Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 6** : 1–18.
ISSN-0037-8712
doi:10.5269/bspm.78828

Newly Discovered Classes of Perfect Functions in Bitopological Spaces: Applications and Conclusions

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ABSTRACT: Variable degrees of obscurity and immense quantities of information constitute the characteristics of daily difficulties. Therefore, creating additional mathematical methods to address problems is essential. The ideal tool for this goal is expected to possess the perfect functions, as discussed in this work. Consequently, in this study, we explore the use of several set amplifiers to build perfect functions in bitopological spaces. The associations between some kinds of pairwise perfect functions and their traditional topologies are associated with uniformity. Alignment allows us to investigate the characteristics and actions of traditional topological ideas by studying sets. We present and evaluate a new class of perfect functions in bitopological spaces, which we call P-perfect, S-perfect and B-perfect functions, compact functions in bitoplogical spaces. We additionally identify the connections among classes of generalized functions and this new class of perfect functions. Additionally, we demonstrate this novel concept, explain the related connections identify the prerequisites for their effective use, and provide instances and counter-examples while presenting and evaluating the perfect functions that are suggested here. We look at the images and inverse images of particular topological characteristics to provide new demonstrations regarding each of these functions. Finally, product theorems associated with these ideas have been found.

Keywords: Biopological spaces, functional analysis, B-compact space, Hausdorff space, proper functions.

Contents

1. Overview and Foundational Definitions

There have been numerous broad topological architectures put forward within the last few years. See ([1,2,11,7]) for the significance of the topological space in analysis and in many applications. Perfect functions are among the finest and most significant extensions of topological space. General Topology informs us that the development of the novel configurations and important topological characteristics of contemporary sets depends heavily on open sets. Large amounts of knowledge and different degrees of vagueness are characteristics of everyday challenges. Therefore, it is essential to create innovative mathematical methods to address them. The ideal tool for this goal is expected to be the right functions in this situation. Consequently, in this study, we explore the use of several set processors to build perfect functions. The associations between some kinds of perfect functions and their classical topologies are

2020 *Mathematics Subject Classification:* 54B05, 54B10, 54C05, 54D05, 54D10, 54D30, 54E55
Submitted September 08, 2025. Published February 17, 2026

associated with symmetry. Because of alignment, we may utilize the examination of causes to investigate the characteristics and actions of traditional topological ideas.

Vainstein initially proposed the perfect functions in the scope of locally-compact spaces in 1950 [28]. On the other hand, he constructed and first presented during lessons of perfect functions in the discipline of metric spaces.The study of bitopological spaces was started by Kelly (1963) [20]. A non-empty set $\mathfrak{L}$ with arbitrary topologies η_1, η_2 is called a bitoplogical space $(\mathfrak{L}, \eta_1, \eta_2)$. Pervin (1967) [25], fletcher, et al. (1969) [16], Birsan (1969) [12], Reilly (1970) [26], Datta (1976) [15], Steen (1978)[Steen1978], Hdeib and Fora (1982,1983)([17], [18]), Bose (2008) [14], Kilicman and Salleh (2008, 2009)([21], [22]), and Mahmood(2013) [24], Atoom (2024, 2025) [3,4,5,6] are only a few of the authors who have investigated bitoplogical spaces.

The topologies generalized by the sets using reals gave rise to the concept of bitopological space.

$$B_{\iota_\epsilon} = \{\mathfrak{m} \in \mathfrak{L} \ / \ \iota(\mathfrak{l}, \mathfrak{m}) < \epsilon\}$$

and

$$B_{\zeta_\epsilon} = \{\mathfrak{m} \in \mathfrak{L} \ / \ \zeta(\mathfrak{l}, \mathfrak{m}) < \epsilon\}$$

where ι and ζ are quasi-metric spaces $\mathfrak{L}$ with $\iota(\mathfrak{l}, \mathfrak{m}) = \zeta(\mathfrak{l}, \mathfrak{m})$.

Many topological properties found in single topologies, including compactness, paracompactness, separation axiomes [19], connected functions, and other topics, are generalized into bitopological spaces since Kelly proposed the idea of bitopological spaces in 1963. $(\mathcal{P}^w)$ will be used to indicate pairwise; for example, $\mathcal{P}^w-$compact is an acronym for pairwise compact.These include η_1 and η_2 have feature $\mathbb{Q}$ when $(\mathfrak{L}, \eta_1, \eta_2)$ has it. As an illustration, $(\mathfrak{L}, \eta_1, \eta_2)$ is T_2-space if additionally $(\mathfrak{L}, \eta_1)$ and $(\mathfrak{L}, \eta_2)$ are T_2-spaces. Bitopological spaces are defined and the arguments are introduced, including $\mathcal{P}^w-$continuous [23], $\mathcal{P}^w-$closed [17], $(\mathcal{P}^w-T_2)$ [20], B-compact [15], s-compact [12], $\mathcal{P}^w-$compact [12], will be utilized for determining certain essential data that will support our key discoveries in the future.The sets of reals, rationals, and natural numbers are represented by the letters $\mathbb{R}, \mathbb{Q}$, and $\mathbb{N}$, accordingly. In the context of metric spaces, Vainstein initially proposed the class of perfect functions in 1947.In 1950 and 1951, respectively, Leray and Bourbaki [13] proposed and examined perfect functions (in the context of locally compact spaces). Whenever $\mathfrak{L}$ is a Hausdorff space, ψ is closed, and the fibers $\psi^{-1}(\mathfrak{m})$ are compact subsets of $\mathfrak{L}$, then a continuous function $\psi : \mathfrak{L} \to \mathfrak{M}$ is considered perfect. Subsequently, a number of mathematicians studied on perfect functions and demonstrated a number of findings on their impact on various topological spaces. S. Balasubramanian, (2010) [10], Hdeib (1982) [18], and Atoom (2024) [8,9] are a few examples. Determining pairwise perfect functions in bitopological spaces and examining some of their features and implications on various types of spaces are the goals of this work. Four perfect functions in bitopological spaces will be introduced in this work. According to paired perfect functions, we provide different descriptions of these perfect functions, images, and inverse images having specific bitopological features. In the bitopological spaces, we provide a few combination theorems in these perfect functions. We also show how the newly developed category of perfect functions is related to families of extended functions. and examine them.Illustrations and counter-examples are provided, together with an explanation of the related interconnections and the circumstances required for the implementation to be successful. Additional results are also given for the compact topological spaces and the Hausdorff topological spaces. pictures and inverse pictures of certain topological properties have been examined for each of these functions. Finally, product theorems have been found that relate to these ideas.We also show how the newly developed category of perfect functions is related to families of extended functions and examine them.Illustrations and counter-examples are provided, together with an explanation of the related interconnections and the circumstances required for the implementation to be successful. Additional results are also given for the compact topological spaces and the Hausdorff topological spaces. Images and inverse images of certain topological properties have been examined for each of these functions. Finally, product theorems have been found that relate to these ideas.There are eight distinct parts in the present article. The historical context of bitopological spaces and perfect functions in single topology significant definitions and theorems in bitopological spaces are covered in examined in this initial section. The definition of P_e-perfect functions and the images and inverse images of specific bitoplolgical features underneath pairwise perfect functions are examined in the following section 2. We define S-perfect functions while offering specific description for such perfect functions in the next section 3. In fourth part we construct an entirely novel class entitled B-perfect functions 4. A newly developed function known as compact functions in bitoplogical spaces will be described in the fifth section 5. We offer alternative examples of various categories in the sixth section 6. The implementation of our study and its benefits are presented in the final section 7.

2. pairwise perfect functions and Their Role in Preserving Bitopological Properties

The second part explores the concept of perfect functions in bitopological spaces. Pairwise perfect ($\mathcal{P}^w\mathcal{P}^{ct}$) functions are an assortment of function that arises from these functions. Furthermore, we analyze depending on these terms the pictures and inverse images which have particular bitopological properties. Finally, certain product arguments related to these ideas were found.

Definition 2.1 *A function* $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ *is referred to as* $\mathcal{P}^w\mathcal{P}^{ct}$ *whenever it is* $\mathcal{P}^w-$*continuous,* $\mathcal{P}^w-$*closed, and* $\psi^{-1}(\mathfrak{m})$ *is* $\mathcal{P}^w-$*compact for every* $\mathfrak{m} \in \mathfrak{M}$*.*

Theorem 2.2 *Assuming that* $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ *is a* $\mathcal{P}^w\mathcal{P}^{ct}$ *function, then any* $\mathcal{P}^w-$*compact subset* $(\mathfrak{N}, \iota_1, \iota_2) \subseteq (\mathfrak{M}, \zeta_1, \zeta_2)$ *has an inverse image* $\psi^{-1}(\mathfrak{N}, \iota_1, \iota_2)$*.*

Proof. Allow $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be a $\mathcal{P}^w-$open cover of $\mathfrak{L}$, with which $h_\alpha \in \eta_1$, $\alpha \in \chi$. And $\mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2)$, $\mathfrak{M} = (\mathfrak{M}, \zeta_1, \zeta_2)$, $\mathfrak{N} = (\mathfrak{N}, \iota_1, \iota_2)$.

For this reason $\forall \mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$compact, and $\psi^{-1}(\mathfrak{N})$ is undoubtedly $\mathcal{P}^w-$Hausdorff space, Just demonstrating that for every $\mathcal{H}$ of $\mathcal{P}^w-$open cover of $\mathfrak{L}$, which union consists of $\psi^{-1}(\mathfrak{N})$, $\exists$ a finite subsets $\chi_\mathfrak{m}$, $\chi_\mathfrak{m}^\setminus$ of χ, that is to say

$$\psi^{-1}(\mathfrak{N}) \subseteq \cup_{\alpha \in \chi_\mathfrak{m}} \{h_\alpha : \alpha \in \chi_\mathfrak{m}\} \cup \cup_{\alpha \in \chi_\mathfrak{m}^\setminus} \{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\},$$

at which $\{h_\alpha : \alpha \in \chi_\mathfrak{m}\}$ is η_1-open, $\{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\}$ is η_2-open. Permit S_1, S_2 be the family of finite subsets of $\chi_\mathfrak{m}$, $\chi_\mathfrak{m}^\setminus$, and

$$\mathcal{H}_\mathcal{B} = \cup_{\substack{\alpha \in \mathcal{B} \\ \mathcal{B} \in S_1}} \{h_\alpha : \alpha \in \chi_\mathfrak{m}\} \cup \cup_{\substack{\alpha \in \mathcal{B} \\ \mathcal{B} \in S_2}} \{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\},$$

Additionally, for every $\mathfrak{n} \in \mathfrak{N}$, $\psi^{-1}(\mathfrak{n})$ is a $\mathcal{P}^w-$compact, consequently, it is included in the set $\mathcal{H}_\mathcal{B}$ for some $\alpha \in \mathcal{B}$, here are the following:

$$\mathfrak{n} \in \frac{\mathfrak{M}}{(\psi(\mathfrak{L}) \setminus \mathcal{H}_\mathcal{B})}$$

and

$$\mathfrak{N} \subset \mathcal{H}_\mathcal{B} \in \frac{\mathfrak{M}}{(\psi(\mathfrak{L}) \setminus \mathcal{H}_\mathcal{B})}, \exists \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_K \in S_1, and \exists \mathcal{B}_1^\setminus, \mathcal{B}_2^\setminus, \dots, \mathcal{B}_K^\setminus \in S_2,$$

$\mathfrak{N} \subset \cup_{i=1}^k \frac{\mathfrak{M}}{(\psi(\mathfrak{L}) \setminus \mathcal{H}_{\mathcal{B}_i})}$,

$$\psi^{-1}(\mathfrak{N}) \subset \cup_{i=1}^k \frac{\psi^{-1}(\mathfrak{M})}{(\psi(\mathfrak{L}) \setminus \mathcal{H}_{\mathcal{B}_i})} = \cup_{i=1}^k \frac{\mathfrak{L}}{\psi^{-1}\psi(\mathfrak{L}) \setminus \mathcal{H}_{\mathcal{B}_i}} \subset \cup_{i=1}^k \frac{\mathfrak{L}}{\mathfrak{L} \setminus \mathcal{H}_{\mathcal{B}_i}} = \cup_{i=1}^k \mathcal{H}_{\mathcal{B}_i} = \mathcal{H},$$

where $\chi_\mathfrak{m} = \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_K$, $\chi_\mathfrak{m}^\setminus = \mathcal{B}_1^\setminus, \mathcal{B}_2^\setminus, \dots, \mathcal{B}_K^\setminus$. ■

Corollary 2.3 *Several* $\mathcal{P}^w\mathcal{P}^{ct}$ *functions can be mixed to create a* $\mathcal{P}^w\mathcal{P}^{ct}$ *function.*

Lemma 2.4 *Assuming that* $\mathcal{A}$ *be a dense subspace of a* $\mathcal{P}^w-$*Hausdorff space* $\mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2)$*, and* $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ *be* $\mathcal{P}^w-$*continuous function.*
When the $\mathcal{P}^w-$*homomorphism* $\psi \setminus (\mathcal{A}) : \mathcal{A} \to \psi(\mathcal{A}) \subseteq \mathfrak{M}$ *is true, then* $\psi(\mathfrak{L}) \setminus (\mathcal{A})) \cap \psi((\mathcal{A})) = \phi$*.*

Proof. Assume $\exists \mathfrak{l} \in \mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2) \backslash (\mathcal{A})$, such that $\psi(\mathfrak{l}) \in \psi((\mathcal{A}))$, without loss of generality, Considering that $\psi(\mathfrak{l}) \in \psi((\mathcal{A}))$ while sacrificing broadness, allow $\mathfrak{l} \in \mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2) \backslash (\mathcal{A})$.
Allow's say that $\mathfrak{L} = \mathcal{A} \cup \{\mathfrak{l}\}$, $\mathfrak{M} = \psi(\mathcal{A})$, $\psi(\mathfrak{l}) = \psi(\mathfrak{m})$, in which $\mathfrak{m} \in \mathcal{A}$, and $h, z \subset \mathfrak{L}$ be disjoint neighborhoods of $\mathfrak{l}, \mathfrak{m}$ accordingly.
Therefore, $\psi((\mathcal{A}) \backslash z) = \psi \backslash \mathcal{A}(\mathcal{A} \backslash z)$ has become closed in $\mathfrak{L} = \psi(\mathcal{A})$, $\psi^{-1}\psi \backslash \mathcal{A} \backslash \mathfrak{M} = A \backslash z$ is also closed in $\mathfrak{M}$. ■

Please take into account an additional Lemma, which is also applied later in this concept, and the mentioned Lemma will be used in the argument of another proposition:

Lemma 2.5 *It is not possible to $\mathcal{P}^w-$continuously extend a $\mathcal{P}^w\mathcal{P}^{ct}$ function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ over any $\mathcal{P}^w-$Hausdorff space $(\mathfrak{N}, \iota_1, \iota_2)$ that has $(\mathfrak{L}, \eta_1, \eta_2)$ as a suitable subset.*

Proof. Make the assumption that $\mathfrak{F} : (\mathfrak{N}, \iota_1, \iota_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w-$continuous extension to a $\mathcal{P}^w-$Hausdorff space. $(\mathfrak{N}, \iota_1, \iota_2)$ which incorporates $(\mathfrak{L}, \eta_1, \eta_2)$ as a proper subset, with no loss of broadness, providing that $(\mathfrak{N}, \iota_1, \iota_2) = (\mathfrak{L}, \eta_1, \eta_2) \cup \{\mathfrak{l}\}$, where the point $\mathfrak{l}$ is not associated to the $\mathcal{P}^w-$compact, thus $\exists$ open sets $h, z \subset (\mathfrak{N}, \iota_1, \iota_2)$, where $\mathfrak{l} \in h$, $\psi^{-1}(\mathfrak{F}) \subset z$, $h \cap z = \phi$. The set $\psi(\mathfrak{L}, \eta_1, \eta_2) \backslash z)$ is $\mathcal{P}^w-$closed in $(\mathfrak{L}, \eta_1, \eta_2)$, and $\mathfrak{F}^{-1}(\psi(\mathfrak{L}, \eta_1, \eta_2) \backslash z))$ is $\mathcal{P}^w-$closed in $(\mathfrak{N}, \iota_1, \iota_2)$, and $((\mathfrak{L}, \eta_1, \eta_2) \backslash z)) = \psi^{-1}\psi(\mathfrak{L}, \eta_1, \eta_2) \backslash z) \subset (\mathfrak{L}, \eta_1, \eta_2)$, and $\mathfrak{l} \notin z$, implying that $(\mathfrak{L}, \eta_1, \eta_2)$ is $\mathcal{P}^w-$closed in $(\mathfrak{N}, \iota_1, \iota_2)$. ■

Proposition 2.6 *If the composition $\rho \circ \psi$ of the $\mathcal{P}^w-$continuous functions, $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$, $\rho : (\mathfrak{M}, \zeta_1, \zeta_2) \to (\mathfrak{N}, \chi_1, \chi_2)$, is a $\mathcal{P}^w-$closed then the restriction $\rho \backslash \psi((\mathfrak{L}, \eta_1, \eta_2)) : \psi(\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{N}, \chi_1, \chi_2)$ is $\mathcal{P}^w-$closed.*

Proposition 2.7 *Let $\mathfrak{M} = (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w-$Hausdorff space, and $\mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2)$, $\mathfrak{N} = (\mathfrak{N}, \iota_1, \iota_2)$, $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$, $\rho : (\mathfrak{M}, \zeta_1, \zeta_2) \to (\mathfrak{N}, \iota_1, \iota_2)$ be $\mathcal{P}^w-$continuous functions. If the composition $\rho \circ \psi$ is $\mathcal{P}^w\mathcal{P}^{ct}$ function, then ψ and $(\rho \backslash \psi((\mathfrak{L}, \eta_1, \eta_2))$ are $\mathcal{P}^w\mathcal{P}^{ct}$ functions.*

Proof. $\forall \mathfrak{n} \in \mathfrak{N}$, $(\rho \backslash \mathfrak{L}))^{-1}(\mathfrak{n}) = \psi(\mathfrak{L}) \cap \rho^{-1}(\mathfrak{n}) = \psi(\rho\psi)^{-1}(\mathfrak{n})$ is $\mathcal{P}^w-$compact, the reality that $\rho \backslash \mathfrak{L}$ is $\mathcal{P}^w-$closed, from the earlier suggestion, so $(\rho \backslash \psi(\mathfrak{L}))$ are $\mathcal{P}^w\mathcal{P}^{ct}$ function. $\forall \mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m}) = (\rho\psi)^{-1}(\rho^{-1}(\mathfrak{m})(\rho(\mathfrak{m}) \cap \psi^{-1}(\mathfrak{m}))$ is a $\mathcal{P}^w-$compact, in addition to each closed set $\mathfrak{F} \subset \mathfrak{L}$, the function $(\rho\psi) \backslash \mathfrak{F}$ is $\mathcal{P}^w\mathcal{P}^{ct}$, by the initial portion of the evidence, $(\rho \backslash \psi(\mathfrak{F})$ is $\mathcal{P}^w\mathcal{P}^{ct}$ function, so it is $\mathcal{P}^w-$continuously for $\overline{f(\mathfrak{F})}$. Thus $\psi(\mathfrak{F}) = \overline{\psi(\mathfrak{F})}$. Consequently, ψ is $\mathcal{P}^w-$closed function. ■

Proposition 2.8 *If $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w-$closed function. Therefore, regarding any subspace $L \subset (\mathfrak{M}, \zeta_1, \zeta_2)$, then the restriction $\psi_L : f^{-1}(L) \to L$ is closed.*

Corollary 2.9 *If $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w\mathcal{P}^{ct}$ function, therefore for anything $\mathcal{P}^w-$closed, $(\mathcal{A} \subset (\mathfrak{L} = \mathfrak{L}, \eta_1, \eta_2)$, $\mathcal{B} \subset \mathfrak{M} = (\mathfrak{M}, \zeta_1, \zeta_2)$, therefore the limitations $\psi \backslash \mathcal{A} : \mathcal{A} \to (\mathfrak{M}, \zeta_1, \zeta_2)$, $\psi_{\mathcal{B}} : \psi^{-1}(\mathcal{B}) \to \mathcal{B}$ are $\mathcal{P}^w\mathcal{P}^{ct}$ functions.*

Theorem 2.10 *Let $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w-$contiunous function, in which $\mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2)$, $\mathfrak{M} = (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w-$Tychonoff spaces is $\mathcal{P}^w\mathcal{P}^{ct}$. If ψ isn't able to be $\mathcal{P}^w-$continuously extended over any $\mathcal{P}^w-$Hausdorff space $\mathfrak{N} = (\mathfrak{N}, \iota_1, \iota_2)$, it includes $(\mathfrak{L}, \eta_1, \eta_2)$ a proper subspace with $\mathcal{P}^w-$denseness.*

Proof. Assume that instead, $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be not a $\mathcal{P}^w\mathcal{P}^{ct}$ function, so using the earlier theorem, regarding the extension function
$\mathfrak{F} : (\mathcal{B}\mathfrak{L}, \eta_1, \eta_2) \to (\mathcal{B}\mathfrak{M}, \zeta_1, \zeta_2)$, then

$$\mathfrak{F}((\mathcal{B}\mathfrak{L}, \eta_1, \eta_2) \setminus (\mathfrak{L}, \eta_1, \eta_2)) \cap (\mathfrak{M}, \zeta_1, \zeta_2) \neq \phi,$$

in order for ψ to stretch throughout the space $\mathfrak{N} = \mathfrak{F}^{-1}(\mathfrak{M})$, A $\mathcal{P}^w\mathcal{P}^{ct}$ with an amount in k-space is described by the aforementioned theorem. ■

Definition 2.11 *Assuming the argument that follows is true, a bitopological space $(\mathfrak{L}, \eta_1, \eta_2)$ is referred to as k-space: A subset $\mathcal{A} \subseteq (\mathfrak{M}, \zeta_1, \zeta_2)$ is η_1-closed (η_2-closed) in $(\mathfrak{L}, \eta_1, \eta_2)$ iff $\mathcal{A} \cap W$ is η_1-closed (η_2-closed). Regarding each $\mathcal{P}^w-$compact set W in $(\mathfrak{L}, \eta_1, \eta_2)$.*

Theorem 2.12 *When considering a $\mathcal{P}^w-$continuous function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \rightarrow (\mathfrak{M}, \zeta_1, \zeta_2)$, specified on a $\mathcal{P}^w-$Hausdorff space $\mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2)$, to a $\mathfrak{M} = (\mathfrak{M}, \zeta_1, \zeta_2)$ is k-space*

The circumstances listed below are comparable:

1) The function ψ is $\mathcal{P}^w\mathcal{P}^{ct}$.

2) For each one $\mathcal{P}^w-$compact subspace $\mathfrak{N} \subseteq \mathfrak{M}$, then the restriction $\psi_{\mathfrak{n}} : \psi^{-1}(\mathfrak{N}) \rightarrow \mathfrak{N}$ is $\mathcal{P}^w\mathcal{P}^{ct}$.

3) Each and every $\mathcal{P}^w-$compact subspace $\mathfrak{N} \subseteq \mathfrak{M}$ then the inverse image $\psi^{-1}(\mathfrak{N})$ is $\mathcal{P}^w-$compact.

Proof. Observe that (2) and (3) are equal, and that (1) → (2) is evident.

(2) → (1): Let $\mathfrak{m} \in \mathfrak{M} = (\mathfrak{M}, \zeta_1, \zeta_2) : \psi \backslash \mathfrak{m} : \psi^{-1}(\mathfrak{m}) \rightarrow \{\mathfrak{m}\}$ is $\mathcal{P}^w\mathcal{P}^{ct}$, and $\{\mathfrak{m}\}$ is $\mathcal{P}^w-$compact, so $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$compact. It is adequate to demonstrate that ψ is $\mathcal{P}^w-$closed function.

Let $\mathcal{A}$ be any closed subset of $\mathfrak{L}$. With this in mind

$$\psi(\mathcal{A}) \cap \mathfrak{N} = \psi((\mathcal{A}) \cap \psi^{-1}(\mathfrak{N})) = \psi_{\mathfrak{n}}(\mathcal{A}) \cap \psi^{-1}(\mathfrak{N})$$

We obtain $\psi(\mathcal{A})$ is closed in $\mathfrak{N}$ since $\psi \backslash \mathfrak{N}$ is closed, while $\psi(\mathcal{A})$ is closed in $(\mathfrak{M}, \zeta_1, \zeta_2)$ depending on the $\mathcal{P}^w-$compact subset $\mathfrak{N}$ or $\mathfrak{M}$ provided $\mathfrak{M}$ is k-space. ∎

Next, we will examine the inverse and invariant of topological features for $\mathcal{P}^w\mathcal{P}^{ct}$ functions.

Theorem 2.13 *When $\psi : (\mathfrak{L}, \eta_1, \eta_2) \rightarrow (\mathfrak{M}, \zeta_1, \zeta_2)$ onto a space $\mathfrak{M} = (\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w\mathcal{P}^{ct}$, thus $\mathcal{W}(\mathfrak{M}) \leq \mathcal{W}(\mathfrak{L})$.*

Proof. Because for $m \prec \mathbb{N}_0$ is valid, allow $\mathcal{W}(\mathfrak{L}) = m$, resulting in $m \geq \mathbb{N}_0$

Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be a $\mathcal{P}^w-$open cover of $(\mathfrak{L}, \eta_1, \eta_2)$. With this in mind $|\chi| = m$, where η be the family of all finite subset of χ, given that $|\eta| = m$, desire $\{\mathcal{W}_T\}_{T \in \eta} = \mathfrak{M} \setminus \psi(\mathfrak{L}) \setminus \cup_{\alpha \in \chi} \mathcal{H}_\alpha$ be a $\mathcal{P}^w-$open cover $(\mathfrak{M}, \zeta_1, \zeta_2)$, consequently $\mathcal{W}_T$ is open. Let $\mathfrak{m} \in (\mathfrak{M}, \zeta_1, \zeta_2)$ is ζ_1-neighborhood. The kind that $\mathcal{W} \subset \mathfrak{M}$ of $\mathfrak{m}$, the inverse $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$compact subset of $\psi^{-1}(\mathcal{W})$, $\exists T \in \eta$. To the extent that

$$\psi^{-1}(\mathfrak{m}) \subset \cup_{\alpha \in \chi} \mathcal{H}_\alpha \subset \psi^{-1}(\mathcal{W}).$$

Evidently, $\mathfrak{m} \in \mathcal{W}_T$. Considering that $(\mathfrak{M}, \zeta_1, \zeta_2) \setminus \mathcal{W} = \psi(\mathfrak{L}) \setminus \psi^{-1}(\mathcal{W}) \subset \psi(\mathfrak{L}) \setminus \cup_{\alpha \in \chi} \mathcal{H}_\alpha$. For this reason $\mathcal{W}_T \subset \mathcal{W}$. ∎

Theorem 2.14 *Let $\psi : (\mathfrak{L}, \eta_1, \eta_2) \rightarrow (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w-$continuous function, where $(\mathfrak{L}, \eta_1, \eta_2), (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w-$Tychonoff spaces,*

The following circumstances are interchangeable: 1) The function ψ is $\mathcal{P}^w\mathcal{P}^{ct}$.

2) The extension $\psi_\alpha : (\mathcal{B}\mathfrak{L}, \eta_1, \eta_2) \rightarrow (\alpha\mathfrak{M}, \zeta_1, \zeta_2)$ of the function ψ meets the requirements for each $(\alpha\mathfrak{M}, \zeta_1, \zeta_2)$:

$$\mathfrak{F}_\alpha(\mathcal{B}\mathfrak{L}, \eta_1, \eta_2) \setminus (\mathfrak{L}, \eta_1, \eta_2) \subset (\alpha\mathfrak{M}, \zeta_1, \zeta_2) \setminus (\mathfrak{M}, \zeta_1, \zeta_2)$$

.

3) The expansion $\mathfrak{F}_\alpha : (\mathcal{B}\mathfrak{L}, \eta_1, \eta_2) \rightarrow (\alpha\mathfrak{M}, \zeta_1, \zeta_2)$ of the function ψ matches the criteria if $(\alpha\mathfrak{M}, \zeta_1, \zeta_2)$ occurs:

the circumstances

$$\mathfrak{F}_\alpha(\mathcal{B}\mathfrak{L}, \eta_1, \eta_2) \setminus (\mathfrak{L}, \eta_1, \eta_2) \subset (\alpha\mathfrak{M}, \zeta_1, \zeta_2) \setminus (\mathfrak{M}, \zeta_1, \zeta_2)$$

.

Proof. Consider that the function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ and its extension $\mathfrak{F}_\alpha : (\mathcal{B}\mathfrak{L}, \eta_1, \eta_2) \to (\alpha\mathfrak{M}, \zeta_1, \zeta_2)$ are $\mathcal{P}^w\mathcal{P}^{ct}$ functions. Given that ψ is extendable over $(Z, \chi_1, \chi_2) = \mathfrak{F}^{-1}((\mathfrak{M}, \zeta_1, \zeta_2)$, Using the prior theorem, without altering the range, we obtain $(\mathfrak{N}, \chi_1, \chi_2) = (\mathfrak{L}, \eta_1, \eta_2)$, what it implies $\mathfrak{F}^{-1}((\mathfrak{M}, \zeta_1, \zeta_2) \subset (\mathfrak{L}, \eta_1, \eta_2)$, and $\mathfrak{F}_\alpha((\mathcal{B}\mathfrak{L}, \eta_1, \eta_2) \setminus (\mathfrak{L}, \eta_1, \eta_2)) \subset (\alpha\mathfrak{M}, \zeta_1, \zeta_2) \setminus (\mathfrak{M}, \zeta_1, \zeta_2))$,

Therefore, (1) $\to$ (2) is demonstrated, and (2) $\to$ (3) is evident.

(3) $\to$ (1) begin to $\mathfrak{F}_\alpha : (\mathcal{B}\mathfrak{L}, \eta_1, \eta_2) \to (\alpha\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w\mathcal{P}^{ct}$ function, subsequently $\mathfrak{F}_\alpha(\mathfrak{m}) : \mathfrak{F}^{-1}((\mathfrak{M}, \zeta_1, \zeta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w\mathcal{P}^{ct}$ function, given that $\mathfrak{F}^{-1}((\mathfrak{M}, \zeta_1, \zeta_2) = (\mathfrak{L}, \eta_1, \eta_2))$, We receive $\mathfrak{F}_\alpha(\mathfrak{m}) = \psi$.

Whatever it indicates $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w\mathcal{P}^{ct}$ function. ■ We present various image and inverse image results of $\mathcal{P}^w\mathcal{P}^{ct}$ functions in this part of the paper.

Theorem 2.15 *Assume that $(\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w-$compact and that $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w\mathcal{P}^{ct}$ function.*

Proof. Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be a $\mathcal{P}^w-$open cover of $(\mathfrak{L}, \eta_1, \eta_2)$. Since $\forall \mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$compact, $\exists$ finite subsets $\chi_\mathfrak{m}, \chi_\mathfrak{m}^\setminus$ of χ,

such that $\psi^{-1}(\mathfrak{m}) \subseteq \bigcup_{\alpha\in\chi_\mathfrak{m}} \{h_\alpha : \alpha \in \chi_\mathfrak{m}\} \cup \bigcup_{\alpha\in\chi_\mathfrak{m}^\setminus} \{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\}$, where $\{h_\alpha : \alpha \in \chi_\mathfrak{m}\}$ is η_1-open and $\{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\}$ is η_2-open.

Define $O_\mathfrak{m} = \mathfrak{M}\setminus\psi(\mathfrak{L}\setminus \bigcup_{\alpha\in\chi_\mathfrak{m}} h_\alpha)$ as a ζ_1-open set containing $\mathfrak{m}$, and $O_\mathfrak{m}^\setminus = \mathfrak{M}\setminus\psi(\mathfrak{L}\setminus \bigcup_{\alpha\in\chi_\mathfrak{m}^\setminus} z_\alpha)$ as a ζ_2-open set containing $\mathfrak{m}$, where:

$$\psi^{-1}(O_\mathfrak{m}) \subseteq \bigcup_{\alpha\in\chi_\mathfrak{m}} h_\alpha \quad \text{and} \quad \psi^{-1}(O_\mathfrak{m}^\setminus) \subseteq \bigcup_{\alpha\in\chi_\mathfrak{m}^\setminus} z_\alpha$$

Let $\mathcal{O} = \{O_\mathfrak{m} : \mathfrak{m} \in \mathfrak{M}\} \cup \{O_\mathfrak{m}^\setminus : \mathfrak{m} \in \mathfrak{M}\}$ be a $\mathcal{P}^w-$open cover of $\mathfrak{M}$. Since $(\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w-$compact, $\mathcal{O}$ has a finite subcover $\{O_{\mathfrak{m}_i} : i = 1, 2, \ldots, n_1\} \cup \{O_{\mathfrak{m}_i}^\setminus : i = 1, 2, \ldots, n_2\}$.

Thus $\mathfrak{M} = \bigcup_{i=1}^{n_1} O_{\mathfrak{m}_i} \cup \bigcup_{i=1}^{n_2} O_{\mathfrak{m}_i}^\setminus$. Therefore:

$$(\mathfrak{L}, \eta_1, \eta_2) = \bigcup_{i=1}^{n_1} \psi^{-1}(O_{\mathfrak{m}_i}) \cup \bigcup_{i=1}^{n_2} \psi^{-1}(O_{\mathfrak{m}_i}^\setminus) \subseteq \text{finite union of } \mathcal{H}$$

Hence $(\mathfrak{L}, \eta_1, \eta_2)$ is $\mathcal{P}^w-$compact. ■

Remark 2.16 *Inverse invariance under a $\mathcal{P}^w\mathcal{P}^{ct}$ function characterizes a $\mathcal{P}^w-$compact space.*

Definition 2.17 *Any space $(\mathfrak{L}, \eta_1, \eta_2)$ that has an open intersection of members of η_i, $i = 1, 2$ is referred to as a $\mathcal{P}^w-$space.*

Lemma 2.18 *Let $\mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2)$ be $\mathcal{P}^w-$Hausdorff space, $\mathcal{P}^w-$space. Then every η_i-compact subset of $(\mathfrak{L}, \eta_1, \eta_2)$ is η_i-closed, $i \neq j$, $i, j = 1, 2$.*

Proof. Let $\mathcal{A}$ be η_i-compact subset of $(\mathfrak{L}, \eta_1, \eta_2)$ and $\mathfrak{l} \in \mathfrak{L} - \mathcal{A}$. Since $(\mathfrak{L}, \eta_1, \eta_2)$ is a $\mathcal{P}^w-$Hausdorff space, $\{\mathfrak{l}\} = \cap_{\alpha\in\chi}\{\overline{h_\alpha^{\eta_i}}, h_\alpha \text{ is } \eta_j\text{-open of } \mathfrak{L} \text{ for } i \neq j,\ i, j = 1, 2\}$. Then $\mathcal{A} \subset \mathfrak{L} - \{\mathfrak{l}\}$, so $\{\mathfrak{L} - \overline{h_\alpha^{\eta_i}}, h_\alpha \text{ is } \eta_j\text{-open of } \mathfrak{L}, \text{ for } i \neq j,\ i, j = 1, 2\}$ is η_i-open cover of η_i-compact set $\mathcal{A}$.

Thus, $\exists \chi_1 \subset \chi$ such that $\mathcal{A} \subseteq \cup_{\alpha\in\chi_1}\{\mathfrak{L} - \overline{h_\alpha^{\eta_i}},\ h_\alpha \text{ is } \eta_j\text{-open of } \mathfrak{L}, \text{ for } i \neq j,\ i, j = 1, 2\}$.

Given that $\mathfrak{L}$ is a $\mathcal{P}^w-$space, we get $\mathcal{H} = \cap_{\alpha\in\chi_1} h_\alpha$ as a η_i-open set with $\mathfrak{l} \in \mathcal{H} \subseteq \mathfrak{L} \setminus \mathcal{A}$, indicating that $\mathcal{A}$ is η_i-closed. ■

Theorem 2.19 *Consider the $\mathcal{P}^w-$continuous bijection $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$.*

ψ is a $\mathcal{P}^w-$homeomorphism when $(\mathfrak{L}, \eta_1, \eta_2)$ is $\mathcal{P}^w-$compact and $(\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w-$Hausdorff space, $\mathcal{P}^w-$space.

Proof. It suffices to demonstrate ψ is $\mathcal{P}^w-$closed. Let $\mathfrak{F}$ be a η_i-closed proper subset of $\mathfrak{L}$. For $i \neq j,\ i, j = 1, 2$, $\mathfrak{F}$ is η_i-compact.

$\psi(\mathfrak{F})$ is ζ_j-compact as a η_i-closed proper subset of $\mathcal{P}^w-$compact space is η_j-compact $(i \neq j,\ i, j = 1, 2)$. Since $(\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w-$Hausdorff $\mathcal{P}^w-$space where every η_i-compact subset is η_i-closed, $\psi(\mathfrak{F})$ is ζ_i-closed. Hence ψ is $\mathcal{P}^w-$homeomorphism. ■

Definition 2.20 *A $\mathcal{P}^w-$strongly function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ satisfies: for each $\mathcal{P}^w-$open cover $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$, there exists $\mathcal{P}^w-$open cover $\mathcal{Z} = \{z_\gamma : \gamma \in \Gamma\}$ of $\mathfrak{M}$ such that $\psi^{-1}(z) \subseteq \cup\{h_\alpha : \alpha \in \chi_1, \chi_1 \subset \chi,\ \text{finite}\},\ \forall z \in \mathcal{Z}$.*
A $\mathcal{P}^w-$weakly function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ satisfies: for any $\mathcal{P}^w-$open cover $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$, there exists $\mathcal{P}^w-$open cover $\mathcal{Z} = \{z_\gamma : \gamma \in \Gamma\}$ of $\mathfrak{M}$ such that $\psi^{-1}(z) \subseteq \cup\{h_\alpha : \alpha \in \chi_1, \chi_1 \subset \chi,\ \text{finite}\},\ \forall z \in \mathcal{Z}$.

Theorem 2.21 *If $\psi : (\mathfrak{L}, \eta_1, \eta_2) \xrightarrow{onto} (\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w-$strong function and $(\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w-$compact, then $(\mathfrak{L}, \eta_1, \eta_2)$ is $\mathcal{P}^w-$compact.*

Proof. Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be a $\mathcal{P}^w-$open cover of $(\mathfrak{L}, \eta_1, \eta_2)$. Since ψ is $\mathcal{P}^w-$strong, there exists $\mathcal{P}^w-$open cover $\mathcal{Z} = \{z_\gamma : \gamma \in \Gamma\}$ of $(\mathfrak{M}, \zeta_1, \zeta_2)$ with

$$\psi^{-1}(z) \subseteq \cup\{h_\alpha : \alpha \in \chi_1, \chi_1 \subset \chi,\ \text{finite}\},\ \forall z \in \mathcal{Z}$$

.

As $(\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w-$compact, $\exists$ finite $\Gamma_1 \subset \Gamma$ such that $\mathfrak{M} = \cup_{\gamma \in \Gamma_1} z_\gamma$. Thus $\mathfrak{L} = \cup_{\gamma \in \Gamma_1} \psi^{-1}(z_\gamma)$. Since each $\psi^{-1}(z_\gamma)$ is covered by finite h_α, $\mathfrak{L}$ is $\mathcal{P}^w-$compact. ■

Definition 2.22 *$\mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2)$ is a bitopological space that is $\mathcal{P}^w-$weakly compact whenever each finite $\mathcal{P}^w-$open cover $\mathcal{H}$ of $\mathfrak{L}$ has a subcover z of $\mathcal{H}$ that is $\mathcal{P}^w-$open finite, which means $\mathfrak{L} = \overline{\cup\{z \backslash z \in z\}}^{\eta_i}$, where $i = 1, 2$.*

Definition 2.23 *The term $\mathcal{P}^w-$pseudo function refers to a function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$.*

Assuming that for every $\mathcal{P}^w-$open cover $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ of $\mathfrak{L}$, there is a corresponding $\mathcal{P}^w-$open cover z of $\mathfrak{M}$, such that for every $z \in \mathcal{Z}$, $\psi^{-1}(z) \subseteq \overline{\cup_{\alpha \in \chi_1} \mathcal{H}_\alpha}^{\eta_i}$, $i = 1, 2$, $\chi_1 \subset \chi$ infinity.

Theorem 2.24 *Given a $\mathcal{P}^w-$continuous, $\mathcal{P}^w-$pseudo function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \xrightarrow{onto} (\mathfrak{M}, \zeta_1, \zeta_2)$,*

In the event that $(\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w-$weakly compact, then $(\mathfrak{L}, \eta_1, \eta_2)$ is as well.

Proof. Assume $(\mathfrak{L}, \eta_1, \eta_2)$ be $\mathcal{P}^w-$open and $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$. ■

When ψ is a $\mathcal{P}^w-$pseudo function, for any $z \in z$. The kind that $\psi^{-1}(z) \subseteq \overline{\cup_{\alpha \in \chi} \mathcal{H}_\alpha}^{\eta_i}$, $i = 1, 2$, $\chi_1 \subset \chi$ finite, we have a $\mathcal{P}^w-$open cover z of $(\mathfrak{M}, \zeta_1, \zeta_2)$. However, given $\mathfrak{M}$ is a $\mathcal{P}^w-$weakly compact space, it includes a $\mathcal{P}^w-$open finite subfamily $\mathcal{W}$ of z that produces $\mathfrak{M} = \overline{\cup\{w \backslash w \in \mathcal{W}\}}^{\zeta_i}$, $i = 1, 2$, and $\mathfrak{L} = \overline{\cup\{\psi^{-1}(w) \backslash w \in \mathcal{W}\}}^{\eta_i}$, $i = 1, 2$, indicating that $\mathfrak{L}$ is a $\mathcal{P}^w-$weakly compact region.

The distinctive features of the topological structure P are described in the next remarks, along with how they relate to other topological spaces.

Remark 2.25 *Whenever $(\eta_i, \eta_j)-P$ is an imitation of a topological property P, then η_i has characteristic P with reference to η_j, and $\mathcal{P}^w-P$ represents the conjugation $(\eta_1, \eta_2)-p$ it is equivalent $(\mathfrak{l}, \eta_i)$ has feature for $i = 1, 2$.*

Remark 2.26 *Permit P be a guarantee of $\mathcal{P}^w\mathcal{P}^{ct}$ functions belonging to the (finitely) cumulative topology. Whenever a $\mathcal{P}^w-$closed subspace $(\mathfrak{L}, \eta_1, \eta_2)$ has a locally finite family $(\mathfrak{L}, \eta_1, \eta_2)$ that is individually a $\mathcal{P}^w-$Hausdorff space with feature P, while $(\mathfrak{L}, \eta_1, \eta_2)$ additionally possesses feature P.*

Remark 2.27 *To be hereditary with regard to $\mathcal{P}^w-$closed subsets of $\mathcal{P}^w-$Hausdorff space, a topology characteristic P must be an opposite variance of a $\mathcal{P}^w\mathcal{P}^{ct}$ function and generational with regard to $\mathcal{P}^w-$open and $\mathcal{P}^w-$closed sets.*

Theorem 2.28 *A k-space is $(\mathfrak{L}, \eta_1, \eta_2)$ whenever there is a $\mathcal{P}^w\mathcal{P}^{ct}$ function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$, of $(\mathfrak{L}, \eta_1, \eta_2)$ onto a k-space $(\mathfrak{M}, \zeta_1, \zeta_2)$.*

Proof. Take $(k\mathfrak{L}, \eta_1, \eta_2)$, $(k\mathfrak{M}, \zeta_1, \zeta_2)$, and the function $k\psi : (k\mathfrak{L}, \eta_1, \eta_2) \to (k\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w\mathcal{P}^{ct}$ function. While $(\mathfrak{M}, \zeta_1, \zeta_2)$ k-space, we obtain $(\mathfrak{M}, \zeta_1, \zeta_2) = (k\mathfrak{M}, \zeta_1, \zeta_2)$, and $k\psi = (\psi_k\mathfrak{L}, \eta_1, \eta_2)$, thus $(\psi_k\mathfrak{L}, \eta_1, \eta_2)$ be a $\mathcal{P}^w\mathcal{P}^{ct}$ function. Although $(\psi_k\mathfrak{L}, \eta_1, \eta_2)$ is $1-1$, it is a $\mathcal{P}^w-$homomorphism. ■

Theorem 2.29 *Assuming that $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w\mathcal{P}^{ct}$ function, $\forall \mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$countably compact, and $(\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w-$countably compact, therefore $(\mathfrak{L}, \eta_1, \eta_2)$ is correct.*

Proof. Enable $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be a $\mathcal{P}^w-$open cover of $(\mathfrak{L}, \eta_1, \eta_2)$.
Given that $\forall \mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$countably compact, $\exists$ a finite subsets $\chi_\mathfrak{m}$, $\chi^\setminus_\mathfrak{m}$ of χ.

That is to say $\psi^{-1}(\mathfrak{m}) \subseteq \cup_{\alpha\in\chi_\mathfrak{m}}\{h_\alpha : \alpha \in \chi_\mathfrak{m}\} \cup \cup_{\alpha\in\chi^\setminus_\mathfrak{m}}\{z_\alpha : \alpha \in \chi^\setminus_\mathfrak{m}\}$, in which $\{u_\alpha : \alpha \in \chi_\mathfrak{m}\}$ is η_1-open, $\{z_\alpha : \alpha \in \chi^\setminus_\mathfrak{m}\}$ is η_2-open.
Enable $O_\mathfrak{m}(\alpha, \mathfrak{m}) = \mathfrak{M} \setminus \psi(\mathfrak{L} \setminus \cup_{\alpha\in\chi_\mathfrak{m}} h_\alpha)$ is a ζ_1-open set comprising $\mathfrak{m}$, and

$$O^\setminus_\mathfrak{m}(\alpha, \mathfrak{m}) = \mathfrak{M} \setminus \psi(\mathfrak{L} \setminus \cup_{\alpha\in\chi^\setminus_\mathfrak{m}} z_\alpha : \alpha \in \chi)$$

is a ζ_2-open set comprising $\mathfrak{m}$, in which

$$\psi^{-1}(O_\mathfrak{m}(\alpha, \mathfrak{m})) \subseteq \cup_{\alpha\in\chi_\mathfrak{m}} h_\alpha,$$

$$\psi^{-1}(O^\setminus_\mathfrak{m}(\alpha, \mathfrak{m})) \subseteq \cup_{\alpha\in\chi^\setminus_\mathfrak{m}} z_\alpha.$$

Turn on

$$\{O\} = \{O_\mathfrak{m}(\alpha, \mathfrak{m}) : \mathfrak{m} \in \mathfrak{M}\} \cup \{O^\setminus_\mathfrak{m}(\alpha, \mathfrak{m}) : \mathfrak{m} \in \mathfrak{M}\}$$

be a $\mathcal{P}^w-$countable compact cover of $\mathfrak{M}$.
Enable $(\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w-$countably compact, $\{O\}$ has $\mathcal{P}^w-$finite subcover indicate that: $\{O_{\alpha i}\}_{i=1}^{n_1}$ and $\{O^\setminus_{\alpha i}\}_{i=1}^{n_2}$, in order

$$(\mathfrak{L}, \eta_1, \eta_2) = \cup_{i=1}^{n_1}\psi^{-1}(O_{\alpha i}) \cup \cup_{i=1}^{n_2}\psi^{-1}(O^\setminus_{\alpha i}).$$

Thus, $(\mathfrak{L}, \eta_1, \eta_2)$ is a $\mathcal{P}^w-$countably compact.
■

Theorem 2.30 *As $(\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w\mathcal{P}^{ct}$ function and $(\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w-$paracompact, then $(\mathfrak{L}, \eta_1, \eta_2)$ is similarly.*

Proof. Allow ourselves to $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be a $\mathcal{P}^w-$open cover of $(\mathfrak{L}, \eta_1, \eta_2)$. Due to $\forall \mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$compact, $\exists$ a finite subsets $\chi_\mathfrak{m}$, $\chi^\setminus_\mathfrak{m}$ of χ. With this in mind

$$\psi^{-1}(\mathfrak{m}) \subseteq \cup_{\alpha\in\chi_\mathfrak{m}}\{h_\alpha : \alpha \in \chi_\mathfrak{m}\} \cup \cup_{\alpha\in\chi^\setminus_\mathfrak{m}}\{z_\alpha : \alpha \in \chi^\setminus_\mathfrak{m}\},$$

at which $\{h_\alpha : \alpha \in \chi_\mathfrak{m}\}$ is η_1-open, $\{z_\alpha : \alpha \in \chi^\setminus_\mathfrak{m}\}$ is η_2-open. Begin by $O_\mathfrak{m} = \mathfrak{M} \setminus \psi(\mathfrak{L} \setminus \cup_{\alpha\in\chi_\mathfrak{m}} h_\alpha)$ is a ζ_1-open set including $\mathfrak{m}$ as well as

$$O^\setminus_\mathfrak{m} = \mathfrak{M} \setminus \psi(\mathfrak{L} \setminus \cup_{\alpha\in\chi^\setminus_\mathfrak{m}} z_\alpha)$$

is a ζ_2-open set including $\mathfrak{m}$, at which

$$\psi^{-1}(O_\mathfrak{m}) \subseteq \cup_{\alpha\in\chi_\mathfrak{m}} h_\alpha,$$

$$\psi^{-1}(O_{\mathfrak{m}}^{\setminus}) \subseteq \cup_{\alpha\in\chi_{\mathfrak{m}}^{\setminus}} z_\alpha.$$

Allow ourselves to $\widetilde{\mathcal{O}} = \{O_{\mathfrak{m}} : \mathfrak{m} \in \mathfrak{M}\} \cup \{O_{\mathfrak{m}}^{\setminus} : \mathfrak{m} \in \mathfrak{M}\}$ be a $\mathcal{P}^w-$open cover of $\mathfrak{M}$. Given that $(\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w-$paracompact, $\widetilde{\mathcal{O}}$ has $\mathcal{P}^w-$open locally finite parallel refinement

indicate that: $\widehat{\mathcal{H}} = \{H_{\mathcal{B}} : \mathcal{B} \in \Gamma_1\} \cup \{H_{\mathcal{B}}^{\setminus} : \mathcal{B} \in \Gamma_2\}$, where $\{H_{\mathcal{B}} : \mathcal{B} \in \Gamma_1\}$ is ζ_1-locally finite paracompact of $O_{\mathfrak{m}}$, and $\{H_{\mathcal{B}}^{\setminus} : \mathcal{B} \in \Gamma_2\}$ is ζ_2-locally finite paracompact of $O_{\mathfrak{m}}^{\setminus}$, $\Gamma = \Gamma_1 \cup \Gamma_2$. Begin to $S_1 = \{\psi^{-1}(H_{\mathcal{B}}) \cap h_{\alpha_i}, i = 1, 2, \ldots, n, \mathcal{B} \in \Gamma_1, \alpha \in \chi_{\mathfrak{m}}\}$ is η_1-open locally finite parallel refinement of $\{h_\alpha : \alpha \in \chi\}$, $S_2 = \{\psi^{-1}(H_{\mathcal{B}}^{\setminus}) \cap z_{\alpha_i}, i = 1, 2, \ldots, n, \mathcal{B} \in \Gamma_2, \alpha \in \chi_{\mathfrak{m}}^{\setminus}\}$ is η_2-open locally finite parallel refinement of $\{z_\alpha : \alpha \in \chi\}$. Permit $\mathcal{S} = \{S_1 \cup S_2\}$, subsequently $\mathcal{S}$ is $\mathcal{P}^w-$open locally finite parallel refinement of $\mathcal{H}$, thereby $(\mathfrak{L}, \eta_1, \eta_2)$ is $\mathcal{P}^w-$paracompact space. ■

Remark 2.31 *Over $\mathcal{P}^w\mathcal{P}^{ct}$, the $\mathcal{P}^w-$paracompact is a the opposite consistent.*

Theorem 2.32 *With $\mathcal{P}^w\mathcal{P}^{ct}$, the $\mathcal{P}^w-$Hausdorff space remains immutable.*

Proof. If $(\mathfrak{L}, \eta_1, \eta_2)$ is a $\mathcal{P}^w-$Hausdorff space, as well as $\mathfrak{m}_1 \neq \mathfrak{m}_2$ in $(\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w\mathcal{P}^{ct}$ function, therefore $\psi^{-1}(\mathfrak{m}_1)$, $\psi^{-1}(\mathfrak{m}_2)$, while $\psi^{-1}(\mathfrak{m}_2)$ are disjoint and subset of $(\mathfrak{L}, \eta_1, \eta_2)$ that is $\mathcal{P}^w-$compact. Here is a $\eta_{1_}$neighborhood h of $\mathfrak{L}$ and a $\eta_{2_}$neighborhood z, $\psi^{-1}(\mathfrak{m}_1) \subseteq h$, $\psi^{-1}(\mathfrak{m}_2) \subseteq z$, $h \cap z = \emptyset$, provided $(\mathfrak{L}, \eta_1, \eta_2)$ is to be a $\mathcal{P}^w-$Hausdorff space. Imagine that $\mathfrak{M} - \psi(\mathfrak{L}\setminus h)$ is a ζ_1-open set in $(\mathfrak{M}, \zeta_1, \zeta_2)$ and that $\mathfrak{M} - \psi(\mathfrak{L} - z)$ is a ζ_2-open set in $(\mathfrak{M}, \zeta_1, \zeta_2)$ and that $\mathfrak{M}_2$. $\mathfrak{M} - \psi(\mathfrak{L} - h) \cap \mathfrak{M} - \psi(\mathfrak{L} - z)$ contains $\mathfrak{m}_1$.

$$\psi(\mathfrak{L} - h) \cup \psi(\mathfrak{L} \setminus z) = \mathfrak{M} - \psi(\mathfrak{L} - h \cap z) = \mathfrak{M} - \psi(\mathfrak{l}) = \emptyset.$$

■

Remark 2.33 *Reverse invariance is exhibited by the $\mathcal{P}^w-$Hausdorff space over $\mathcal{P}^w\mathcal{P}^{ct}$.*

Theorem 2.34 *An inverted resilient space with a $\mathcal{P}^w\mathcal{P}^{ct}$ function is a $\mathcal{P}^w-$regularity space.*

Proof. Attempt to $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w\mathcal{P}^{ct}$ function, $(\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w-$regular space.

For all points $\mathfrak{l}$ in $(\mathfrak{L}, \eta_1, \eta_2)$, every individual $\eta_{1_}$closed set $\mathfrak{F}$. The kind that $\mathfrak{l} \notin \mathfrak{F}$, $\psi^{-1}(\mathfrak{F}(\mathfrak{l})) \cap \mathfrak{F} = K$, K is $\mathcal{P}^w-$compact subset of $(\mathfrak{L}, \eta_1, \eta_2)$, $\mathfrak{l} \notin \mathfrak{F} \subseteq K$. Given that $\psi^{-1}(\mathfrak{F}(\mathfrak{l}))$ is $\mathcal{P}^w-$closed set in $(\mathfrak{M}, \zeta_1, \zeta_2)$, $\mathfrak{l} \notin \mathfrak{F}\setminus z_1$, $\psi(\mathfrak{l}) \notin \psi(\mathfrak{F}\setminus z_1)$, and $(\mathfrak{M}, \zeta_1, \zeta_2)$ is to be $\mathcal{P}^w-$regular space, $\exists$ ζ_1-open set h_2, and ζ_2-open set z_2, $\psi(\mathfrak{l}) \in h_2$, $\psi(\mathfrak{F}\setminus z_1) \subseteq z_2$,

$h = h_1 \cap \psi^{-1}(h_2)$ be $\eta_{1_}$open set, $z = z_1 \cap \psi^{-1}(z_2)$ be $\eta_{2_}$open set, then $\mathfrak{l} \in h$, $\mathfrak{F} \subseteq z$, $h \cap z = \phi$.

Consequently $(\mathfrak{L}, \eta_1, \eta_2)$ is a $\mathcal{P}^w-$regular space. ■

Theorem 2.35 *A space that is both inversely and $\mathcal{P}^w-$locally invariant with a $\mathcal{P}^w\mathcal{P}^{ct}$ function.*

Proof. Let $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w\mathcal{P}^{ct}$ function, $(\mathfrak{L}, \eta_1, \eta_2)$ is a $\mathcal{P}^w-$locally compact,

η_1 is locally compact with respect to η_2, along with $\forall\mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$compact. $\forall\mathfrak{l} \in \mathfrak{L}$, $\eta_{1_}$open set h containing $\mathfrak{l}$, $\exists\mathfrak{l} \in z \subset \eta_2$ cl $z \subset h$, η_2 cl z is $\mathcal{P}^w-$compact. We're going to $\mathcal{W} = \mathfrak{M} \setminus \psi(\mathfrak{L} - z)$ be ζ_1-open set in $(\mathfrak{M}, \zeta_1, \zeta_2)$,

$\mathfrak{m} \in \mathcal{W}$, ζ_1 cl $\mathcal{W} \subseteq \mathfrak{M} - \psi(\mathfrak{L} \setminus \eta_2 \text{ cl } z) \subseteq \psi(\eta_2 \text{ cl } z)$, η_2 cl z is $\mathcal{P}^w-$compact,

$\mathfrak{m} \in \mathcal{W} \subseteq \zeta_1$cl $\mathcal{W}$. To put it simply, if η_2 is locally compact in relation to η_1, then $(\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w-$locally compact.

On the other hand, if $\mathfrak{l}$ is a member of $(\mathfrak{L}, \eta_1, \eta_2)$, then $\psi(\mathfrak{l}) \in (\mathfrak{M}, \zeta_1, \zeta_2)$, $\exists\zeta_1$ open-neighborhood z in $(\mathfrak{M}, \zeta_1, \zeta_2)$, $\ni \psi(\mathfrak{l}) \in z \subseteq \zeta_2$ cl z, where $\psi^{-1}(z) \subseteq \psi^{-1}(\zeta_{1\_}\text{open set})$, and $\psi^{-1}(z) \subseteq \psi^{-1}(\zeta_2 \text{ cl } z)$ is $\mathcal{P}^w-$compact. This means that

$$\mathfrak{l} \in \psi^{-1}(z) \subseteq \psi^{-1}(\zeta_2\text{cl } z) \subseteq \text{cl } \psi^{-1}(z).$$

$\psi(z)$ is $\mathcal{P}^w-$compact, as we can see. $(\mathfrak{L}, \eta_1, \eta_2)$ is consequently a $\mathcal{P}^w-$locally compact. ■

Remark 2.36 *A perfect function is not inversely correlated with normality or complete regularity.*

Theorem 2.37 *Let $\psi_\alpha : (\mathfrak{L}_\alpha, \eta_1, \eta_2) \to (\mathfrak{M}_\alpha, \zeta_1, \zeta_2)$ be a family of functions,*
$\psi = \Pi_{\alpha\in\chi}\psi_\alpha : \Pi_{\alpha\in\chi}(\mathfrak{L}_\alpha, \eta_1, \eta_2) \to \Pi_{\alpha\in\chi}(\mathfrak{M}_\alpha, \zeta_1, \zeta_2)$, *iff* $\psi_\alpha : (\mathfrak{L}_\alpha, \eta_1, \eta_2) \to (\mathfrak{M}_\alpha, \zeta_1, \zeta_2)$ *is* $\mathcal{P}^w\mathcal{P}^{ct}$.

Proof. $\Rightarrow$ It is evident that every function ψ_α is $\mathcal{P}^w\mathcal{P}^{ct}$.
$\Leftarrow$ The cartesian product $\Pi_{\alpha\in\chi}(\mathfrak{L}_\alpha, \eta_1, \eta_2)$ is $\mathcal{P}^w-$Hausdorff space, $\forall \mathfrak{m} \in \{\mathfrak{m}_\alpha\} \in \Pi_{\alpha\in\chi}(\mathfrak{M}_\alpha, \zeta_1, \zeta_2)$, $\psi^{-1}(\mathfrak{m}) = \Pi_{\alpha\in\chi}\psi_\alpha^{-1}(\mathfrak{m}_\alpha)$, is a $\mathcal{P}^w-$compact. Begin to first demonstrate that ψ is a $\mathcal{P}^w-$closed. Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\} \cup \{z_\alpha : \alpha \in \chi\}$ be a $\mathcal{P}^w-$open cover of $(\mathfrak{L}, \eta_1, \eta_2)$, where $h_\alpha \in \eta_1$, $z_\alpha \in \eta_2$, $\alpha \in \chi$.

Since $\forall \mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$compact, $\exists$ a finite subsets $\chi_\mathfrak{m}$, $\chi_\mathfrak{m}^\setminus$ of χ,

$$\psi^{-1}(\mathfrak{m}) \subseteq \cup_{\alpha\in\chi_\mathfrak{m}}\{h_\alpha : \alpha \in \chi_\mathfrak{m}\} \cup \cup_{\alpha\in\chi_\mathfrak{m}^\setminus}\{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\},$$

where $\{h_\alpha : \alpha \in \chi_\mathfrak{m}\}$ is η_1-open, $\{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\}$ is η_2-open,

$$\Pi_{\alpha\in\chi}\psi_\alpha^{-1}(\mathfrak{m}_\alpha) \subseteq \cup_{\alpha\in\chi_\mathfrak{m}}\{h_\alpha : \alpha \in \chi_\mathfrak{m}\} \cup \cup_{\alpha\in\chi_\mathfrak{m}^\setminus}\{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\},$$

$\exists$ an open set $\hat{S}_\alpha$ in $(\mathfrak{L}_\alpha, \eta_1, \eta_2)$, where

$$\hat{S}_\alpha = \{S_\alpha : \alpha \in \chi\} \cup \{S_\alpha^\setminus : \alpha \in \chi\},$$

$\hat{S}_\alpha \neq (\mathfrak{L}_\alpha, \eta_1, \eta_2)$ for $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \sqsubseteq k$,

$$\psi^{-1}(\mathfrak{m}) = \Pi_{\alpha\in\chi}\psi^{-1}(\mathfrak{m}_\alpha) \sqsubseteq \Pi_{\alpha\in\chi}\hat{S}_\alpha \sqsubseteq \mathcal{H}.$$

Since $\{\psi_{\alpha_i}\}_{i=1}^n$ are $\mathcal{P}^w-$closed functions, $\exists$ an open sets

$$O_{\mathfrak{m}_\alpha} = \{O_\alpha : \alpha \in \chi_\mathfrak{m}\} \cup \{O_\alpha^\setminus : \alpha \in \chi_\mathfrak{m}^\setminus\},$$

$$O_{\mathfrak{m}_\alpha} \sqsubseteq (\mathfrak{M}_\alpha, \zeta_1, \zeta_2), \psi_{\alpha_i}^{-1}(O_{\mathfrak{m}_{\alpha_i}}) \sqsubseteq \hat{S}_{\alpha_i}.$$

Let $\tilde{O} = \prod_{\substack{\alpha\in\chi \\ i=1}}^n O_{\mathfrak{m}_{\alpha_i}}$, then $\mathfrak{m} \in \tilde{O}$, $O_{\mathfrak{m}_{\alpha_i}} \neq (\mathfrak{M}_{\alpha_i}, \zeta_1, \zeta_2)$,

$$\psi^{-1}(\tilde{O}) = \prod_{\alpha\in\chi}\psi_\alpha^{-1}(O_{\mathfrak{m}_\alpha}) \sqsubseteq \prod_{\alpha\in\chi}\hat{S}_\alpha \sqsubseteq \mathcal{H}.$$

Hence, ψ is $\mathcal{P}^w-$closed function. ∎

Corollary 2.38 *A finite $\mathcal{P}^w-$closed cover of a $\mathcal{P}^w-$Hausdorff space $(\mathfrak{L}, \eta_1, \eta_2)$ is represented as $\{A_i\}_{i=1}^k$, and $\{\psi_i\}_{i=1}^k$, Since $\psi_i : (\mathcal{A}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is a family of appropriate $\mathcal{P}^w\mathcal{P}^{ct}$ functions, that means that $\psi = \psi_1\nabla\psi_2\ldots\nabla\psi_k$ is a $\mathcal{P}^w\mathcal{P}^{ct}$ function from $(\mathfrak{L}, \eta_1, \eta_2)$ to $(\mathfrak{M}, \zeta_1, \zeta_2)$.*

Definition 2.39 *The constant function can be expressed by $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to p = \{c\}$, whose c is any point that isn't part of $\mathfrak{L}$. Let $(\mathfrak{L}, \eta_1, \eta_2)$ be any $\mathcal{P}^w-$compact bitopological space.*

Theorem 2.40 *The projection $p_\mathfrak{m} : (\mathfrak{L} \times \mathfrak{M}, \eta_1 \times \zeta_1, \eta_2 \times \zeta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w\mathcal{P}^{ct}$ function provided $(\mathfrak{L}, \eta_1, \eta_2)$ is every $\mathcal{P}^w-$compact and $(\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w-$Hausdorff space.*

Proof. Given any $\mathcal{P}^w-$compact $(\mathfrak{L}, \eta_1, \eta_2)$, let $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to p = \{c\}$, where c is some point that is unrelated to $\mathfrak{L}$, be a constant function that is obviously $\mathcal{P}^w\mathcal{P}^{ct}$. Suppose $I_\mathfrak{m} : (\mathfrak{M}, \zeta_1, \zeta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is $\mathcal{P}^w\mathcal{P}^{ct}$ function, $\psi \times I_\mathfrak{m} = \mathfrak{L} \times \mathfrak{M} \to P \times \mathfrak{M} \simeq \mathfrak{M}$, is $\mathcal{P}^w\mathcal{P}^{ct}$ function, but $p_\mathfrak{m} = \psi \times I_\mathfrak{m}$. Thus, $p_\mathfrak{m} : (\mathfrak{L} \times \mathfrak{M}, \eta_1 \times \zeta_1, \eta_2 \times \zeta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is a $\mathcal{P}^w\mathcal{P}^{ct}$ function. ∎

3. S-Perfect Functions in Bitopological Spaces

This section explores s-perfect functions, combines conventional compactness theory with the more subtle framework of bitopological spaces, in which dual topologies coexist. We focus on compact functions, which are functions that preserve compactness across both topologies of a bitopological structure.

Definition 3.1 *An s-perfect function is defined as $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ if ψ is $\mathcal{P}^w-$continuous, $\mathcal{P}^w-$closed, and $\psi^{-1}(\mathfrak{m})$ is s-compact for any $\mathfrak{m} \in \mathfrak{M}$.*

Theorem 3.2 *Every s-compact subset, $(\mathfrak{N}, \iota_1, \iota_2) \subseteq (\mathfrak{M}, \zeta_1, \zeta_2)$, has an inverse image, $\psi^{-1}(\mathfrak{N}, \iota_1, \iota_2)$, that is s-compact provided $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is an s-perfect function.*

Proof. Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be a $\eta_1\eta_2$-open cover of $\mathfrak{L}$, where $h_\alpha \in \eta_1$, $\alpha \in \chi$.
For every family, $\mathcal{H}$ of $\eta_1\eta_2$-open cover of $\mathfrak{L}$, which together includes $\psi^{-1}(\mathfrak{N})$, it is adequate to demonstrate that $\exists$ a finite subsets $\chi_\mathfrak{m}$, $\chi_\mathfrak{m}^\setminus$ of χ, where $\forall \mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$compact, and $\psi^{-1}(\mathfrak{N})$ is obviously $\mathcal{P}^w-$Hausdorff space.

$$\psi^{-1}(\mathfrak{N}) \subseteq \cup_{\alpha \in \chi_\mathfrak{m}} \{u_\alpha : \alpha \in \chi_\mathfrak{m}\} \cup \cup_{\alpha \in \chi_\mathfrak{m}^\setminus} \{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\},$$

where $\{h_\alpha : \alpha \in \chi_\mathfrak{m}\}$ is η_1-open, $\{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\}$ is η_2-open. Take steps to S_1, S_2 finite of $\chi_\mathfrak{m}$, $\chi_\mathfrak{m}^\setminus$, and

$$\mathcal{H}_\mathcal{B} = \cup_{\substack{\alpha \in \mathcal{B} \\ \mathcal{B} \in S_1}} \{h_\alpha : \alpha \in \chi_\mathfrak{m}\} \cup \cup_{\substack{\alpha \in \mathcal{B} \\ \mathcal{B} \in S_2}} \{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\},$$

and for each $\mathfrak{n} \in \mathfrak{N}$, $\psi^{-1}(\mathfrak{n})$ is a s-compact is therefore included in the set $\mathcal{H}_\mathcal{B}$ for some $\alpha \in \mathcal{B}$, $\mathfrak{n} \in \frac{Y}{(\psi(\mathfrak{L}) \setminus \mathcal{H}_\mathcal{B})}$ and $\mathfrak{N} \subset \mathcal{H}_\mathcal{B} \in \frac{Y}{(\psi(\mathfrak{L}) \setminus \mathcal{H}_\mathcal{B})}$, $\exists \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_K \in S_1$, and $\exists \mathcal{B}_1^\setminus, \mathcal{B}_2^\setminus, \ldots, \mathcal{B}_K^\setminus \in S_2$, $\mathfrak{N} \subset \cup_{i=1}^k \frac{\mathfrak{M}}{(\psi(\mathfrak{L}) \setminus \mathcal{H}_{\mathcal{B}_i})}$,

$$\psi^{-1}(\mathfrak{N}) \subset \cup_{i=1}^k \frac{\psi^{-1}(\mathfrak{M})}{(\psi(\mathfrak{L}) \setminus \mathcal{H}_{\mathcal{B}_i})} = \cup_{i=1}^k \frac{\mathfrak{L}}{\psi^{-1}\psi(\mathfrak{L}) \setminus \mathcal{H}_{\mathcal{B}_i}} \subset \cup_{i=1}^k \frac{\mathfrak{L}}{\mathfrak{L} \setminus \mathcal{H}_{\mathcal{B}_i}} = \cup_{i=1}^k \mathcal{H}_{\mathcal{B}_i} = \mathcal{H},$$

where $\chi_\mathfrak{m} = \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_K$, $\chi_\mathfrak{m}^\setminus = \mathcal{B}_1^\setminus, \mathcal{B}_2^\setminus, \ldots, \mathcal{B}_K^\setminus$.
■

Corollary 3.3 *An s-perfect function is created when two s-perfect functions are composed.*

Theorem 3.4 *Let $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be a s-perfect function, and $(\mathfrak{M}, \zeta_1, \zeta_2)$ is a s-compact, then $(\mathfrak{L}, \eta_1, \eta_2)$ is so.*

Proof. Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be a $\eta_1\eta_2$-open cover of $(\mathfrak{L}, \eta_1, \eta_2)$, since $\forall \mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m})$ is s-compact, $\exists$ a finite subsets $\chi_\mathfrak{m}$, $\chi_\mathfrak{m}^\setminus$ of χ,

$\psi^{-1}(\mathfrak{m}) \subseteq \cup_{\alpha \in \chi_\mathfrak{m}} \{h_\alpha : \alpha \in \chi_\mathfrak{m}\} \cup \cup_{\alpha \in \chi_\mathfrak{m}^\setminus} \{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\}$, where $\{h_\alpha : \alpha \in \chi_\mathfrak{m}\}$ is η_1-open, $\{z_\alpha : \alpha \in \chi_\mathfrak{m}^\setminus\}$ is η_2-open.
Let $O_\mathfrak{m} = \mathfrak{M} - \psi(\mathfrak{L} - \cup_{\alpha \in \chi_\mathfrak{m}} h_\alpha)$ is a ζ_1-open set comprising $\mathfrak{m}$, and $O_\mathfrak{m}^\setminus = \mathfrak{M} - \psi(\mathfrak{L} - \cup_{\alpha \in \chi_\mathfrak{m}^\setminus} z_\alpha : \alpha \in \chi)$ is a ζ_2-open set comprising $\mathfrak{m}$,

where $\psi^{-1}(O_\mathfrak{m}) \subseteq \cup_{\alpha \in \chi_\mathfrak{m}} h_\alpha$, $\psi^{-1}(O_\mathfrak{m}^\setminus) \subseteq \cup_{\alpha \in \chi_\mathfrak{m}^\setminus} z_\alpha$.

Let $\widetilde{O} = \{O_\mathfrak{m} : \mathfrak{m} \in \mathfrak{M}\} \cup \{O_\mathfrak{m}^\setminus : \mathfrak{m} \in \mathfrak{M}\}$ be a $\mathcal{P}^w-$open cover of $\mathfrak{m}$. Since $(\mathfrak{m}, \zeta_1, \zeta_2)$ is s-Compact $\widetilde{O}$ has a finite subcover say $\{O_{\mathfrak{M}_i}, i = 1, 2, \ldots, n_1\} \cup \{O_{\mathfrak{m}_i}^\setminus : i = 1, 2, \ldots, n_2\}$,

$\mathfrak{M} = \cup_{i=1}^{n_1}(O_{\mathfrak{m}_i}) \cup \cup_{i=1}^{n_2}(O_{\mathfrak{m}_i}^\setminus)$. Thus $(\mathfrak{L}, \eta_1, \eta_2) = \cup_{i=1}^{n_1} \psi^{-1}(O_{\mathfrak{m}_i}) \cup \cup_{i=1}^{n_2} \psi^{-1}(O_{\mathfrak{m}_i}^\setminus) \subseteq$ combination of finite of $\mathcal{H}$. Hence $(\mathfrak{L}, \eta_1, \eta_2)$ is s-Compact.
■

As corollaries, we obtain the following findings by applying the comparable techniques in Theorem 3.4:

Corollary 3.5 *Considering an s-perfect function, a $\mathcal{P}^w-$compact space is inversely consistent.*

Corollary 3.6 *For s-perfect, the $\mathcal{P}^w-$Hausdorff space is inversely stable.*

Corollary 3.7 *Considering an s-perfect function, a $\mathcal{P}^w-$regularity space is inversely robust.*

Definition 3.8 *A function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is known as an s-strongly (s-weakly) function if, for each $\eta_1\eta_2$-open cover $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$, we have a $\eta_1\eta_2$-open cover $\mathcal{Z} = \{z_\gamma : \gamma \in \Gamma\}$ of $\mathfrak{M}$, such that $\psi^{-1}(z) \subseteq \cup\{h_\alpha : \alpha \in \chi_1, \chi_1 \subset \chi, finite\}$, $\forall z \in \mathcal{Z}$.*

Theorem 3.9 *Let $\psi : (\mathfrak{L}, \eta_1, \eta_2) \xrightarrow{onto} (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w-$closed function, and $\psi^{-1}(\mathfrak{m})$ is s-compact for all $\mathfrak{m} \in \mathfrak{M}$, then ψ is s-weak function.*

Proof. Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be a $\eta_1\eta_2$-open cover of $(\mathfrak{L}, \eta_1, \eta_2)$. For $\mathfrak{m} \in (\mathfrak{M}, \zeta_1, \zeta_2)$, $\psi^{-1}(\mathfrak{m})$ is s-compact. Thus, there is $\chi_1 \subset \chi$ finite, $\psi^{-1}(\mathfrak{m}) \subseteq \cap_{\alpha\in\chi_1} h_\alpha$, $o_{\mathfrak{m}} = \mathfrak{M} - \psi(\mathfrak{L} - \cup_{\alpha\in\chi_1} h_\alpha)$, then $o_{\mathfrak{m}}$ is $\eta_1\eta_2$-open in $\mathfrak{M}$. Define $\mathcal{O} = \{o_{\mathfrak{m}} : \mathfrak{m} \in \mathfrak{M}\}$,
then $\mathcal{O}$ is $\eta_1\eta_2$-open cover of $\mathfrak{M}$,
hence $\psi^{-1}(o_{\mathfrak{m}})$ is seen in a limited number of members of $\mathcal{H}$, thus ψ is s-weak function. ■

Theorem 3.10 *Let $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w-$continuous s-strong function, and let $K \in \eta_1 \cup \eta_2$ be s-compact in $(\mathfrak{M}, \zeta_1, \zeta_2)$.*
Then $\psi^{-1}(K)$ is s-compact in $(\mathfrak{L}, \eta_1, \eta_2)$.

Proof. Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be a $\eta_1\eta_2$-open cover $\psi^{-1}(K)$, $\mathcal{W} = \mathcal{H} \cup \{\mathfrak{L} - \psi^{-1}(K)\}$,
then $\mathcal{W}$ is a $\eta_1\eta_2$-open cover of $(\mathfrak{L}, \eta_1, \eta_2)$. Because ψ is s-strong function, there exists $\eta_1\eta_2$-open cover $z = \{z_\gamma : \gamma \in \Gamma\}$ of $(\mathfrak{M}, \zeta_1, \zeta_2)$, $\psi^{-1}(K)$ is seen in a limited number of members of $\mathcal{H}$, but K is s-compact, so K contains limited number of members of z. Hence, $\psi^{-1}(K)$ is s-compact in $(\mathfrak{L}, \eta_1, \eta_2)$. ■

Corollary 3.11 *Let $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w-$continuous s-weak function, and let $K \in \eta_1 \cup \eta_2$ be s-compact in $(\mathfrak{M}, \zeta_1, \zeta_2)$.*
Then $\psi^{-1}(K)$ is s-compact in $(\mathfrak{L}, \eta_1, \eta_2)$.

Definition 3.12 *Let $\mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2)$ be a bitopological space. η_1-locally compact with respect η_2.*
$\forall \mathfrak{l} \in \mathfrak{L}$, $\exists$ η_1-open set $\mathcal{H}$ comprising $\mathfrak{l}$ with $\overline{\mathcal{H}}^{\eta_2}$ is s-compact.

Definition 3.13 *Let $\mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2)$ be a bitopological space is called B_s-locally compact*
if η_1 is s-locally compact with deference η_2, η_2 is s-locally compact with respect η_1.

Theorem 3.14 *Consider the $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be s-compact function, and $\mathfrak{M}$ is B_s-locally compact, then ψ is s-weak function.*

Proof. Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ $\eta_1\eta_2$-open cover of $\mathfrak{L}$, $\mathfrak{m} \in (\mathfrak{M}, \zeta_1, \zeta_2)$,
since $(\mathfrak{M}, \zeta_1, \zeta_2)$ is B_s-locally compact, $\exists$ a ζ_1-open set $\mathcal{W}_{\mathfrak{m}}$, ζ_2-open set $z_{\mathfrak{m}}$, comprising $\mathfrak{m}$,
such that $\overline{\mathcal{W}_{\mathfrak{m}}}^{\eta_2}$ and $\overline{z_{\mathfrak{m}}}^{\eta_1}$ are s-compact, $\psi^{-1}(\overline{\mathcal{W}_{\mathfrak{m}}}^{\eta_2}) \subseteq \cup_{\alpha\in\chi_1} h_\alpha$, $\chi_1 \subset \chi$ finite, and $\psi^{-1}(\overline{z_{\mathfrak{m}}}^{\eta_1}) \subseteq \cup_{\alpha\in\chi_2} h_\alpha$, $\chi_2 \subset \chi$ finite,
then $O = \{\mathcal{W}_{\mathfrak{m}} : \mathfrak{m} \in \mathfrak{M}\} \cup \{z_{\mathfrak{m}} : \mathfrak{m} \in \mathfrak{M}\}$ is $\zeta_1\zeta_2$-open cover, and ψ is s-compact function. ■

Definition 3.15 *Let $\mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2)$ be a bitopological space is called K_s-space if $\mathcal{A} \subseteq \mathfrak{L}$ is η_i-open (η_i-closed), if $\psi\mathcal{A} \cap K$ is η_i-open (η_i-closed) in K, $i = 1, 2$, for each s-compact set K in $(\mathfrak{L}, \eta_1, \eta_2)$.*

Theorem 3.16 *If $\mathfrak{L} = (\mathfrak{L}, \eta_1, \eta_2)$ is B_s-locally compact, then $\mathfrak{L}$ is K_s-space.*

Proof. Let $\mathcal{A}$ be a subset of $\mathfrak{L}$, and for any s-compact subset K in $\mathfrak{L}$, we have $\mathcal{A} \cap K$, is a η_i-open, $i = 1, 2$, $\mathfrak{l} \in \mathcal{A}$, $\exists$ a η_i-open set (in $\mathfrak{L}$) containing $\mathfrak{l}$, $\overline{z}^{\eta_j}$ is s-compact for $i \neq j$, $i = 1, 2$. Now $\mathcal{A} \cap \overline{z}^{\eta_j}$ η_i-open, $i \neq j$, $i = 1, 2$, $\mathcal{A} \cap z = (\mathcal{A} \cap \overline{z}^{\eta_j}) \cap z$ is η_i-open, and $\mathfrak{l} \in \mathcal{A} \cap z \subseteq \mathcal{A}$, so $\mathcal{A}$ is η_i-open, $i = 1, 2$. ■

Theorem 3.17 *Let $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be $\mathcal{P}^w-$continuous function, and $\mathfrak{M}$ is B_s-locally compact, $\mathcal{P}^w-$Hausdorff, Consequently, the following are comparable:*
(1) ψ is s-weak function.
(2) ψ is s-compact function.
(3) ψ is s-perfect function.

Proof. (1) $\to$ (2): From theorem 3.11 we get the result.

(2) $\to$ (3):
It suffices to demonstrate that ψ is $\mathcal{P}^w-$closed. Let $\mathfrak{N}$ be a s-compact subset of $(\mathfrak{M}, \zeta_1, \zeta_2)$ and $\psi \backslash \mathfrak{n} : \psi^{-1}(\mathfrak{N}) \to \mathfrak{N}$, $\mathcal{A}$ be η_i-closed subset of $\psi^{-1}(\mathfrak{N})$, then $\mathcal{A}$ is s-compact set in $\mathfrak{L}$, but ψ is $\mathcal{P}^w-$continuous, so $\psi \backslash \mathfrak{n}(\mathcal{A})$ is s-compact in $\mathfrak{M}$, then $\psi \backslash \mathfrak{n}(\mathcal{A})$ is a η_i-closed for $i = 1, 2$, then ψ is $\mathcal{P}^w-$closed, hence ψ is s-perfect function.

(3) $\to$ (1):
Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ $\eta_1\eta_2$-open cover of $\mathfrak{L}$. Since ψ is s-perfect, $\forall \mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m}) \subseteq \cup \{h_\alpha : \alpha \in \chi\}$, but $(\mathfrak{m}, \zeta_1, \zeta_2)$ is B_s-locally compact, $\exists$ a η_1-open set $\mathcal{H}_\mathfrak{m}$, containing $\mathfrak{m}$, η_2-open set $z_\mathfrak{m}$ containing $\mathfrak{m}$, $\overline{\mathcal{H}_\mathfrak{m}}^{\eta_2}$ and $\overline{z_\mathfrak{m}}^{\eta_1}$ are s-compact, so $\psi^{-1}(\overline{\mathcal{H}_\mathfrak{m}}^{\eta_2}) \subseteq \cup_{\alpha \in \chi_1} h_\alpha$, $\chi_1 \subset \chi$ finite, and $\psi^{-1}(\overline{z_\mathfrak{m}}^{\eta_1}) \subseteq \cup_{\alpha \in \chi_2} h_\alpha$, $\chi_2 \subset \chi$ finite, thus $O = \{\mathcal{H}_\mathfrak{m} : \mathfrak{m} \in \mathfrak{M}\} \cup \{z_\mathfrak{m} : \mathfrak{m} \in \mathfrak{M}\}$ is $\zeta_1\zeta_2$-open cover of $\mathfrak{M}$. Hence ψ is s-weak function. ■

Corollary 3.18 *If there exists a s-perfect function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$, of $(\mathfrak{L}, \eta_1, \eta_2)$ onto a k-space $(\mathfrak{M}, \zeta_1, \zeta_2)$, then $(\mathfrak{L}, \eta_1, \eta_2)$ is a k-space.*

Theorem 3.19 *Let $(\mathfrak{M}, \zeta_1, \zeta_2)$ be a T_1-space such that each point in $\mathfrak{M}$ has finite ζ_1 or ζ_2-open base, then $(\mathfrak{L}, \eta_1, \eta_2)$ is s-compact iff $\pi : (\mathfrak{L} \times \mathfrak{M}, \eta_1 \times \zeta_1, \eta_2 \times \zeta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is s-weak function.*

Proof. Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be a $(\eta_1 \times \zeta_1)(\eta_2 \times \zeta_2)$-open cover of $(\mathfrak{L} \times \mathfrak{M}, \eta_1 \times \zeta_1, \eta_2 \times \zeta_2)$.

For $\mathfrak{m} \in \mathfrak{M}$, $\exists$ a finite $\zeta_1\zeta_2$-base $\{z_i(\mathfrak{m}) : i \in \chi\}$ of $\mathfrak{m}$. Let $\mathfrak{l} \in \mathfrak{L}$,
$\exists$ $H_i(\mathfrak{l}, h) \subseteq \eta_1 \cup \eta_2$ with $(\mathfrak{l}, \mathfrak{m}) \in H_i(\mathfrak{l}, h) \times z_i(\mathfrak{m}) \subseteq \mathcal{H}$, for $h \in \mathcal{H}$, $i \in \chi$.

Let $H_i(h) = \cup\{H_i(\mathfrak{l}, h) : H_i(\mathfrak{l}, h) \times z_i(\mathfrak{m}) \subseteq \mathcal{H}\}$, and $H = \{H_i(h) : h \in \mathcal{H}, i \in \chi\}$ is a $\eta_1\eta_2$-open cover of $\mathfrak{L}$, but $\mathfrak{L}$ is s-compact, therefore H has a finite subcover

$$H^* = \{H_i^*(h) : h \in \mathcal{H}, i \in \chi^*\},$$

and

$$\pi^{-1}(\cap_{i \in \chi^*} z_i(\mathfrak{m})) \subseteq \cup_{i \in \chi^*} (H_i^*(h) \times z_i(\mathfrak{m})) \subseteq \cup_{i \in \chi^*} h_\alpha.$$

Let $z_i = \cap_{i \in \chi^*} z_i(\mathfrak{m})$, $z = \{z_i : i \in \chi^*\}$, then z is a finite $\zeta_1\zeta_2$-open cover of $\mathfrak{M}$.
Hence π is s-weak function. ■

4. Structural Invariance Under B-Perfect functions

This section defines and examines B-perfect functions, which are a structured class of functions in bitopological spaces that combine continuity, closedness, and fiberwise compactness.

Definition 4.1 *A function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is called B-perfect, if ψ is $\mathcal{P}^w-$continuous, $\mathcal{P}^w-$closed, and for each $\mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m})$ is B-compact.*

As a result of applying the comparable techniques in section3, we have the following findings:

Corollary 4.2 *If $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is a B-perfect function, then every B-compact subset $(\mathfrak{N}, \iota_1, \iota_2) \subseteq (\mathfrak{M}, \zeta_1, \zeta_2)$, the inverse image $\psi^{-1}(\mathfrak{N}, \iota_1, \iota_2)$ is a B-compact.*

Corollary 4.3 *A B-perfect function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ can't be $\mathcal{P}^w-$continuously extended over any $\mathcal{P}^w-$Hausdorff space $(\mathfrak{N}, \iota_1, \iota_2)$, that contains $(\mathfrak{L}, \eta_1, \eta_2)$ as a proper subset.*

Corollary 4.4 *Let $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be a B-perfect function, and $(\mathfrak{M}, \zeta_1, \zeta_2)$ is a B-compact, then $(\mathfrak{L}, \eta_1, \eta_2)$ is so.*

Corollary 4.5 *The $\mathcal{P}^w-$compact space is invariant under B-perfect.*

Corollary 4.6 *A $\mathcal{P}^w-$compact space is inverse invariant under B-perfect function.*

Theorem 4.7 *Suppose we are given a family of $\mathcal{P}^w-$continuous functions $\{\psi_i\}_{\alpha\in\chi}$, where $\psi_\alpha : (\mathfrak{L}_\alpha, \eta_1, \eta_2) \to (\mathfrak{M}_\alpha, \zeta_1, \zeta_2)$, if there exists an $\alpha_0 \in \chi$, such that ψ_{α_0} is a B-perfect function, and $(\mathfrak{M}_\alpha, \zeta_1, \zeta_2)$ is $\mathcal{P}^w-$Hausdorff space, for every $\alpha \in \chi\backslash\{\alpha_0\}$, then the function diagonal $\chi_\alpha \in \chi$, ψ_α is B-perfect function.*

Proof. Consider the diagonal $h = \psi\nabla g$ of a B-perfect function, $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be a $\mathcal{P}^w-$continuous function, $\rho : (\mathfrak{M}, \zeta_1, \zeta_2) \to (\mathfrak{N}, \chi_1, \chi_2)$ to $\mathcal{P}^w-$Hausdorff space $(\mathfrak{N}, \chi_1, \chi_2)$. The combination may then offer the diagonal h, $(\mathfrak{L}, \eta_1, \eta_2) \xrightarrow{id\times\triangle\mathfrak{m}} (\mathfrak{L}\times\mathfrak{N}, \eta_1\times\chi_1, \eta_2\times\chi_2) \xrightarrow{\psi\times id_{\mathfrak{N}}} (\mathfrak{M}\times\mathfrak{N}, \zeta_1\times\chi_1, \zeta_2\times\chi_2)$, the function $id\times\Delta\rho$ is a B-perfect. Hence, h is B-perfect. ■

Corollary 4.8 *If the cartesian product $\psi = \prod_{\alpha\in\chi}\psi_\alpha$, where $\psi_\alpha : (\mathfrak{L}_\alpha, \eta_1, \eta_2) \to (\mathfrak{M}_\alpha, \zeta_1, \zeta_2)$, $\mathfrak{L}_\alpha \neq \phi$ for $\alpha \in \chi$ is $\mathcal{P}^w-$closed, then all functions ψ_α are $\mathcal{P}^w-$closed.*

The following interesting characterization of B-perfect:

Theorem 4.9 *For a $\mathcal{P}^w-$continuous function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{N}, \chi_1, \chi_2)$, defined on a $\mathcal{P}^w-$Hausdorff space $(\mathfrak{L}, \eta_1, \eta_2)$. The circumstances listed below are comparable:*

1) The function ψ is a B-perfect.

2) For every $\mathcal{P}^w-$Hausdorff space $(\mathfrak{M}, \zeta_1, \zeta_2)$, the cartesian product $\psi\times id_{\mathfrak{M}}$ is B-perfect.

3) For every $\mathcal{P}^w-$Hausdorff space $(\mathfrak{M}, \zeta_1, \zeta_2)$, the cartesian product $\psi\times id_{\mathfrak{M}}$ is $\mathcal{P}^w-$closed.

Proof. The implications of 1) $\to$ 2) $\to$ 3) are clear, and we wish to demonstrate that all of ψ's fibers are B-Compact since 3) $\to$ 1. Let a $\mathfrak{n}_0 \in (\mathfrak{N}, \chi_1, \chi_2)$, and $(\mathfrak{M}_\alpha, \zeta_1, \zeta_2)$ is $\mathcal{P}^w-$Hausdorff space, the restriction $\rho_0 = \rho\backslash\{\mathfrak{n}_0\}\times\mathfrak{M} : \psi^{-1}(\mathfrak{n}_0)\times(\mathfrak{M}, \zeta_1, \zeta_2) \to \{\mathfrak{n}_0\}\times(\mathfrak{M}, \zeta_1, \zeta_2)$, of the $\mathcal{P}^w-$closed function $\rho = \psi\times \mathrm{id}_{\mathfrak{M}}$ is $\mathcal{P}^w-$closed.

Thus, the composition $p_0\rho_0$, where $p_0 : \{\mathfrak{n}_0\}\times(\mathfrak{M}, \zeta_1, \zeta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is B-projection, and $\mathcal{P}^w-$closed. $p_0\rho_0$ the projection $p : \psi^{-1}(\mathfrak{n}_0)\times(\mathfrak{M}, \zeta_1, \zeta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$,

so that $\psi^{-1}(\mathfrak{n}_0) \subset (\mathfrak{L}, \eta_1, \eta_2)$ being $\mathcal{P}^w-$Hausdorff space-the $\mathcal{P}^w-$compactness of $\psi^{-1}(\mathfrak{n}_0)$, hence ψ is B-perfect. ■

Theorem 4.10 *Let P be a topology with attributes that are stable during Cartesian multiplication by a $\mathcal{P}^w-$compact space and heritable in relation to a $\mathcal{P}^w-$closed set. If there exists a B-perfect $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$, where $(\mathfrak{L}, \eta_1, \eta_2)$, $(\mathfrak{M}, \zeta_1, \zeta_2)$ be $\mathcal{P}^w-$Tychonoff spaces, and $(\mathfrak{M}, \zeta_1, \zeta_2)$ that has the property P, then the space $(\mathfrak{L}, \eta_1, \eta_2)$ also has the property P.*

Proof. The diagonal $\psi\Delta\rho : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}\times\mathfrak{N}, \zeta_1\times\chi_1, \zeta_2\times\chi_2)$ is both $\mathcal{P}^w-$homomorphism and B-perfect function, hence $(\mathfrak{L}, \eta_1, \eta_2)$ is $\mathcal{P}^w-$homomorphism to a $\mathcal{P}^w-$closed of $(\mathfrak{M}\times\mathfrak{N}, \zeta_1\times\chi_1, \zeta_2\times\chi_2)$, thus has the property P. ■

5. Closed Projections and Perfectness: Structural Properties of Bitopological Compact Functions

This section studies compact functions across bitopological spaces, focusing on their interactions with continuity, perfectness, and product space properties.

Definition 5.1 *A $\mathcal{P}^w-$continuous function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is called compact function, iff $\psi : (\mathfrak{L}, \eta_1) \to (\mathfrak{M}, \zeta_1)$ and $\psi : (\mathfrak{L}, \eta_2) \to (\mathfrak{M}, \zeta_2)$ are compact functions.*

Theorem 5.2 *Let $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ be $\mathcal{P}^w-$continuous function,*

where $(\mathfrak{M}, \zeta_1, \zeta_2)$ is a locally compact, Hausdorff space.
Then the following are equivalent:
(1) ψ is a compact function.
(2) ψ is a $\mathcal{P}^w\mathcal{P}^{ct}$ function.

Proof. (1) $\to$ (2): Demonstrating that ψ is a $\mathcal{P}^w-$closed function is sufficient.

$\psi : (\mathfrak{L}, \eta_1) \to (\mathfrak{M}, \zeta_1)$ and $\psi : (\mathfrak{L}, \eta_2) \to (\mathfrak{M}, \zeta_2)$ are closed functions.
Let $\mathfrak{F}$ be any closed subset in $(\mathfrak{L}, \eta_1)$, and $\mathfrak{m}$ be a cluster point $\psi(\mathfrak{F})$ in $(\mathfrak{M}, \zeta_1)$.
Since $(\mathfrak{M}, \zeta_1, \zeta_2)$ is a locally compact $\exists$ a ζ_1-open set $\mathfrak{G}$ containing $\mathfrak{m}$ and $\overline{\mathfrak{G}^{\zeta_1}}$ is compact.
$\psi(\mathfrak{F}) \cap \overline{\mathfrak{G}^{\zeta_1}}$ cannot be compact, since if it's true, then $\psi(\mathfrak{F}) \cap \overline{\mathfrak{G}^{\zeta_1}}$ is closed and $U = \mathfrak{G} - \psi(\mathfrak{F}) \cap \overline{\mathfrak{G}^{\zeta_1}}$ is an open set and $U \cap \psi(\mathfrak{F}) = \phi$, which is contradiction.

Hence, $\mathfrak{m} \in \overline{\psi(\mathfrak{F})}$. Since $\overline{\mathfrak{G}^{\zeta_1}}$ is compact. $\psi^{-1}(\overline{\mathfrak{G}^{\zeta_1}}) \cap \mathfrak{F}$ is compact. Thus $\psi(\mathfrak{F} \cap \psi^{-1}(\overline{\mathfrak{G}^{\zeta_1}}))$ is compact that contradicts itself. Then $\mathfrak{m} \in \psi(\mathfrak{F})$, $\psi(\mathfrak{F})$ is closed.

Likewise, we can demonstrate that $\psi : (\mathfrak{L}, \eta_2) \to (\mathfrak{M}, \zeta_2)$ is closed function.

(2) $\to$ (1): simple. ∎

Theorem 5.3 *A function $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is s-perfect, iff ψ is $\mathcal{P}^w\mathcal{P}^{ct}$ and compact function.*

Proof. Sufficient to demonstrate that $\psi^{-1}(\mathfrak{m})$ is s-compact iff $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$compact and compact, (since ψ is $\mathcal{P}^w-$closed, $\mathcal{P}^w-$continuous) Let $\mathcal{H} = \{h_\alpha : \alpha \in \chi\}$ be any $\mathcal{P}^w-$open, η_1-open or η_2-open cover of $\mathfrak{L}$ is s-compact iff it is $\mathcal{P}^w-$compact and compact.
For each $\mathfrak{m} \in \mathfrak{M}$, $\psi^{-1}(\mathfrak{m})$ is $\mathcal{P}^w-$compact and compact, there is a finite $\chi_1 \subset \chi$, $\psi^{-1}(\mathfrak{m}) \subseteq \cap_{\alpha \in \chi_1} h_\alpha$.
Let $O_\mathfrak{m} = \mathfrak{M} - \psi(\mathfrak{L} - \cup_{\alpha \in \chi_1} h_\alpha)$ is ζ_1-open set or a ζ_2-open set, it/s $\zeta_1\zeta_2$-open in $\mathfrak{M}$. Define $\mathcal{O} = \{O_\mathfrak{m} : \mathfrak{m} \in \mathfrak{M}\}$ is $\zeta_1\zeta_2$-open cover of $\mathfrak{M}$. ψ is therefore s-perfect since it contains a member of $\mathcal{H}$ $(\psi^{-1}(O_\mathfrak{m}))$. ∎

On the other hand, we obtain the outcome by employing a similar approach.

Theorem 5.4 *Let $(\mathfrak{L}, \eta_1, \eta_2)$, $(\mathfrak{M}, \zeta_1, \zeta_2)$ be any bitopological spaces. If $(\mathfrak{L}, \eta_1, \eta_2)$ is compact, then $\pi : (\mathfrak{L} \times \mathfrak{M}, \eta_1 \times \zeta_1, \eta_2 \times \zeta_2) \to (\mathfrak{M}, \zeta_1, \zeta_2)$ is closed.*

Proof. If $(\mathfrak{L}, \eta_1, \eta_2)$ is compact, then $(\mathfrak{L}, \eta_1)$ is compact, $(\mathfrak{L}, \eta_2)$ is compact, thus $\pi_1 : (\mathfrak{L} \times \mathfrak{M}, \eta_1 \times \zeta_1) \to (\mathfrak{M}, \zeta_1)$, $\pi_2 : (\mathfrak{L} \times \mathfrak{M}, \eta_2 \times \zeta_2) \to (\mathfrak{M}, \zeta_2)$ are closed, thus π is closed. ∎

Corollary 5.5 *Let $(\mathfrak{L}, \eta_1, \eta_2)$ and $(\mathfrak{M}, \zeta_1, \zeta_2)$ are s-compact (compact), then $(\mathfrak{L} \times \mathfrak{M}, \eta_1 \times \zeta_1, \eta_2 \times \zeta_2)$ is s-compact (compact).*

Corollary 5.6 *The product of s-compact and $\mathcal{P}^w-$compact is a $\mathcal{P}^w-$compact.*

6. Alternatives Examples

Here are some examples of different types of perfect functions in bitopological spaces.

Example 6.1 *Assume that* $\psi : (\mathbb{R}, \eta_h, \eta_{ind}) \to (\mathbb{R}, \eta_h, \eta_{ind})$ *be the identity function, where* η_h *and* η_{ind} *are the usual and indiscrete topologies, respectively, then* ψ *is* $\mathcal{P}^w\mathcal{P}^{ct}$ *function not B-perfect function. Given that* $(\mathbb{R}, \eta_h, \eta_{ind})$ *is* $\mathcal{P}^w-$*compact but not compact, and therefore not B-compact,* ψ *is* $\mathcal{P}^w\mathcal{P}^{ct}$ *not flawless.*

Example 6.2 *Let* $\psi : (\mathbb{R}, \eta_\psi, \eta_d) \to (\mathbb{R}, \eta_\psi, \eta_d)$ *be the identity function, where* η_ψ *and* η_d *are denoted the cofinite topology on* $\mathbb{R}$ *and discrete topologies, respectively.*
then ψ *is* $\mathcal{P}^w\mathcal{P}^{ct}$ *not s-perfect. Since* $(\mathbb{R}, \eta_\psi, \eta_d)$ *is* $\mathcal{P}^w-$*compact, and not s-compact, take* $\{\mathfrak{l}\}$ *is* $\eta_1\eta_2$*-open set has not a finite subcover, hence* ψ *is* $\mathcal{P}^w\mathcal{P}^{ct}$ *not s-perfect.*

Example 6.3 *Let* $\psi : (\mathfrak{L}, \eta_1, \eta_2) \to (\mathfrak{L}, \eta_1, \eta_2)$ *be the identity function, then* ψ *is B-perfect function not* $\mathcal{P}^w\mathcal{P}^{ct}$ *function, not s-perfect function.*
Let $\mathfrak{L} = [0, 1]$, $\eta_1 = \{\phi, \mathfrak{L}, \{0\}\} \cup \{(0, a), a \in \mathfrak{L}\}$, $\eta_2 = \{\phi, \mathfrak{L}, \{1\}\} \cup \{(a, 1), a \in \mathfrak{L}\}$, *then* $(\mathfrak{L}, \eta_1, \eta_2)$ *is B-compact, since for any* η_1*-open cover of* $\mathfrak{L}$*, or any* η_2*-open cover of* $\mathfrak{L}$*, must contain* $\mathfrak{L}$ *as member.* $(\mathfrak{L}, \eta_1, \eta_2)$ *is neither* $\mathcal{P}^w-$*compact, not s-compact, for the* $\mathcal{P}^w-$*open cover* $\{\{0\} \cup (a, 1], a \in \mathfrak{L}, a \neq 0\}$ *of* $\mathfrak{L}$ *has not finite.*
Hence ψ *is B-perfect function not* $\mathcal{P}^w\mathcal{P}^{ct}$ *function, not s-perfect function.*

Example 6.4 *Consider the production function,* $\pi : (\mathbb{R} \times \mathbb{R}, \eta_h \times \eta_h, \eta_s \times \eta_s) \to (\mathfrak{L}, \eta_h, \eta_s)$*, then* π *is not closed, since* $(\mathbb{R}, \eta_h, \eta_s)$ *is not compact.*

Example 6.5 *Let* $\mathfrak{L} = (\mathbb{R}, \eta_h, \eta_{ind})$*, then it's* $\mathcal{P}^w-$*compact, but not compact. However* $\pi : (\mathbb{R} \times \mathbb{R}, \eta_h \times \eta_h, \eta_{ind} \times \eta_{ind}) \to (\mathfrak{L}, \eta_h, \eta_{ind})$ *is not closed.*

7. Dual Topologies, How Perfect Functions Architect Tomorrow's Predictive Systems

Perfect functions in bitopological spaces are more than simply abstract mathematics; they are tools with real world applications. These functions, which combine continuity, closedness, and compactness, assist in representing systems in which several structures reside together, such as a city's road network covered with Wi-Fi signals or a healthcare dataset recording both genes and symptoms. Next how they're shaping predictions potentially changing our future:
A $\mathcal{P}^w\mathcal{P}^{ct}$ function operates as a clever organizer, reorganizing the chaos while preserving each category's essential structure. In machine learning, this ensures that data is compressed. Is the data crowded? s-perfect functions demonstrate as a filter, preserving the "shape" of the data even when parts of it get lost or fuzzy.

Compact functions serve as traffic controllers, detecting weak bridges (non-compact zones) before they break down. They could identify dead zones in 5G networks by analyzing how signals shift from real towers to virtual channels. Closedness in $\mathcal{P}^w\mathcal{P}^{ct}$ functions assures that suggested network architectures do not contain unrealistic shortcuts, such as a GPS route that somehow transports you across a river. Smart cities may employ these features to balance power grids and traffic lights in real time, avoiding blackouts and delays during an event.

Doctors are frequently confronted with confuses: a patient's genes point to heart disease, although their symptoms tell elsewhere. B-perfect functions could represent these overlapping "clues" as two topologies, retaining linkages between genetic markers and clinical symptoms even when data is limited. This aids in predicting which patients will have difficulties. Wearables might apply S-perfect functions to detect abnormal heartbeats and sleep patterns, indicating potential dangers without requiring frequent doctor visits.

Perfect functions are more than basically math; they connect theory and reality. They now help us predict network failures and diseases risks. Tomorrow, they could change the way we construct AI, secure data, and even populate Mars. The secret is their capacity to maintain structure under chaos. As data grows more chaotic and systems become more complicated, these functions will quietly power the algorithms that keep our world running, making the future a bit less unpredictable, one topology at a time.

8. Conclusions

The connections between the topological spaces generated by functions and the perfect functions in those spaces were examined in this study. In accordance with the notion of perfect functions that is provided here, the study established the prerequisites for harmonizing the compact space and other functions. We looked at the connection between these two ideas and used several types of perfect functions to describe them. One other purpose of this work was to spotlight some advanced properties of the perfect functions and some of the peculiarities of the cartesian process of multiplication of these functions in unexpected conditions. In addition, dominant elements of these principles and some instructive situations were thoroughly investigated. We identified their main characteristics in general and made clear the requirements for establishing comparable links between them. We talked about their main traits and demonstrated how they work together. Additionally, the study highlighted these functions characteristics and included numerous instances. Investigations into the various futures of these functions will begin with these functions. Future studies might look into investigating more variations of these functions.

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