Moreover, P is discontinuous at ω^* iff $\lim\limits_{\omega\to\omega^*} m_4(\omega, \omega^*) \neq 0$.*

Proof: The proof follows easily by following the proof of Theorem 4.4. □

Corollary 4.1 *Let P be a selfmapping on a c.m.s. W. If P satisfies the conditions (a_4) then P has a contractive fixed point.*

5. Supportive Examples

Now, we present some illustrative examples in support of our findings.

Example 5.1 *Let (W, ϱ) be a usual metric space, where*

(1) $W = [0, 2]$ and $P : W \to W$ such that

$$P\omega = \begin{cases} \dfrac{\omega+1}{2}, & \text{if } 0 \le \omega \le 1, \\ 0, & \text{if } \omega > 1. \end{cases}$$

If we take $\delta = 1 - \varepsilon, n_\varepsilon = 1$ for each $\omega \in W$ and $\varepsilon > 0$, then P satisfies hypothesis (D_4), also all the assumptions of Theorem 4.1 are true, hence P has a contractive fixed point at $\omega = 1$.

(2) $W = [0, 4]$ and $P : W \to W$ by

$$P\omega = \begin{cases} \omega/3, & \text{if } \omega \text{ is rational}, \\ 0, & \text{otherwise}. \end{cases}$$

Then P satisfies condition (D_4) with $\delta(\varepsilon) = 2\varepsilon/3$ and $n_\varepsilon = 1$ for each $\omega \in W$ and $\varepsilon > 0$, and condition (d_4) with $\varphi(\ell) = \ell/3$. Thus all the assumptions of Theorem 4.4 are ture and P has a contractive fixed point at $\omega = 0$. Note that in this example P and P^2 are not continuous at any point except at $\omega = 0$.

6. Deduced Results

Recently, Proinov [29] established a generalization of the Banach contraction principle (BCP) which includes several other generalizations of the BCP due to Geraghty [10], Dutta and Chaudhary [9], Wardowski [31], Jleli and Samet [12], and many more. In this section, we show that the result of Proinov [29, Theorem 3.1] can be derived from Theorem 4.3 and consequently from our findings.

Theorem 6.1 *Let (W, ϱ) be a c.m.s. and $P : W \to W$ be a mapping satisfying*

$$\psi(\varrho(P\omega, P\nu)) \le \varphi(\varrho(\omega, \nu)) \text{ for all } \omega, \nu \in W, \tag{6.1}$$

where the functions $\psi, \varphi : (0, \infty) \to \mathbb{R}$ satisfy the following conditions:

(i) ψ is nondecreasing;

(ii) for any $\ell > 0$, $\varphi(\ell) < \psi(\ell)$;

(iii) for any $\varepsilon > 0$, $\limsup_{\ell \to \varepsilon^+} \varphi(\ell) < \psi(\varepsilon^+)$.

Then P has a contractive fixed point.

Proof: First we show that a mapping satisfying condition (6.1) also satisfies (a_1). If not then for given $\varepsilon > 0$, there exist $\omega, \nu \in W$ such that

$$\varepsilon \le \varrho(\omega, \nu) < \varepsilon + \delta \implies \varrho(P\omega, P\nu) \ge \varepsilon \text{ for all } \delta > 0. \tag{6.2}$$

Let $S = \{(\omega, \nu) \in W \times W$ for which (6.2) holds$\}$. Then $\varrho(\omega, \nu) > \varepsilon$ for all $(\omega, \nu) \in S$. Contrary, let $\varrho(\omega, \nu) = \varepsilon$ for some $(\omega, \nu) \in S$ then from (6.1) and (6.2), we get

$$\begin{aligned} \psi(\varepsilon) &\le \psi(\varrho(P\omega, P\nu)) \\ &\le \varphi(\varrho(\omega, \nu)) < \psi(\varrho(\omega, \nu)) \\ &\le \psi(\varepsilon), \end{aligned}$$

a contradiction. Now, we consider two cases here:

Case I When S is infinite set. Then there exists a sequence $\{(\omega_n, \nu_n)\}$ in S such that $\varrho(\omega_n, \nu_n) \to \varepsilon^+$. Then from the condition (6.1), we have

$$\psi(\varrho(P\omega_n, P\nu_n)) \leq \varphi(\varrho(\omega_n, \nu_n)).$$

Let $t_n = \varrho(\omega_n, \nu_n)$ for $n \in \mathbb{N}$. Making lim sup, and using property (iii), we get

$$\psi(\varepsilon^+) \leq \limsup_{t_n \to \varepsilon^+} \varphi(t_n) < \psi(\varepsilon^+),$$

a contradiction. Thus S cannot be infinite.

Case II Let S be a finite set and $t = \min \varrho(\omega, \nu)$ for $(\omega, \nu) \in S$. . In this case, since $\varrho(\omega, \nu) > \varepsilon$, there always exists $\delta = t - \varepsilon > 0$ such that antecedent of (a_1) does not satisfy and the condition (a_1) holds obviously. Hence, S is an empty set in this case.

Thus, we conclude that P satisfies condition (a_1) and by Theorem 4.3, it has a contractive fixed point. □

For different choices for ψ and φ in Theorem 6.1, we get different generalizations of the BCP.

1. Taking $\psi(t) = t$, we get the result of Boyd and Wong [5].
2. Taking $\varphi(t) = \alpha(t)\psi(t)$ and $\psi(t) = t$, where $\alpha : (0, \infty) \to (0, 1)$ with $\limsup_{t \to \varepsilon^+} \alpha(t) < 1$, then we get the well-known result of Geraghty [10].
3. Taking $\varphi(t) = \psi(t) - \tau$, where $\tau > 0$, in Theorem 6.1, we get an improved version of Wardowski's [31] result.
4. Taking $\varphi(t) = \psi(t)^\alpha$, where $\alpha \in (0, 1)$, we get a generalized version of Jleli and Samet's [12] result.
5. Setting $\varphi(t) = \psi(t) - \phi(t)$, where $\phi : (0, \infty) \to (0, \infty)$ satisfying $\liminf_{t \to \varepsilon^+} > 0$ for any $\varepsilon > 0$, we get an generalized version of Dutta and Chaudhary's [9] result.

Thus, all these results can be derived from our findings.

7. Conclusion

In this paper, we have introduced some new classes of MKCs which subsume several existing classes as special cases. We established some existence results for these mappings, providing answers to open questions posed in [13] and the Rhoades problem related to the existence of contractive mappings that may exhibit discontinuity at the fixed point. Additionally, we presented illustrative examples to support our findings and deduced several results as direct consequences of our work. Our findings generalize numerous results found in [2,3,4,5,11,19,21,23,24,25,30]. These findings can be extended to more general settings of metric spaces, such as semimetric spaces, b-metric spaces, partial metric spaces, and others.

Acknowledgments

The authors gratefully acknowledge the reviewers for their constructive comments and valuable suggestions.

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[1] *Department of Mathematics and Applied Mathematics*

University of Johannesburg, Kingsway Campus, Auckland Park

Johannesburg 2006, South Africa

E-mail address: deepakkhantwal15@gmail.com, dkhantwal@uj.ac.za (D. Khantwal)

and

[2] *National Institute for Theoretical and Computational Sciences (NITheCS), South Africa*

E-mail address: pant.rajendra@gmail.com, rpant@uj.ac.za (R. Pant)