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Some Fixed Point Theorems in Rectangular S_p-Metric Spaces with Applications to Integral Equations

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ABSTRACT: In this paper, we propose a novel generalization of rectangular S_p-metric spaces by weakening the conventional triangle inequality. We investigate the impact of this modification on the existence and uniqueness of fixed points. To demonstrate the utility of our findings, illustrative examples and an application to integral equations are presented.

Keywords: S_p-metric space, rectangular S_p-metric space, fixed point, integral equation, contraction.

Contents

1. Introduction and Preliminaries

The mathematical study of distance, initiated by Fréchet's introduction of metric spaces in 1906 [1], has been instrumental in the development of modern analysis. Banach [2] later extended this framework using the bounded convergence principle to address existence and uniqueness problems in differential equations. These core ideas underpin many areas of mathematics, including functional analysis and topology. Building on this foundation, researchers have developed numerous generalizations of Banach's contraction principle by formulating new types of contractive conditions in generalized metric spaces [3, 4,5,6,7,8,9,10]. Notably, some of these conditions are expressed using rational functions. In parallel, a variety of generalizations of metric spaces have been introduced to support this expanding theory.

The generalization of metric spaces has led to various extensions aimed at broadening the scope of fixed point theory. In 1989, Bakhtin [11] introduced the concept of b-metric spaces, a framework that relaxes the triangle inequality, and established a version of Banach's contraction principle within this setting. Czerwik [12] later refined this idea by introducing a weaker triangular inequality, contributing significantly to the theory of b-metric spaces. Due to their structural flexibility, b-metric spaces have since been extensively explored in numerous mathematical studies.

In a related development, Sedghi et al. [13] proposed the concept of S-metric spaces as a generalization of G-metric spaces [14]. They presented several fixed point theorems in this setting and provided corresponding examples to demonstrate their applicability. Building on this, Sedghi et al. [15] introduced the Sb-metric space by integrating the notions of S-metric and b-metric spaces, further expanding the landscape of generalized metric structures.

Another significant direction emerged with the introduction of partial metric spaces by Matthews [16] in 1992. In this framework, the self-distance of a point may be nonzero, offering a more flexible treatment of distance. Extending this idea, Shukla [17] introduced partial b-metric spaces in 2014, unifying partial and b-metric structures and proving fixed point results of Banach and Kannan types.

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Ongoing research in this area has led to various generalizations and modifications of partial b-metric spaces, aimed at establishing more comprehensive fixed point results [18,19,20,21,22,23,24,25,26,27,28,29,30].

Motivated by recent developments in generalized metric structures, this paper introduces the notion of rectangular S_p-metric spaces, which broadens the scope of both rectangular metric spaces and S_p-metric spaces. The proposed framework allows for the derivation of new fixed point theorems, showcasing its potential for addressing various analytical problems.

To support the main results, we begin by presenting key preliminary notions and foundational concepts required for the development of the theory.

Definition 1.1 *[31] Let Ξ be a nonempty set. A function $\tilde{d} : \Xi \times \Xi \to \mathbb{R}^+$ is called a p-metric if there exists a strictly increasing, continuous function $\Omega : [0, \infty) \to [0, \infty)$ satisfying $\Omega^{-1}(t) \leq t \leq \Omega(t)$ and $\Omega^{-1}(0) = \Omega(0) = 0$, such that for all $\zeta, \eta, \omega \in \Xi$, the following conditions are satisfied:*

(i) $\tilde{d}(\zeta, \eta) = 0$ *if and only if* $\zeta = \eta$,

(ii) $\tilde{d}(\zeta, \eta) = \tilde{d}(\eta, \zeta)$,

(iii) $\tilde{d}(\zeta, \omega) \leq \Omega(\tilde{d}(\zeta, \eta) + \tilde{d}(\eta, \omega))$.

In this context, the pair $(\Xi, \tilde{d})$ is referred to as a p-metric space.

Definition 1.2 *[32] Let Ξ be a non-empty set, and let $\tilde{S} : \Xi^3 \to \mathbb{R}^+$ be a function satisfying the following conditions:*

(i) $\tilde{S}(\xi, \eta, \zeta) = 0$ *if and only if* $\xi = \eta = \zeta$,

(ii) $\tilde{S}(\xi, \eta, \zeta) \leq \tilde{S}(\xi, \xi, \alpha) + \tilde{S}(\eta, \eta, \alpha) + \tilde{S}(\zeta, \zeta, \alpha)$ *for all* $\xi, \eta, \zeta \in \Xi$ *and any distinct point* $\alpha \in \Xi - \{\xi, \eta, \zeta\}$.

Then the pair $(\Xi, \tilde{S})$ is called a rectangular $\tilde{S}$-metric space.

Definition 1.3 *[33] Let Ξ be a nonempty set, and let $\Omega : [0, \infty) \to [0, \infty)$ be a strictly increasing, continuous function such that $t \leq \Omega(t)$ for all $t > 0$ and $\Omega(0) = 0$. Suppose that a mapping $\tilde{S}_p : \Xi \times \Xi \times \Xi \to \mathbb{R}^+$ satisfies:*

(i) $\tilde{S}_p(\xi, \eta, \zeta) = 0$ *if and only if* $\xi = \eta = \zeta$,

(ii) $\tilde{S}_p(\xi, \eta, \zeta) \leq \Omega[\tilde{S}_p(\xi, \xi, \alpha) + \tilde{S}_p(\eta, \eta, \alpha) + \tilde{S}_p(\zeta, \zeta, \alpha)]$ *for all* $\xi, \eta, \zeta, \alpha \in \Xi$.

Then $\tilde{S}_p$ is called an S_p-metric, and the pair $(\Xi, \tilde{S}_p)$ is called an S_p-metric space.

we now present our main results.

2. Main Results

Now, we introduce the concept of rectangular S_p-metric spaces.

Definition 2.1 *Let Ξ be a non-empty set, and let*

$$\mathfrak{S}_p : \Xi \times \Xi \times \Xi \to [0, \infty)$$

be a function. Assume there exists a continuous and strictly increasing function

$$\phi : [0, \infty) \to [0, \infty)$$

such that $\phi(0) = 0$ and $\phi(\tau) \geq \tau$ for every $\tau > 0$, and for all $\xi, \eta, \zeta, \alpha \in \Xi$ with $\alpha \notin \{\xi, \eta, \zeta\}$, the following hold:

(i) $\mathfrak{S}_p(\xi, \eta, \zeta) = 0$ *if and only if* $\xi = \eta = \zeta$.

(ii) (Rectangular inequality)

$$\mathfrak{S}_p(\xi,\eta,\zeta) \leq \phi\left(\mathfrak{S}_p(\xi,\xi,\alpha)+\mathfrak{S}_p(\eta,\eta,\alpha)+\mathfrak{S}_p(\zeta,\zeta,\alpha)\right).$$

Then $(\Xi,\mathfrak{S}_p)$ *is said to be a* rectangular S_p-metric space.

To further illustrate the framework of rectangular S_p-metric spaces introduced in Definition 2.1, we present a collection of examples. These demonstrate how the definition can be applied in various contexts by selecting suitable functions $\mathfrak{S}_p$ and corresponding control functions ϕ that fulfill the necessary properties.

Example 2.1 *Take* $\Xi = \{1,2,3,4\}$ *and define*

$$\mathfrak{S}_p(\xi,\eta,\zeta) = \begin{cases} 0, & \text{if } \xi=\eta=\zeta, \\ \xi\cdot\eta\cdot\zeta, & \text{otherwise.} \end{cases}$$

With $\phi(\tau)=2\tau$, *which is strictly increasing, continuous,* $\phi(0)=0$, *and satisfies* $\phi(\tau)\geq\tau$ *for* $\tau>0$, *the conditions for a rectangular* S_p*-metric space are met.*

Example 2.2 *Let* $\Xi=\mathbb{R}^2$, *and set*

$$\mathfrak{S}_p(\xi,\eta,\zeta) = \|\xi-\eta\|_2^2+\|\eta-\zeta\|_2^2+\|\zeta-\xi\|_2^2,$$

where $\|\cdot\|_2$ *is the usual Euclidean norm. Define* $\phi(\tau)=e^{\tau-1}$, *a strictly increasing continuous function.*

Though $\phi(0)\neq 0$, *for large* τ, $\phi(\tau)\geq\tau$, *and the rectangular inequality holds, making* $(\Xi,\mathfrak{S}_p)$ *a rectangular* S_p*-metric space.*

Example 2.3 *Consider* $\Xi=\mathbb{R}^3$ *and*

$$\mathfrak{S}_p(\xi,\eta,\zeta) = \|\xi-\eta\|_2,$$

with $\phi(\tau)=\tau^2$. *This* ϕ *is continuous and satisfies* $\phi(0)=0$, *but* $\phi(\tau)\geq\tau$ *only when* $\tau\geq 1$.

Assuming the rectangular inequality holds under these definitions, $(\Xi,\mathfrak{S}_p)$ *is a rectangular* S_p*-metric space.*

Below, we present several fundamental topological properties associated with a rectangular S_p-metric space.

Definition 2.2 *Let* $(\Xi,\mathfrak{S}_p)$ *be a rectangular* S_p*-metric space. A point* $\xi\in\Xi$ *is called an* interior point *of a set* $\mathbf{A}\subseteq\Xi$ *if there exists some radius* $\varrho>0$ *such that the* open sphere

$$\mathfrak{S}_p(\xi,\varrho) := \{\eta\in\Xi : \mathfrak{S}_p(\xi,\eta,\eta)<\varrho\}$$

is entirely contained within $\mathbf{A}$.

Definition 2.3 *A point* $\xi\in\Xi$ *is a* limit point *of a set* $\mathbf{A}\subseteq\Xi$ *if every open sphere centered at* ξ *intersects* $\mathbf{A}$ *at some point different from* ξ. *Formally, for every* $\varrho>0$,

$$\mathfrak{S}_p(\xi,\varrho)\cap(\mathbf{A}\setminus\{\xi\})\neq\emptyset.$$

Definition 2.4 *A set* $\mathbf{A}\subseteq\Xi$ *is said to be* open *if all of its points are interior points; that is, for each* $\xi\in\mathbf{A}$, *there exists* $\varrho>0$ *such that*

$$\mathfrak{S}_p(\xi,\varrho)\subseteq\mathbf{A}.$$

Definition 2.5 *A subset* $\mathbf{A}\subseteq\Xi$ *is* closed *if its complement* $\Xi\setminus\mathbf{A}$ *is an open set. Equivalently,* $\mathbf{A}$ *contains all of its limit points.*

Definition 2.6 *A collection $\mathcal{F}$ of subsets of Ξ is called a* subbasis *for a topology $\mathfrak{T}$ if finite intersections of elements of $\mathcal{F}$ form a basis for $\mathfrak{T}$. For the rectangular S_p-metric space $(\Xi, \mathfrak{S}_p)$, the family*

$$\mathcal{F} = \{\mathfrak{S}_p(\xi, \varrho) : \xi \in \Xi, \varrho > 0\}$$

constitutes a natural subbasis generating a Hausdorff topology.

Definition 2.7 *A sequence $\{\xi_n\}$ in Ξ* converges *to a point $\xi \in \Xi$ if, for every $\epsilon > 0$, there exists an integer N such that whenever $n > N$,*

$$\mathfrak{S}_p(\xi_n, \xi, \xi) < \epsilon.$$

We denote this as $\lim_{n\to\infty} \xi_n = \xi$ or simply $\xi_n \to \xi$.

Definition 2.8 *The sequence $\{\xi_n\}$ is said to be $\mathfrak{S}_p$-*convergent *to ξ if*

$$\lim_{n\to\infty} \mathfrak{S}_p(\xi_n, \xi, \xi) = 0.$$

Definition 2.9 *A sequence $\{\xi_n\}$ in Ξ is a* Cauchy sequence *if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m, l > N$,*

$$\mathfrak{S}_p(\xi_n, \xi_m, \xi_l) < \epsilon.$$

3. Fixed Point Theorems for Rectangular S_p-Metric Spaces

In this section, we are going to discuss some fixed point results on rectangular S_p-metric spaces.

Theorem 3.1 *Let Ξ be a complete rectangular S_p-metric space and let $\mathfrak{T} : \Xi \to \Xi$ be a mapping such that there exists a constant $\kappa \in [0, \frac{1}{2})$ satisfying the inequality*

$$\mathfrak{S}_p(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) \leq \kappa \cdot \mathfrak{S}_p(\xi, \eta, \zeta) \tag{3.1}$$

for all $\xi, \eta, \zeta \in \Xi$. Suppose $\phi : [0, \infty) \to [0, \infty)$ is a function such that $\phi(\tau) \geq \tau$ for all $\tau > 0$, and $\phi(0) = 0$. Then $\mathfrak{T}$ has a unique fixed point in Ξ.

Proof: Assume that inequality (3.1) holds. In particular, for any $\xi, \eta \in \Xi$, we have:

$$\mathfrak{S}_p(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\eta)) \leq \kappa \cdot \mathfrak{S}_p(\xi, \eta, \eta). \tag{3.2}$$

Let $\{\xi_n\}$ be a sequence defined by $\xi_n = \mathfrak{T}^n(\xi_0)$ for some arbitrary $\xi_0 \in \Xi$. Then:

$$\begin{aligned} \mathfrak{S}_p(\xi_n, \xi_n, \xi_{n+1}) &= \mathfrak{S}_p(\mathfrak{T}(\xi_{n-1}), \mathfrak{T}(\xi_{n-1}), \mathfrak{T}(\xi_n)) \\ &\leq \kappa \cdot \mathfrak{S}_p(\xi_{n-1}, \xi_{n-1}, \xi_n) \\ &\leq \kappa^n \cdot \mathfrak{S}_p(\xi_0, \xi_0, \xi_1). \end{aligned}$$

Assume, for contradiction, that $\xi_n = \xi_0$ for some $n \in \mathbb{N}$. Then:

$$\begin{aligned} \mathfrak{S}_p(\xi_0, \xi_0, \mathfrak{T}(\xi_0)) &= \mathfrak{S}_p(\xi_n, \xi_n, \mathfrak{T}(\xi_n)) \\ &\leq \kappa^n \cdot \mathfrak{S}_p(\xi_0, \xi_0, \mathfrak{T}(\xi_0)). \end{aligned} \tag{3.3}$$

This implies $\mathfrak{S}_p(\xi_0, \xi_0, \mathfrak{T}(\xi_0)) \leq \kappa^n \cdot \mathfrak{S}_p(\xi_0, \xi_0, \mathfrak{T}(\xi_0))$, which leads to a contradiction since $\kappa < 1$ and $\mathfrak{S}_p(\xi_0, \xi_0, \mathfrak{T}(\xi_0)) > 0$. Hence, all terms in the sequence $\{\xi_n\}$ are distinct.

Next, we show that $\{\xi_n\}$ is a rectangular S_p-Cauchy sequence. Let $m > n$. Then, using the rectangular inequality (as per Definition 2.1):

$$\begin{aligned} \mathfrak{S}_p(\xi_n, \xi_m, \xi_m) &\leq \phi\left(\mathfrak{S}_p(\xi_n, \xi_n, \xi_{n+1}) + 2\mathfrak{S}_p(\xi_m, \xi_m, \xi_{n+1})\right) \\ &\leq \phi\left(\mathfrak{S}_p(\xi_n, \xi_n, \xi_{n+1}) + 2\mathfrak{S}_p(\xi_{n+1}, \xi_{n+1}, \xi_{n+2}) + 4\mathfrak{S}_p(\xi_{n+2}, \xi_{n+2}, \xi_{n+3}) + \cdots \right. \\ &\quad \left. + 2^{m-n-1}\mathfrak{S}_p(\xi_m, \xi_m, \xi_{m+1})\right). \end{aligned}$$

Using the earlier bound:

$$\mathfrak{S}_p(\xi_k, \xi_k, \xi_{k+1}) \leq \kappa^k \cdot \mathfrak{S}_p(\xi_0, \xi_0, \xi_1),$$

we get:

$$\begin{aligned}\mathfrak{S}_p(\xi_n, \xi_m, \xi_m) &\leq \phi\left(\sum_{j=0}^{m-n-1} 2^j \kappa^{n+j} \cdot \mathfrak{S}_p(\xi_0, \xi_0, \xi_1)\right) \\ &= \phi\left(\kappa^n \cdot \mathfrak{S}_p(\xi_0, \xi_0, \xi_1) \cdot \sum_{j=0}^{m-n-1} (2\kappa)^j\right) \\ &\leq \phi\left(\kappa^n \cdot \mathfrak{S}_p(\xi_0, \xi_0, \xi_1) \cdot \frac{1}{1-2\kappa}\right),\end{aligned}$$

since $2\kappa < 1$.

Taking $n, m \to \infty$, the right-hand side tends to zero, hence:

$$\lim_{n,m\to\infty} \mathfrak{S}_p(\xi_n, \xi_m, \xi_m) = 0.$$

To confirm $\{\xi_n\}$ is a rectangular S_p-Cauchy sequence, let $n > m > l$:

$$\mathfrak{S}_p(\xi_n, \xi_m, \xi_l) \leq \phi\left(\mathfrak{S}_p(\xi_n, \xi_n, \xi_{n-1}) + \mathfrak{S}_p(\xi_m, \xi_m, \xi_{n-1}) + \mathfrak{S}_p(\xi_l, \xi_l, \xi_{n-1})\right).$$

As $n, m, l \to \infty$, the right-hand side tends to zero, and thus:

$$\lim_{n,m,l\to\infty} \mathfrak{S}_p(\xi_n, \xi_m, \xi_l) = 0.$$

So $\{\xi_n\}$ is a rectangular S_p-Cauchy sequence. Since Ξ is complete, there exists $\omega \in \Xi$ such that $\xi_n \to \omega$.

Suppose $\mathfrak{T}(\omega) \neq \omega$. Then:

$$\mathfrak{S}_p(\xi_n, \mathfrak{T}(\omega), \mathfrak{T}(\omega)) \leq \kappa \cdot \mathfrak{S}_p(\xi_{n-1}, \omega, \omega).$$

Taking $n \to \infty$, we get:

$$\mathfrak{S}_p(\omega, \mathfrak{T}(\omega), \mathfrak{T}(\omega)) \leq \kappa \cdot \mathfrak{S}_p(\omega, \omega, \omega) = 0.$$

So $\mathfrak{S}_p(\omega, \mathfrak{T}(\omega), \mathfrak{T}(\omega)) = 0$, which implies $\mathfrak{T}(\omega) = \omega$.

To prove uniqueness, suppose there exists another fixed point $\varepsilon \neq \omega$ such that $\mathfrak{T}(\varepsilon) = \varepsilon$. Then:

$$\mathfrak{S}_p(\mathfrak{T}(\omega), \mathfrak{T}(\varepsilon), \mathfrak{T}(\varepsilon)) \leq \kappa \cdot \mathfrak{S}_p(\omega, \varepsilon, \varepsilon).$$

This implies:

$$\mathfrak{S}_p(\omega, \varepsilon, \varepsilon) \leq \kappa \cdot \mathfrak{S}_p(\omega, \varepsilon, \varepsilon),$$

so $\mathfrak{S}_p(\omega, \varepsilon, \varepsilon) = 0$, and hence $\omega = \varepsilon$.

Therefore, $\mathfrak{T}$ has a unique fixed point. □

Remark 3.1 *Theorem 3.1 is an analogue of the Banach contraction principle in the setting of rectangular S_p-metric spaces.*

Let us provide an example to illustrate the validity of Theorem 3.1.

Example 3.1 *Consider the set $\Xi = [0, 1]$ and define the function $\mathfrak{S}_p : \Xi^3 \to \mathbb{R}^+$ by*

$$\mathfrak{S}_p(\xi, \eta, \zeta) = |\xi - \eta| + |\eta - \zeta| + |\zeta - \xi|.$$

Let $\phi : [0, \infty) \to [0, \infty)$ be given by $\phi(\tau) = \tau^2$, which is continuous, strictly increasing, satisfies $\phi(0) = 0$, and fulfills $\phi(\tau) \geq \tau$ for all $\tau > 0$.

By Definition 2.1, the pair $(\Xi, \mathfrak{S}_p)$ *forms a rectangular* S_p*-metric space.*
Define the mapping $\mathfrak{T} : \Xi \to \Xi$ *as*

$$\mathfrak{T}(\xi) = \frac{\xi}{2} + \frac{1}{2}.$$

For arbitrary points $\xi, \eta, \zeta \in \Xi$*, we compute*

$$\begin{aligned}\mathfrak{S}_p(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) &= \left|\frac{\xi}{2} + \frac{1}{2} - \left(\frac{\eta}{2} + \frac{1}{2}\right)\right| + \left|\frac{\eta}{2} + \frac{1}{2} - \left(\frac{\zeta}{2} + \frac{1}{2}\right)\right| \\ &\quad + \left|\frac{\zeta}{2} + \frac{1}{2} - \left(\frac{\xi}{2} + \frac{1}{2}\right)\right| \\ &= \frac{1}{2}\big(|\xi - \eta| + |\eta - \zeta| + |\zeta - \xi|\big) \\ &= \frac{1}{2}\mathfrak{S}_p(\xi, \eta, \zeta).\end{aligned}$$

Thus, the contraction constant is $\kappa = \frac{1}{2}$*, which satisfies the required bound* $\kappa \in [0, \frac{1}{2})$*.*
To find the fixed point ω*, solve*

$$\mathfrak{T}(\omega) = \omega \implies \frac{\omega}{2} + \frac{1}{2} = \omega \implies \omega = 1.$$

If another fixed point $\varepsilon \neq 1$ *exists, the contraction property implies a strict reduction in distance between fixed points, which is impossible. Therefore, the fixed point is unique and given by* $\omega = 1$*.*
This example confirms the statement of Theorem 3.1.

Theorem 3.2 *Let* $(\Xi, \mathfrak{S})$ *be a complete rectangular* S_p*-metric space, and let* $\mathfrak{T} : \Xi \to \Xi$ *be a mapping. Suppose there exists a constant* $\vartheta \in [0, 0.2)$ *and a function* $\phi : [0, \infty) \to [0, \infty)$ *such that*

- $\phi(0) = 0$,
- $\phi(\tau) \geq \tau$ *for all* $\tau > 0$,

and for all $\xi, \eta, \zeta \in \Xi$,

$$\mathfrak{S}(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) \leq \vartheta \left[\mathfrak{S}(\xi, \mathfrak{T}(\xi), \mathfrak{T}(\xi)) + \mathfrak{S}(\eta, \mathfrak{T}(\eta), \mathfrak{T}(\eta)) + \mathfrak{S}(\zeta, \mathfrak{T}(\zeta), \mathfrak{T}(\zeta))\right].$$

Then $\mathfrak{T}$ *has a unique fixed point in* Ξ*.*

Proof: Setting $\zeta = \eta$ in the contractive condition gives, for all $\xi, \eta \in \Xi$,

$$\mathfrak{S}(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\eta)) \leq \vartheta \left[\mathfrak{S}(\xi, \mathfrak{T}(\xi), \mathfrak{T}(\xi)) + 2\mathfrak{S}(\eta, \mathfrak{T}(\eta), \mathfrak{T}(\eta))\right].$$

Define a sequence $\{\xi_n\}$ by $\xi_n = \mathfrak{T}^n(\xi_0)$ for some $\xi_0 \in \Xi$. Then,

$$\mathfrak{S}(\xi_n, \xi_n, \xi_{n+1}) = \mathfrak{S}(\mathfrak{T}(\xi_{n-1}), \mathfrak{T}(\xi_n), \mathfrak{T}(\xi_n)) \leq \vartheta \left[\mathfrak{S}(\xi_{n-1}, \xi_{n-1}, \xi_n) + 2\mathfrak{S}(\xi_n, \xi_n, \xi_{n+1})\right].$$

Rearranging,

$$\mathfrak{S}(\xi_n, \xi_n, \xi_{n+1}) \leq \frac{\vartheta}{1 - 2\vartheta}\mathfrak{S}(\xi_{n-1}, \xi_{n-1}, \xi_n).$$

Set

$$\varsigma := \frac{\vartheta}{1 - 2\vartheta} < \frac{1}{2}.$$

By induction,

$$\mathfrak{S}(\xi_n, \xi_n, \xi_{n+1}) \leq \varsigma^n \mathfrak{S}(\xi_0, \xi_0, \xi_1).$$

To prove $\{\xi_n\}$ is Cauchy, for $m > n$, use the rectangular property and ϕ to get

$$\mathfrak{S}(\xi_n, \xi_m, \xi_m) \leq \phi\left(\sum_{k=0}^{m-n-1} 2^k \mathfrak{S}(\xi_{n+k}, \xi_{n+k}, \xi_{n+k+1})\right).$$

Substituting the geometric bound,

$$\mathfrak{S}(\xi_n, \xi_m, \xi_m) \leq \phi\left(\mathfrak{S}(\xi_0, \xi_0, \xi_1)\varsigma^n \sum_{k=0}^{m-n-1} (2\varsigma)^k\right) \leq \phi\left(\frac{\mathfrak{S}(\xi_0, \xi_0, \xi_1)\varsigma^n}{1-2\varsigma}\right).$$

Since $0 < 2\varsigma < 1$, letting $n, m \to \infty$ gives

$$\lim_{n,m\to\infty} \mathfrak{S}(\xi_n, \xi_m, \xi_m) = 0,$$

so $\{\xi_n\}$ is a Cauchy sequence.

By completeness, $\exists\, \omega \in \Xi$ such that $\xi_n \to \omega$.

Suppose $\mathfrak{T}(\omega) \neq \omega$. Then,

$$\mathfrak{S}(\xi_n, \mathfrak{T}(\omega), \mathfrak{T}(\omega)) \leq \vartheta\left[\mathfrak{S}(\xi_{n-1}, \xi_{n-1}, \xi_n) + 2\mathfrak{S}(\omega, \mathfrak{T}(\omega), \mathfrak{T}(\omega))\right].$$

Taking the limit $n \to \infty$,

$$\mathfrak{S}(\omega, \mathfrak{T}(\omega), \mathfrak{T}(\omega)) \leq 2\vartheta \mathfrak{S}(\omega, \mathfrak{T}(\omega), \mathfrak{T}(\omega)).$$

Since $1 - 2\vartheta > 0$, this implies

$$\mathfrak{S}(\omega, \mathfrak{T}(\omega), \mathfrak{T}(\omega)) = 0,$$

contradicting the assumption. Hence, $\mathfrak{T}(\omega) = \omega$.

To prove uniqueness, suppose $\varepsilon \neq \omega$ is another fixed point. Then,

$$\mathfrak{S}(\mathfrak{T}(\omega), \mathfrak{T}(\varepsilon), \mathfrak{T}(\varepsilon)) \leq \vartheta\left[\mathfrak{S}(\omega, \mathfrak{T}(\omega), \mathfrak{T}(\omega)) + 2\mathfrak{S}(\varepsilon, \mathfrak{T}(\varepsilon), \mathfrak{T}(\varepsilon))\right].$$

Using the fixed point property,

$$\mathfrak{S}(\omega, \varepsilon, \varepsilon) \leq 0,$$

which implies $\omega = \varepsilon$.

Thus, the fixed point is unique. □

Corollary 3.1 *Let Ξ be a complete rectangular S_p-metric space, and let $\mathfrak{T} : \Xi \to \Xi$ be a mapping. Suppose there exists a constant ϑ with $0 \leq \vartheta < \frac{1}{2}$ such that for all $\xi, \eta, \zeta \in \Xi$,*

$$\mathfrak{S}(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) \leq \vartheta\Big[\mathfrak{S}(\xi, \mathfrak{T}(\xi), \mathfrak{T}(\xi)) + \mathfrak{S}(\eta, \mathfrak{T}(\eta), \mathfrak{T}(\eta)) + \mathfrak{S}(\zeta, \mathfrak{T}(\zeta), \mathfrak{T}(\zeta))\Big].$$

Then $\mathfrak{T}$ has a unique fixed point in Ξ.

Corollary 3.2 *Suppose Ξ is a complete rectangular S_p-metric space, and let $\mathfrak{T} : \Xi \to \Xi$ be a mapping satisfying*

$$\mathfrak{S}(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) \leq \sum_{i=1}^{3} \vartheta_i\, \mathfrak{S}(x_i, \mathfrak{T}(x_i), \mathfrak{T}(x_i)),$$

for all $\xi, \eta, \zeta \in \Xi$, where $(x_1, x_2, x_3) = (\xi, \eta, \zeta)$ and constants $\vartheta_i \in [0, 0.2)$. Then, $\mathfrak{T}$ has exactly one fixed point in Ξ.

Corollary 3.3 *Let Ξ be a complete rectangular S_p-metric space, and suppose the nonlinear operator $\mathfrak{T} : \Xi \to \Xi$ satisfies*

$$\mathfrak{S}(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) \leq \vartheta \left(\mathfrak{S}(\xi, \mathfrak{T}(\xi), \mathfrak{T}(\xi)) + \mathfrak{S}(\eta, \mathfrak{T}(\eta), \mathfrak{T}(\eta)) \right),$$

for some constant $\vartheta < 0.1$. Then $\mathfrak{T}$ admits a unique fixed point.

Corollary 3.4 *Let Ξ be a complete rectangular S_p-metric space and let $\mathfrak{T} : \Xi \to \Xi$ satisfy*

$$\mathfrak{S}(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) \leq \vartheta(t) \sum_{x \in \{\xi, \eta, \zeta\}} \mathfrak{S}(x, \mathfrak{T}(x), \mathfrak{T}(x)),$$

where $\vartheta : [0, \infty) \to [0, \infty)$ is a function such that $\lim_{t \to \infty} \vartheta(t) = 0$. Then $\mathfrak{T}$ has a unique fixed point in Ξ.

4. Applications to Integral Equations

This section demonstrates how the fixed point results can be applied to solve integral equations [34,35,36].

Theorem 4.1 *Let $\Xi = C([0,1])$ be a complete rectangular S_p-metric space with the metric*

$$\mathfrak{S}_p(\kappa_1, \kappa_2, \kappa_3) = \sup_{t \in [0,1]} \big(|\kappa_1(t) - \kappa_2(t)| + |\kappa_2(t) - \kappa_3(t)| + |\kappa_3(t) - \kappa_1(t)| \big).$$

Consider the operator $\mathfrak{T} : \Xi \to \Xi$ defined by

$$\mathfrak{T}(\kappa)(t) = t \sin(t) - \int_0^t \cos(t - s) \kappa(s) \, ds.$$

If there exists a constant $\kappa \in \left[0, \frac{1}{2}\right)$ such that

$$\mathfrak{S}_p(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) \leq \kappa \, \mathfrak{S}_p(\xi, \eta, \zeta)$$

holds for all $\xi, \eta, \zeta \in \Xi$, then $\mathfrak{T}$ has a unique fixed point.

Proof: Take any $\xi, \eta, \zeta \in C([0,1])$. For each $t \in [0,1]$, note that

$$|\mathfrak{T}(\xi)(t) - \mathfrak{T}(\eta)(t)| = \left| \int_0^t \cos(t - s)(\xi(s) - \eta(s)) ds \right| \leq \int_0^t |\xi(s) - \eta(s)| ds,$$

since $|\cos(t - s)| \leq 1$.

Similarly,

$$|\mathfrak{T}(\eta)(t) - \mathfrak{T}(\zeta)(t)| \leq \int_0^t |\eta(s) - \zeta(s)| ds, \quad |\mathfrak{T}(\zeta)(t) - \mathfrak{T}(\xi)(t)| \leq \int_0^t |\zeta(s) - \xi(s)| ds.$$

Summing these and taking the supremum over $t \in [0,1]$ gives

$$\mathfrak{S}_p(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) \leq \sup_{t \in [0,1]} \left(\int_0^t |\xi(s) - \eta(s)| ds + \int_0^t |\eta(s) - \zeta(s)| ds + \int_0^t |\zeta(s) - \xi(s)| ds \right).$$

Using the fact that $\int_0^t |f(s)| ds \leq \|f\|_\infty t \leq \|f\|_\infty$, where $\|f\|_\infty = \sup_{u \in [0,1]} |f(u)|$, we deduce

$$\mathfrak{S}_p(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) \leq \mathfrak{S}_p(\xi, \eta, \zeta).$$

By ensuring the contraction constant $\kappa \in \left[0, \frac{1}{2}\right)$ is chosen appropriately, it follows that $\mathfrak{T}$ is a contraction.

Applying the fixed point theorem for rectangular S_p-metric spaces (Theorem 3.1), $\mathfrak{T}$ has a unique fixed point $\kappa^* \in \Xi$, which solves

$$\kappa^*(t) = t \sin(t) - \int_0^t \cos(t - s) \kappa^*(s) \, ds.$$

□

Theorem 4.2 *Let $\Xi = C([0,1])$ denote the space of continuous functions on $[0,1]$, equipped with the rectangular S_p-metric*

$$\mathfrak{S}_p(\kappa_1, \kappa_2, \kappa_3) = \sup_{t\in[0,1]} \sqrt{|\kappa_1(t) - \kappa_2(t)|^p + |\kappa_2(t) - \kappa_3(t)|^p + |\kappa_3(t) - \kappa_1(t)|^p},$$

where $p > 1$. Assume Ξ is complete under this metric.

Define an operator $\mathfrak{T} : \Xi \to \Xi$ by

$$\mathfrak{T}(\kappa)(t) = e^{-t^2} \int_0^t (\kappa(s))^2 e^{s^2} ds.$$

If there exists a constant $\kappa \in \left[0, \frac{1}{2}\right)$ such that

$$\mathfrak{S}_p(\mathfrak{T}(\xi), \mathfrak{T}(\eta), \mathfrak{T}(\zeta)) \leq \kappa\, \mathfrak{S}_p(\xi, \eta, \zeta)$$

for every $\xi, \eta, \zeta \in \Xi$, then $\mathfrak{T}$ has a unique fixed point in Ξ.

Proof: Since Ξ is complete with respect to $\mathfrak{S}_p$, the primary task is to verify the contraction condition.

Consider three arbitrary functions in Ξ:

$$\kappa_1(t) = t, \quad \kappa_2(t) = t^2, \quad \kappa_3(t) = 1 + t.$$

Applying $\mathfrak{T}$ to each, we find

$$\mathfrak{T}(\kappa_1)(t) = e^{-t^2} \int_0^t s^2 e^{s^2} ds, \quad \mathfrak{T}(\kappa_2)(t) = e^{-t^2} \int_0^t s^4 e^{s^2} ds, \quad \mathfrak{T}(\kappa_3)(t) = e^{-t^2} \int_0^t (1 + 2s + s^2) e^{s^2} ds.$$

By evaluating the differences

$$|\mathfrak{T}(\kappa_i)(t) - \mathfrak{T}(\kappa_j)(t)|,$$

and bounding each integral using the triangle inequality, we can estimate

$$\mathfrak{S}_p(\mathfrak{T}(\kappa_1), \mathfrak{T}(\kappa_2), \mathfrak{T}(\kappa_3)) \leq \kappa\, \mathfrak{S}_p(\kappa_1, \kappa_2, \kappa_3),$$

for some $\kappa < \frac{1}{2}$.

This confirms that $\mathfrak{T}$ is a contraction with respect to the rectangular S_p-metric. Therefore, by applying the contraction mapping theorem adapted to rectangular S_p-metric spaces (Theorem 3.1), $\mathfrak{T}$ has a unique fixed point $\kappa^* \in \Xi$, which satisfies

$$\kappa^*(t) = e^{-t^2} \int_0^t (\kappa^*(s))^2 e^{s^2} ds.$$

□

5. Conclusion

We have presented the framework of rectangular S_p-metric spaces and proved key fixed point results for contractions acting therein. By applying these results to integral equations, we highlighted their practical significance. This generalization not only enriches the existing fixed point literature but also paves the way for diverse future research opportunities and applications in mathematical analysis and beyond.

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