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Convergence of a Modified Iterated Lavrentiev Scheme Under Weaker Assumptions

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ABSTRACT: Inverse problems commonly arise in various scientific and engineering applications and are known for their instability with respect to data perturbations, making their numerical treatment both delicate and mathematically demanding. Lavrentiev regularization, particularly its iterative form, is a widely used technique for solving such problems, especially those governed by monotone operators. However, the classical iterative Lavrentiev method requires the computation of the Fréchet derivative at each iteration step, which is computationally expensive and often depends on strong assumptions for convergence analysis. In this study, we propose a simplified variant of the iterative Lavrentiev scheme, in which the Fréchet derivative is computed only once at the initial approximation u_0. We establish convergence and derive error estimates under a relaxed nonlinear condition that is weaker than those typically assumed in the literature. Numerical experiments are presented to confirm the practical applicability of the method.

Key Words: Nonlinear Ill-Posed Problems, Iterative Regularization, Monotone Operators.

Contents

1. Introduction

For the past two decades, mathematicians have been conducting research on solving inverse problems involving nonlinear operators. Addressing these problems is important because many physical phenomena can be modeled in this form

$$F(u) = v, \tag{1.1}$$

where $F : D(F) \subset H \to H$, and F is a nonlinear monotone operator with domain $D(F) \subset H$ and H is a real Hilbert space with the inner product $\langle ., . \rangle$ and the norm $\|.\|$ respectively [2,3,4,5,12]. Here, we are interested for obtaining a stable approximate solution for the exact solution $u^\dagger$ of (1.1). An operator F is said to be monotone if it satisfies the condition

$$\langle F(u_1) - F(u_2), u_1 - u_2 \rangle \geq 0, \forall u_1, u_2 \in D(F). \tag{1.2}$$

In general, the inverse problem described by equation (1.1) is ill-posed, implying that small perturbations in the input data v can lead to substantial deviations in the corresponding solution $u^\dagger$. In real-world scenarios, the exact data v is rarely available, and instead, one typically has access to perturbed measurements $\tilde{v}$, satisfying the noise bound $\|v - \tilde{v}\| \leq \delta$, where $\delta > 0$ quantifies the noise level. Solving equation (1.1) directly with noisy data $\tilde{v}$ often results in highly inaccurate approximations of the true

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solution $u^\dagger$, primarily due to the instability inherent in the problem. Therefore, the development of stable solution techniques becomes imperative, and regularization methods are essential tools in this context. Among these, Tikhonov regularization [4,6,11] is a classical and widely adopted approach. Given the monotonicity of the operator F, Lavrentiev regularization emerges as a particularly appealing method due to its simplicity in implementation. Numerous adaptations of regularization strategies tailored to monotone operators can be found in the literature [2,9,10,13,15,16,17,19,20,21,22,23,24]. Of particular interest are the simplified regularization techniques [15,16,18,19], which have gained prominence owing to their theoretical and numerical tractability. These methods are often based on relaxed assumptions, allowing for rigorous convergence analysis while maintaining practical applicability. In this work, we focus on a modified version of the Lavrentiev iterative scheme, expressed as follows:

$$\tilde{u}_{k+1} = u_0 + (K + \alpha_k I)^{-1} \left(\tilde{v} - F(\tilde{u}_k) + K(\tilde{u}_k - u_0)\right), \tag{1.3}$$

where $\tilde{u}_0 = u_0$ is the known initial guess for the exact solution $u^\dagger$, and $K = F'(u_0)$ denotes the Fréchet derivative of F at u_0. The sequence α_k consists of positive real numbers converging to zero, and there exists a constant $\mu > 1$ such that

$$1 \leq \frac{\alpha_k}{\alpha_{k+1}} \leq \mu \quad \text{for all } k \in \mathbb{N}. \tag{1.4}$$

The advantage of equation (1.3) over standard schemes [8,9,13] is that the Fréchet derivative is computed only at the known initial point, rather than at every step of the iterations. This simplification provides computational benefits compared to traditional methods.

We derive the convergence and convergence analysis under a-posteriori stopping rule and derive error estimates accordingly. In the case of exact data v is available, the iteration (1.3) becomes,

$$u_{k+1} = u_0 + (K + \alpha_k I)^{-1} \left(v - F(\tilde{u}_k) + K(u_k - u_0)\right), \tag{1.5}$$

For our study, the paper is organized as follows. In Section 2, we outline the fundamental assumptions necessary for our analysis. In Section 3, we present the convergence analysis using exact data v. The convergence analysis with noisy data $\tilde{v}$ is provided in Section 4. Numerical examples and conclusion are given in Section 5 and 6 respectively.

2. Basic assumptions and results

In the study of inverse problems certain conditions are required for the operator F, which are given below. Through out this paper we assume that F is Fréchet differentiable in $B_r(u^\dagger) = \{u \in H : \|u - u^\dagger\| < r\} \subset D(F), r > 0$.
Also assume that K is positive and self-adjoint operator. We use following results whenever required.

Theorem 2.1 *For a positive and self-adjoint operator K and any $\alpha > 0$, we have the following*

$$\|(K + \alpha I)^{-1}\| \leq \frac{1}{\alpha}, \tag{2.1}$$

and

$$\|(K + \alpha I)^{-1} K\| \leq 1. \tag{2.2}$$

Proof: [12].

To establish the convergence analysis, the following assumption is required.
Assumption-I: There exist a constant $C > 0$ such that,

$$\|F(u_1) - F(u_2) - K(u_1 - u_2)\| \leq C\|u_1 - u_2\|\|K(u_1 - u_2)\|, \forall u_1, u_2 \in B_r(u^\dagger).$$

We note that the above Assumption-I differs from the standard assumptions typically used. Similar assumptions have been successfully applied in Newton-type and Landweber methods [5,7,14]. However, to the best of our knowledge, there has been little research on incorporating this new assumption into the Lavrentiev method, making this our first attempt in that direction. Moreover, we note that Assumption-I is strictly weaker than the conditions used to establish convergence and error estimates in earlier studies such as [15,16].

The iterative scheme (1.3) was analyzed under the following assumption in [16].
L1 There exist a constant $k_0 > 0$ and $\phi(x, x_0, h) \in H$ satisfying

$$(F'(x) - F'(x_0))h = F'(x_0)\phi(x, x_0, h)$$

with

$$\|\phi(x, x_0, h)\| \leq k_0\|h\|\|x - x_0\|, \forall x, h \in B_{\frac{r}{2}}(x_0), \frac{k_0 r}{2} < 1.$$

Suppose that assumption L1 holds.Then,

$$\begin{aligned}
\|F(u_1) - F(u_2) - K(u_1 - u_2)\| &= \int_0^1 F'(u_1 + t(u_1 - u_2) - K(u_1 - u_2))dt \\
&\leq \int_0^1 \|K\|\|\phi(u_2 + t(u_1 - u_2), u_0, u_1 - u_2)\|dt \\
&\leq \|K\|\frac{k_0 r}{2}\|u_1 - u_2\| \\
&= a\|u_1 - u_2\|, a = \frac{\|K\|k_0 r}{2} \\
&\leq a\|u_1 - u_2\| + \|K(u_1 - u_2)\| \\
&= \|u_1 - u_2\|\|K(u_1 - u_2)\|\Big(\frac{a}{\|K(u_1 - u_2)\|} + \frac{1}{\|u_1 - u_2\|}\Big),
\end{aligned}$$

for $u_1 \neq u_2$ and $u_1 - u_2 \notin Ker(K)$. Therefore,

$$\|F(u_1) - F(u_2) - K(u_1 - u_2)\| \leq C\|u_1 - u_2\|\|K(u_1 - u_2)\|, C = \Big(\frac{a}{\|K(u_1 - u_2)\|} + \frac{1}{\|u_1 - u_2\|}\Big).$$

Hence the Assumption I is weaker than L1.
For getting the results in more precise way we assume that there is a constant $\alpha_0 > 0$ such that $\|K\| \leq \alpha_0$. Moreover, We adopt the following choice for the regularization parameter α_k. Choose the first positive number N such that

$$\alpha_N \leq \frac{(c_0 - 1)\delta^{1/2}}{\|u_0 - u^\dagger\|} < \alpha_k, 0 \leq k < N, c_0 > 1, \quad \text{and} \quad \alpha_0 > \frac{(c_0 - 1)\delta^{1/2}}{\|u_0 - u^\dagger\|} \tag{2.3}$$

Existence of such α_k guaranteed by the fact that $\lim_{k\to\infty} \alpha_k = 0$.

In the following theorem we prove that the iterations in (1.3) well defined $\forall 0 \leq k \leq N$.

Theorem 2.2 *Let the assumption I hold and N be chosen as in (2.3). If* $\max\Big\{\frac{1}{c_0-1} + 2C\eta\|u_0 - u^\dagger\|, 2C\mu\|u_0 - u^\dagger\|\Big\} < 1$, *then*

$$\|\tilde{u}_k - u^\dagger\| \leq 2\|u_0 - u^\dagger\|, \forall 0 \leq k \leq N. \tag{2.4}$$

and

$$\|K(\tilde{u}_k - u^\dagger)\| \leq \eta\alpha_k\|u_0 - u^\dagger\|, \quad \textit{where} \quad \eta = \frac{\frac{c_0\mu}{c_0-1}}{1 - 2C\mu\|u_0 - u^\dagger\|} \tag{2.5}$$

$\forall 0 \leq k \leq N$. *In particular if* $r > 2\|u_0 - u^\dagger\|$, *then* $\tilde{u}_k \in B_r(u^\dagger), 0 \leq k \leq N$.

Proof:
We will prove the result using mathematical induction. The result is trivial when $k = 0$. Suppose that the result is true for $0 \leq k < N$

$$\begin{aligned}
\tilde{u}_{k+1} &= u_0 + (K + \alpha_k I)^{-1}(K(\tilde{u}_k - u_0) + \tilde{v} - F(\tilde{u}_k)) \\
\tilde{u}_{k+1} - u^\dagger &= (K + \alpha_k I)^{-1}\left(\tilde{v} - F(\tilde{u}_k) + K(u_0 - u^\dagger) + \alpha_k(u_0 - u^\dagger) + K(\tilde{u}_k - u_0)\right).
\end{aligned}$$

$$\tilde{u}_{k+1} - u^\dagger = (K+\alpha_k I)^{-1}\left(\tilde{v} - v + \alpha_k(u_0 - u^\dagger) - (F(\tilde{u}_k) - F(u^\dagger) - K(\tilde{u}_k - u^\dagger)\right) \tag{2.6}$$

Using assumptions and induction

$$\begin{aligned}
\|\tilde{u}_{k+1} - u^\dagger\| &\leq \|(K+\alpha_k I)^{-1}(\tilde{v} - v)\| + \|(K+\alpha_k I)^{-1}\alpha_k(u_0 - u^\dagger)\| \\
&+ \|(K+\alpha_k I)^{-1}(F(\tilde{u}_k) - F(u^\dagger) - K(\tilde{u}_k - u^\dagger)\| \\
&\leq \frac{\delta}{\alpha_k} + \|u_0 - u^\dagger\| + \frac{C\|\tilde{u_k} - u^\dagger\|\|K(\tilde{u_k} - u^\dagger)\|}{\alpha_k} \\
&\leq \frac{\|u_0 - u^\dagger\|\delta^{1/2}}{c_0 - 1} + \|u_0 - u^\dagger\| + \frac{2C\|u_0 - u^\dagger\|\eta\alpha_k\|u_0 - u^\dagger\|}{\alpha_N} \\
&= \|u_0 - u^\dagger\|\left(\frac{\delta^{1/2}}{c_0 - 1} + 1\right) + 2c\eta\|u_0 - u^\dagger\|^2 \\
&= \|u_0 - u^\dagger\|\left(\frac{1}{c_0 - 1} + 1 + 2c\eta\|u_0 - u^\dagger\|\right) \\
&\leq 2\|u_0 - u^\dagger\|.
\end{aligned}$$

Hence by mathematical induction on k,

$$\|\tilde{u}_k - u^\dagger\| \leq 2\|u_0 - u^\dagger\|, \forall \quad 0 \leq k \leq N.$$

Also from (2.6),

$$\begin{aligned}
K(\tilde{u}_{k+1} - u^\dagger) &= -K(K+\alpha_k I)^{-1}(F(\tilde{u}_k) - F(u^\dagger) - K(\tilde{u}_k - u^\dagger)) - K(K+\alpha_k I)^{-1}(v - \tilde{v}) \\
&+ \alpha_k K\left(K + \alpha_k I)^{-1}\right)(u_0 - u^\dagger)\right).
\end{aligned}$$

Therefore by assumptions

$$\begin{aligned}
\|K(\tilde{u}_{k+1} - u^\dagger)\| &\leq C\|\tilde{u}_k - u^\dagger\|\|K(\tilde{u}_k - u^\dagger)\| + \delta + \alpha_k\|u_0 - u^\dagger\| \\
&\leq C\|\tilde{u}_k - u^\dagger\|\|K(\tilde{u}_k - u^\dagger)\| + \frac{\alpha_k\delta^{1/2}}{c_0 - 1}\|u_0 - u^\dagger\| + \alpha_k\|u_0 - u^\dagger\|.
\end{aligned}$$

By induction on k, we will get

$$\begin{aligned}
\|K(\tilde{u}_{k+1} - u^\dagger)\| &\leq 2C\eta\alpha_k\|u_0 - u^\dagger\|^2 + \frac{\alpha_k\delta^{1/2}}{c_0 - 1}\|u_0 - u^\dagger\| + \alpha_k\|u_0 - u^\dagger\| \\
&= \left(2C\eta\|u_0 - u^\dagger\| + \frac{\delta^{1/2}}{c_0 - 1} + 1\right)\alpha_k\|u_0 - u^\dagger\| \\
&\leq \left(2C\eta\|u_0 - u^\dagger\| + \frac{1}{c_0 - 1} + 1\right)\|u_0 - u^\dagger\|\mu\alpha_{k+1} \\
&\leq \eta\alpha_{k+1}\|u_0 - u^\dagger\|, \eta = \frac{\frac{c_0\mu}{c_0 - 1}}{1 - 2C\mu\|u_0 - u^\dagger\|}.
\end{aligned}$$

Hence

$$\|K(\tilde{u}_k - u^\dagger)\| \leq \eta\alpha_k\|u_0 - u^\dagger\|, \forall \quad 0 \leq k \leq N.$$

Thus from (2.4), for any $r > 2\|u_0 - u^\dagger\|, \tilde{u}_k \in B_r(u^\dagger), \forall k, 0 \leq k \leq N$.

2.1. A-posteriori stopping rule

We propose the following stopping rule to terminate the iteration:
Let $\tilde{N}$ be the first positive integer satisfying

$$\|F(\tilde{u}_{\tilde{N}}) - \tilde{v}\| < \tau_0\delta^{1/2}, \tau_0 > 1 \quad \text{is a constant}.$$

Choose the stopping index to be the same N as defined in equation (2.3). In the following theorem, we establish the existence of such a stopping index N.

Theorem 2.3 *Under the hypothesis of Theorem 2.2, we have*

$$\|F(\tilde{u}_N) - \tilde{v}\| \leq \tau_0 \delta^{1/2}, \tau_0 = 1 + 2\eta(c_0 - 1)\delta^{1/2}. \tag{2.7}$$

Proof:

$$\begin{aligned}
\|F(\tilde{u}_N) - \tilde{v}\| &= \|F(\tilde{u}_N) - v - K(\tilde{u}_N - u^\dagger) + K(\tilde{u}_N - u^\dagger) - \tilde{v} + v\| \\
&\leq C\|\tilde{u}_N - u^\dagger\|\|K(\tilde{u}_N - u^\dagger)\| + \eta\alpha_N\|(u_0 - u^\dagger)\| + \delta \\
&\leq \eta\frac{(c_0-1)\delta^{1/2}}{\|(u_0 - u^\dagger)\|}(2C\|(u_0 - u^\dagger)\| + 1) + \delta \\
&\leq \delta^{1/2}(1 + 2\eta(c_0 - 1)\delta^{1/2}), \quad \text{since} \quad 2C\|(u_0 - u^\dagger)\| \leq 1 \\
&= \tau_0\delta^{1/2}.
\end{aligned}$$

Thus,

$$\|F(\tilde{u}_N) - \tilde{v}\| \leq \tau_0 \delta^{1/2}, \tau_0 = 1 + 2\eta(c_0 - 1)\delta^{1/2}.$$

3. Convergence analysis with exact data

In this section we derive the estimates with exact data v.

Lemma 3.1 *Let the assumption I hold and N be chosen as in 2.3. Suppose that* $\max\Big\{2C\zeta\|u_0 - u^\dagger\|, 2C\mu\|u_0 - u^\dagger\|\Big\} < 1$, *then* $\forall k$

$$\|u_k - u^\dagger\| \leq 2\|u_0 - u^\dagger\| \tag{3.1}$$

and

$$\|K(u_k - u^\dagger)\| \leq \zeta\alpha_k\|u_0 - u^\dagger\|, \tag{3.2}$$

where $\zeta = \frac{\mu}{1 - 2C\mu\|u_0 - u^\dagger\|}$. *Moreover if* $r > 2\|u_0 - u^\dagger\|, u_k \in B_r(u^\dagger), \forall 0 \leq k \leq N$.

Proof:
We prove the results by using induction on k.
If $k = 0$, the result is trivial. Suppose that the result is true for $0 \leq k < N$. We have from (1.5),

$$\begin{aligned}
u_{k+1} &= x_0 + (K + \alpha_k)^{-1}(F(u_k) - u - K(u_k - u_0)) \\
u_{k+1} - u^\dagger &= (K + \alpha_k I)^{-1}\left(K(u_k - u^\dagger) + \alpha_k(u_0 - u^\dagger) + v - F(u_k)\right).
\end{aligned}$$

Thus,

$$u_{k+1} - u^\dagger = -(K + \alpha_k I)^{-1}\left(F(u_k) - F(u^\dagger) + K(u_k - u^\dagger)\right) - \alpha_k(K + \alpha_k)^{-1}(u_0 - u^\dagger). \tag{3.3}$$

Taking norm on both sides,

$$\begin{aligned}
\|u_{k+1} - u^\dagger\| &\leq \|(K + \alpha_k I)^{-1}\left(F(u_k) - F(u^\dagger) + K(u_k - u^\dagger)\right)\| + \alpha_k\|(K + \alpha_k)^{-1}(u_0 - u^\dagger)\| \\
&\leq \frac{C}{\alpha_k}\|u_k - u^\dagger\|\|K(u_k - u^\dagger)\| + \|u_0 - u^\dagger\| \\
&\leq 2C\zeta(\|u_0 - u^\dagger\| + 1)\|u_0 - u^\dagger\|.
\end{aligned}$$

By the assumption of the theorem, we will get, $\|u_{k+1} - u^\dagger\| \leq 2\|u_0 - u^\dagger\|$.
Hence

$$\|u_k - u^\dagger\| \leq 2\|u_0 - u^\dagger\|, \forall 0 \leq k \leq N,$$

and for any $r > 2\|u_0 - u^\dagger\|, u_k \in B_r(u^\dagger) \forall k$.
We have from (3.3),

$$\begin{aligned} K(u_{k+1} - u^\dagger) &= K(K + \alpha_k I)^{-1}\left(K(u_k - u^\dagger) + \alpha_k(u_0 - u^\dagger) + v - F(u_k)\right) \\ &= K(K + \alpha_k I)^{-1}\left(F(u_k) - v - K(u_k - u^\dagger)\right) + \alpha_k K(K + \alpha_k I)^{-1}(u_0 - u^\dagger) \\ \|K(u_{k+1} - u^\dagger)\| &\leq C\|u_k - u^\dagger\|\|K(u_k - u^\dagger\|) + \alpha_k\|u_0 - u^\dagger\| \\ &\leq 2C\|u_0 - u^\dagger\| + \eta\alpha_k\|u_0 - u^\dagger\| + \alpha_k\|u_0 - u^\dagger\| \\ &\leq \left(2C\eta\|u_0 - u^\dagger\| + 1\right)\|u_0 - u^\dagger\|\alpha_N \\ &\leq \left(2C\zeta\|u_0 - u^\dagger\| + 1\right)\|u_0 - u^\dagger\|\mu\alpha_{k+1} \\ &= \left(2C\zeta\mu\|u_0 - u^\dagger\| + \mu\right)\|u_0 - u^\dagger\|\alpha_{k+1} \\ &\leq \zeta\alpha_{k+1}\|u_0 - u^\dagger\|, \zeta = \frac{\mu}{1 - 2c\mu\|u_0 - u^\dagger\|}. \end{aligned}$$

Thus by induction,

$$\|K(u_k - u^\dagger)\| \leq \zeta\alpha_k\|u_o - u^\dagger\|, 0 \leq k \leq N.$$

Lemma 3.2 *Let $\{\theta_K\}$ be a sequence of positive real numbers such that*

$$\theta_{k+1} \leq a\theta_k + b_k, \quad \text{where} \quad a > 0 \quad \text{and} \quad b_k \tag{3.4}$$

be a sequence of positive real numbers satisfying $\frac{b_k}{b_{k+1}} < s, s > 1$. If $sa < 1$ and $\theta_0 \leq \frac{sb_0}{1-a}$, then $\theta_k \leq \frac{sb_k}{1-a} \forall k$.

Proof:
We prove the result using induction. Clearly the result is true for $k = 0$. Assume the result is true for k. We have,

$$\begin{aligned} \theta_{k+1} &\leq a\theta_k + b_k \\ &\leq \frac{asb_k}{1 - sa} + b_k \\ &= \frac{sb_{k+1}}{1 - s}. \end{aligned}$$

Thus the result follows.
In the following theorem we prove that the iterations defined in (1.5) is well defined.

In the following theorem we prove that the $u_k \to u^\dagger$ as $k \to \infty$.

Theorem 3.3 *Suppose that assumptions I holds and that $\max\{2C\zeta\|u_0 - u^\dagger\|, 2C\mu\|u_0 - u^\dagger\|\} < 1$ and let $\mu\zeta C\|u_0 - u^\dagger\| < 1$, and u_0 is choosen such that $u^\dagger - u_0 \in N(K)^\perp$ then*

$$\|u_k - u^\dagger\| \leq c_1\gamma_k, \forall 0 \leq k \leq N, \tag{3.5}$$

where $\gamma_k = \|\alpha_k(K + \alpha_k I)^{-1}(u_0 - u^\dagger)\|$ and $c_1 = \frac{2\mu}{1 - C\mu\zeta\|u_0 - u^\dagger\|}$. Furthermore as $k \to \infty, u_k \to u^\dagger$.

Proof: We have from (3.3),

$$u_{k+1} - u^\dagger = -(K + \alpha_k I)^{-1}(F(u_k) - F(u^\dagger) - K(u_k - u^\dagger) + \alpha_k(K + \alpha_k I)^{-1}(u_0 - u^\dagger)$$

Thus

$$\begin{aligned} \|u_{k+1} - u^\dagger\| &= \|(K + \alpha_k I)^{-1}(F(u_k) - F(u^\dagger) - K(u_k - u^\dagger)\| + \|\alpha_k(K + \alpha_k I)^{-1}(u_0 - u^\dagger)\| \\ &\leq \frac{C}{\alpha_k}\|(u_k - u^\dagger\|\|K(u_k - u^\dagger)\| + \gamma_k \\ &\leq \frac{C}{\alpha_k}\alpha_k\zeta\|u_0 - u^\dagger\|\|u_k - u^\dagger\| + \gamma_k \\ \frac{\|u_{k+1} - u^\dagger\|}{2} &\leq C\zeta\|u_0 - u^\dagger\|\frac{\|u_k - u^\dagger\|}{2} + \frac{\gamma_k}{2} \\ &\leq C\zeta\|u_0 - u^\dagger\|\frac{\|u_k - u^\dagger\|}{2} + \gamma_k \end{aligned}$$

Thus,

$$\frac{\|u_{k+1}-u^\dagger\|}{2} \leq C\zeta\|u_0-u^\dagger\|\frac{\|u_k-u^\dagger\|}{2}+\gamma_k. \tag{3.6}$$

Thus (3.6) is of the form

$\theta_{k+1} \leq a\theta_k + b_k$, where $\theta_{k+1} = \frac{\|u_{k+1}-u^\dagger\|}{2}, b_k = \gamma_k, a = C\zeta\|u_0-u^\dagger\|$. Now let $\{E_\lambda, \lambda \geq 0\}$ be the spectral family generated by the operator K. By the spectral theorem of the self-adjoint operator we have $K = \int_0^{\|K\|} \lambda dE_\lambda$.
Therefore,

$$(K+\alpha_k I)^{-2}(u_0-u^\dagger) = \int_0^{\|K\|} \frac{1}{(\lambda++\alpha_k)^2} dE_\lambda(u_0-u^\dagger).$$

Now,

$$\begin{aligned} b_k^2 &= \|\alpha_k(K+\alpha_k I)^{-1}(u_0-u^\dagger)\|^2 \\ &= \alpha_k^2 \left\langle (K+\alpha_k I)^{-2}(u_0-u^\dagger), u_0-u^\dagger \right\rangle \\ &\leq \frac{\alpha_k^2}{\alpha_{k+1}^2}\left\langle \int_0^{\|K\|}\frac{\alpha_{k+1}^2}{\lambda+\alpha_{k+1}^2}dE_\lambda(u_0-u^\dagger), u_0-u^\dagger \right\rangle \\ &\leq \mu^2 b_{k+1}^2. \end{aligned}$$

Hence $b_k \leq \mu b_{k+1}$. In the Lemma 3.2, take $s=\mu$. Note that $b_0 = \|\alpha_0(K+\alpha_0 I)^{-1}(u_0-u^\dagger)\|$. Hence,

$$\begin{aligned} b_0 &\geq \frac{\alpha_0(u_0-u^\dagger)}{\|K+\alpha_0 I\|} \\ &\geq \frac{\alpha_0(u_0-u^\dagger)}{2\alpha_0} \\ &= \frac{\|u_0-u^\dagger\|}{2} = \theta_0. \end{aligned}$$

Therefore,

$$\theta_0 \leq b_0 < \frac{b_0\mu}{1-C\zeta\mu\|u_0-u^\dagger\|}.$$

Hence by Lemma 3.2,

$$\theta_k \leq \frac{\mu}{1-C\zeta\mu\|u_o-u^\dagger\|}\gamma_k, \forall k.$$

Hence

$$\|u_k-u^\dagger\| \leq \frac{2\mu}{1-C\zeta\mu\|u_o-u^\dagger\|}\gamma_k = c_1\gamma_k, \forall k.$$

Since $u^\dagger - u_0 \in N(K)^\perp = \overline{R(K)}$, there exists some element $w \in D(F)$ such that $u^\dagger - u_0 = Kw$. Hence,

$$\begin{aligned} \gamma_k &= \alpha_k\|(K+\alpha_k I)^{-1}(u_0-u^\dagger)\| \\ &= \alpha_k\|(K+\alpha_k I)^{-1}Kw\| \\ &\leq \alpha_k\|w\| \to 0 \quad \text{as} \quad k\to\infty. \end{aligned}$$

Hence by 3.5, $u_k \to u^\dagger$ as $k \to \infty$.

4. Convergence analysis for noisy data

In this section we will prove our main result. First we find the estimate between $\|\tilde{u}_k - u_k\|$.Next we state the following lemma.

Lemma 4.1 *Let $\{\theta_K\}$ be a sequence of positive real numbers such that*

$$\theta_{k+1} \leq a\theta_k + b, a, b > 0. \tag{4.1}$$

If $a < 1$ and $\theta_0 \leq \frac{b}{1-a}$, then $\theta_k \leq \frac{b}{1-a} \forall k$.

Proof: [15].

Theorem 4.2 *Under the hypothesis of Theorem 2.2 and 3.1 and if $C\alpha_0\|w\|(\zeta+\eta) < 1$, and u_0 is chosen such that $u^\dagger - u_0 \in N(K)^\perp$ then,*

$$\|\tilde{u}_k - u_k\| \leq c_2 \frac{\delta}{\alpha_k}, c_2 = \frac{1}{1 - C\alpha_0\|w\|(\zeta+\eta)} \tag{4.2}$$

Proof:
From (1.3) and (1.5),

$$\begin{aligned} \tilde{u}_{k+1} - u_{k+1} &= u_0 + (K + \alpha_k I)^{-1}(F(\tilde{u_k}) - \tilde{v} - K(\tilde{u_k} - u_0)) - u_0 \\ &+ (K + \alpha_k I)^{-1}(F(u_k) - v - K(u_k - u_0)) \\ &= (K + \alpha_k I)^{-1}(F(\tilde{u_k}) - F(u_k) - K(\tilde{u_k} - u_k)) + (K + \alpha_k I)^{-1}(v - \tilde{v}). \end{aligned}$$

Now,

$$\begin{aligned} \|\tilde{u}_{k+1} - u_{k+1}\| &\leq \frac{1}{\alpha_k} C\|\tilde{u_k} - u_k\|\|K(\tilde{u}_k - u_k)\| + \frac{\delta}{\alpha_k} \\ &\leq \frac{1}{\alpha_k} C\|\tilde{u}_k - u_k\|\Big(\|K(\tilde{u_k} - u^\dagger)\| + \|K(u_k - u^\dagger\|\Big) + \frac{\delta}{\alpha_k} \\ &\leq \frac{1}{\alpha_k} C\|\tilde{u}_k - u_k\|\Big(\alpha_k\eta\|u_0 - u^\dagger\| + \alpha_k\zeta\|u_0 - u^\dagger\|\Big) + \frac{\delta}{\alpha_k} \\ &= \frac{C\alpha_k\|\tilde{u}_k - u_k\|\|u^\dagger - u_0\|(\zeta+\eta) + \delta}{\alpha_k} \\ &= \frac{C\alpha_k\|\tilde{u}_k - u_k\|\|Kw\|(\zeta+\eta) + \delta}{\alpha_k}. \end{aligned}$$

Thus,

$$\alpha_k\|\tilde{u}_{k+1} - u_{k+1}\| \quad \leq \quad C\alpha_0\|w\|(\zeta+\eta)\alpha_k\|\tilde{u_k} - u_k\| + \delta.$$

Since $\alpha_{k+1} < \alpha_k$ we get,

$$\alpha_{k+1}\|\tilde{u}_{k+1} - u_{k+1}\| \leq C\alpha_0\|w\|(\zeta+\eta)\alpha_k\|\tilde{u_k} - u_k\| + \delta. \tag{4.3}$$

Take $\theta_{k+1} = \alpha_{k+1}\|\tilde{u}_{k+1} - u_{k+1}\|, a = C\alpha_0\|w\|(\zeta+\eta)$, and $b = \delta$. Then (4.3) transforms to the form of (4.1). Now,

$$\theta_0 = \alpha_0\|\tilde{u}_0 - u_0\| = 0 \leq \frac{\delta}{1 - C\alpha_0\|w\|(\zeta+\eta)}.$$

Thus, by lemma (4.1)

$$\alpha_k\|\tilde{u}_k - u_k\| \leq \frac{\delta}{1 - C\alpha_0\|w\|(\zeta+\eta)}.$$

Therefore,

$$\|\tilde{u}_k - u_k\| \leq c_2 \frac{\delta}{\alpha_k}, c_2 = \frac{1}{1 - C\alpha_0\|w\|(\zeta+\eta)}.$$

Next we will prove our main result.

Theorem 4.3 *Under the assumptions of Theorem 4.2, the sequence $\tilde{u}_N$ converges to the exact solution $u^\dagger$ as $\delta \to 0$, and the following error estimate holds:*

$$\|\tilde{u}_N - u^\dagger\| = O(\delta^{1/2}). \tag{4.4}$$

Proof:

$$\begin{aligned}
\|\tilde{u}_N - u^\dagger\| &\leq \|\tilde{u}_N - u_N\| + \|u_N - u^\dagger\| \\
&\leq \left(\frac{c_2\delta}{\alpha_N} + c_1\gamma_N\right) \\
&\leq c_3\left(\frac{\delta}{\alpha_N} + \gamma_N\right), c_3 = \max\{c_1, c_2\}.
\end{aligned}$$

Thus

$$\|\tilde{u}_N - u^\dagger\| \leq c_3\left(\frac{\delta}{\alpha_N} + \gamma_N\right). \tag{4.5}$$

Now,

$$\frac{\delta}{\alpha_N} \leq \frac{\mu\delta}{\alpha_{N-1}} \leq \frac{\mu\delta^{1/2}\|u_0 - u^\dagger\|}{c_0 - 1}. \tag{4.6}$$

Also $\gamma_N = \alpha_N\|(K+\alpha_N I)^{-1})(u_0 - u^\dagger)\| = \alpha_N\|(K+\alpha_N I)^{-1})Kw\| \leq \alpha_N\|w\|$.
By the choice of regularization parameter α_N,

$$\gamma_N \leq \alpha_N\|w\| \leq \frac{(c_0-1)\delta^{1/2}\|w\|}{\|u_0 - u^\dagger\|}. \tag{4.7}$$

Substituting (4.6) and (4.7) in (4.5),we get

$$\begin{aligned}
\|\tilde{u}_N - u^\dagger\| \leq c_3\left(\frac{\delta}{\alpha_N} + \gamma_N\right) &\leq c_3\left(\frac{\mu\delta^{1/2}\|u_0 - u^\dagger\|}{c_0 - 1} + \frac{(c_0-1)\delta^{1/2}\|w\|}{\|u_0 - u^\dagger\|}\right) \\
&= c_3\left(\frac{\mu\|u_0 - u^\dagger\|}{c_0 - 1} + \frac{(c_0-1)\|w\|}{\|u_0 - u^\dagger\|}\right)\delta^{1/2} \to 0 \quad \text{as} \quad \delta \to 0.
\end{aligned}$$

Hence $\tilde{u}_N \to u^\dagger$ as $k \to \infty$ and the estimate

$$\|\tilde{u}_N - u^\dagger\| = O(\delta^{1/2}).$$

5. Numerical Examples

In this section, we implement the numerical scheme using two different examples. The computations were performed in MATLAB on a PC equipped with a 2.5 GHz Intel Core i3 processor and 4GB of RAM.

Example 5.1 *Consider the following nonlinear integral operator equation defined on $H = L^2[0,1]$ [8,9].*

$$F(u) = B(u) + (\arctan u)^3 = \int_0^1 e^{-|x-y|}u(y)dy + (\arctan u)^3. \tag{5.1}$$

For the data,

$$v(x) = \begin{cases} 0 & \frac{1}{3} \le x \le \frac{2}{3} \\ 2 + (\arctan 1)^3 - e^{x-1} - e^{-x} & \text{otherwise} \end{cases}$$

the solution is

$$u(x) = \begin{cases} 0 & \frac{1}{3} \le x \le \frac{2}{3} \\ 1 & \text{otherwise.} \end{cases}$$

The Fréchet derivative of F is

$$F^{'}(u)h = \frac{3(\arctan u)^2}{(1+u^2)}h + \int_0^1 \exp(-|x-y|)h(y)dy \tag{5.2}$$

To verify assumption there exists $\alpha_0 > 0$, from [8,9] we have the result

$$\|F'(u)\| \le 1 + \frac{\sqrt{2}}{\pi}. \tag{5.3}$$

Now, we proceed to verify assumption I,

$$F(u) - F(v) \quad = \quad \int_0^1 \exp(-|x-y|)(u(y) - v(y))dy + (\arctan u)^3 - (\arctan v)^3.$$

By mean value theorem of Fréchet derivative we have,

$$(\arctan u)^3 - (\arctan v)^3 = c_1(w)(u-v), c_1(w) = \frac{(3\arctan w(x))^2}{1+w^2(x)} \quad \text{for some} \quad w(x).$$

Thus,

$$\begin{aligned} F(u) - F(v) &= \int_0^1 \exp(-|x-y|)(u(y) - v(y))dy + c_1(w)(u-v) \\ &= K(u-v) + (c_1(w) - c_1(u_0))(u-v) \\ \|F(u) - F(v)\| &\le \|K(u-v)\| + \|(c_1(w) - c_1(u_0))(u-v)\| \\ &\le \|K(u-v)\| + \|(c_1(w) - c_1(u_0))\|\|(u-v)\|. \end{aligned}$$

Thus

$$\begin{aligned} \|F(u) - F(v) - K(u-v)\| &\le \|F(u) - F(v)\| + \|K(u-v)\|. \\ &\le 2 \quad \|K(u-v)\| + \|(c_1(w) - c_1(u_0))\|\|(u-v)\|. \\ &\le 2\|K(u-v)\| + \|(c_1(w) - c_1(u_0))\|\|(u-v)\|. \\ &= \|K(u-v)\| \left(2 + \frac{\|(c_1(w) - c_1(u_0))\|}{\|K(u-v)\|}\|u-v\|\right) \\ &\le C\|K(u-v)\|\|u-v\| \quad \text{where,} \end{aligned}$$

$$C \ge \frac{2}{\|u-v\|} + \frac{\|(c_1(w) - c_1(u_0))\|}{\|K(u-v)\|}.$$

Thus Assumption I is verified.
Let u_0 be the initial guess of the exact solution $u^\dagger$. Existence of w such that $u^\dagger - u_0 = Kw$, is as follows.

$$u^\dagger(x) - u_0(x) = Kw(x) \quad \text{for some} \quad w(x) \in D(F).$$

Therefore,

$$\begin{aligned} u^\dagger(x) - u_0(x) &= Kw(x) \\ &= \frac{3(\arctan u_0(x))^2}{1+u_0^2(x)}w(x) + \int_0^1 e^{-|x-y|}w(y)dy, \end{aligned}$$

which can be written by

$$w(x) \quad = \quad \frac{1+u_0^2(x)}{3(\arctan u_0(x))^2}(u^\dagger(x)-u_0(x)) - \frac{1+u_0^2(x)}{3(\arctan u_0(x))^2}\int_0^1 e^{-|x-y|}w(y)dy.$$

For any $u_0(x)$ such that $\arctan u_0(x) \neq 0$, $-\frac{1+u_0^2(x)}{3(\arctan u_0(x))^2}$ is always a real number. Therefore

$$w(x) = g(x) + \lambda \int_0^1 e^{-|x-y|}w(y) \tag{5.4}$$

is a Fredelhom integral equation of second kind, with $g(x) = \frac{1+u_0^2(x)}{3(\arctan u_0(x))^2}(u^\dagger(x)-u_0(x))$ and $\lambda = -\frac{1+u_0^2(x)}{3(\arctan u_0(x))^2}$. For proving existence and uniqueness of solution for (5.4), we use the following theorem in [1].

Theorem 5.2 *An integral equation of the form*

$$f(x) = g(x) + \lambda \int_0^1 K(x,y)f(y),$$

where $K : [a,b]\times[a,b] \to \mathbb{R}$ *and* $g : (a,b) \to \mathbb{R}$ *are continuous and if* $\sup_{a\le x\le 1}\int_a^b |\lambda K(x,y)| < 1$, *there exist a unique solution* $f : [a,b] \to \mathbb{R}$.

Here we have,

$$\begin{aligned} K(x,y) &= e^{-|x-y|} \\ \sup_{0\le x\le 1}\int_0^1 |K(x,y)| &= \sup_{a\le x\le 1}\int_0^1 e^{-|x-y|} \\ &= \sup_{0\le x\le 1}\{e^{-x}\int_0^x e^y dy + e^x \int_x^1 e^{-y}dy\} \\ &= \sup_{0\le x\le 1}(2-e^{-x}-e^{x-1}). \end{aligned}$$

The maximum value of $2-e^{-x}-e^{x-1}$ is 0.7869, which occurs at $x = 0.5$. Therefore

$$\sup_{0\le x\le 1}\int_0^1 e^{-|x-y|}dy \le 0.7869.$$

In our Example 5.1 take $u_0(x) = 1.23$, therefore $\lambda = -1.0618$,
Therefore, $\sup_{a\le x\le 1}\int_a^b |\lambda K(x,y)| < 0.7869 \times 1.0618 \approx 0.83554 < 1$.
So there is a unique solution $w(x)$ satisfying (5.4). Hence all assumptions have been verified.
The operator $F(u)$ in (5.1) and $F'(u)$ in (5.2) is discretized by Trapezoidal rule with N=100 points. We take $\alpha_0 = 1.5$. For our analysis, we use $0.1, 0.01, 0.001$, and 0.001 of data errors. The computational results are presented in Table 1. The approximate solutions corresponding to data errors of $0.1, 0.01, 0.001$, and 0.0001 are shown in Figures 1, 2, 3, and 4, respectively. Figure 9 illustrates the perturbed data with an error of 0.01. The error behavior is depicted in Figure 10. These results demonstrate that the proposed method produces effective computational outcomes.

Table 1: Computational results for Example 1

δ	N	α_N	$\|u_N - u^\dagger\|$	Expected ($O(\delta^{1/2})$)
0.1	5	0.0024	0.2111	0.3162
0.01	6	0.00048	0.0791	0.1
0.001	7	0.00009	0.0325	0.03162
0.0001	9	0.0000034	0.000056	0.01

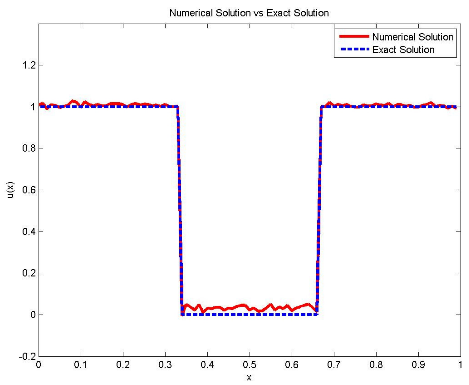

Figure 1: solution when $\delta = 0.1$

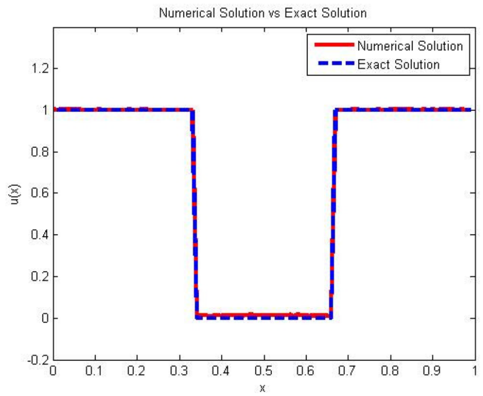

Figure 2: solution when $\delta = 0.01$

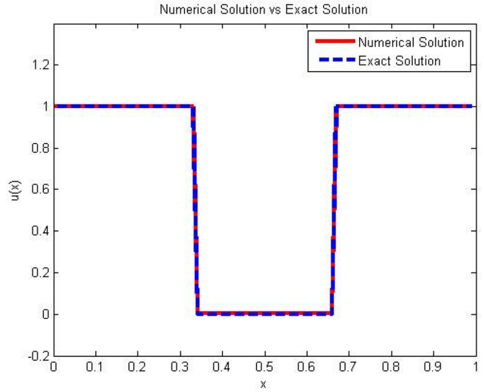

Figure 3: solution when $\delta = 0.001$

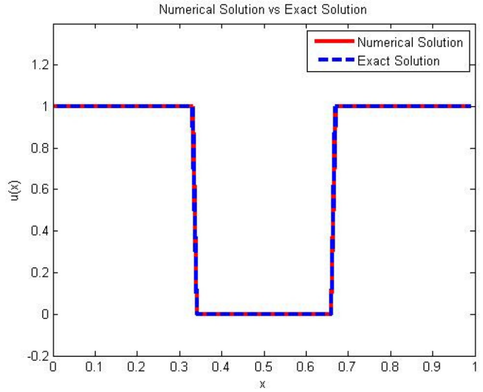

Figure 4: solution when $\delta = 0.0001$

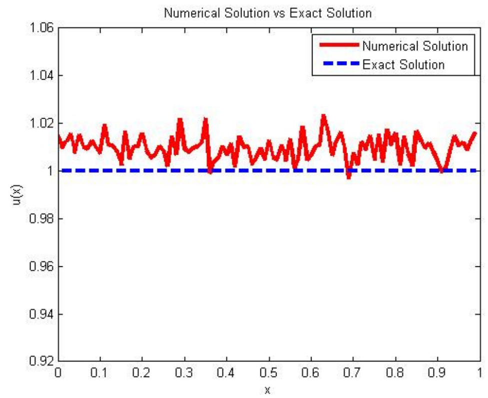

Figure 5: Solution when $\delta = 0.1$

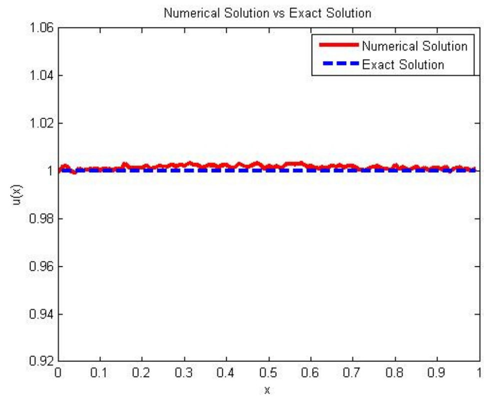

Figure 6: Solution when $\delta = 0.01$

Example 5.3

In this example, for the operator (5.1), the exact solution is $u(x) = 1$. We select the initial guess as $u_0(x) = 1.4$. In this example also we consider data errors of $0.1, 0.01, 0.001$, and 0.0001. The computational results are shown in Table 2. The approximate solutions corresponding to these data errors are displayed in Figures 5, 6, 7, and 8, respectively. Figure 11 presents the perturbed data with an error of 0.01, while the error behavior is shown in Figure 12. These results also indicate that the proposed method yields efficient computational outcomes.

Table 2: Computational results for Example 2

δ	N	α_N	$\|u_N - u^\dagger\|$	Expected $(O(\delta^{1/2}))$
0.1	3	0.0121	0.01123	0.3162
0.01	4	0.0011	0.0174	0.1
0.001	4	0.0011	0.0171	0.03162
0.0001	5	0.00098	0.00536	0.01

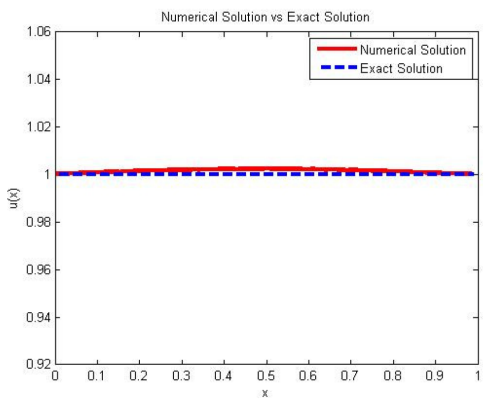

Figure 7: Solution when $\delta = 0.001$

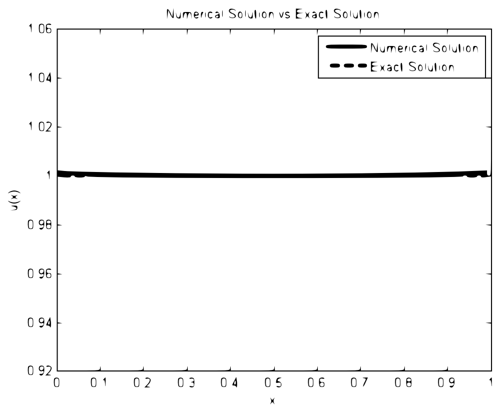

Figure 8: Solution when $\delta = 0.0001$

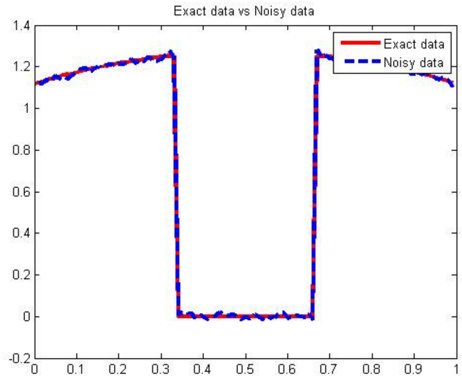

Figure 9: Exact data and noisy data when $\delta = 0.01$

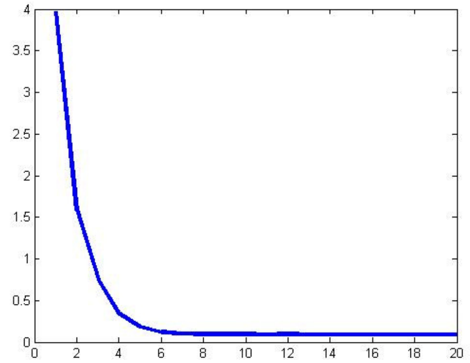

Figure 10: Error behavior when $\delta = 0.01$

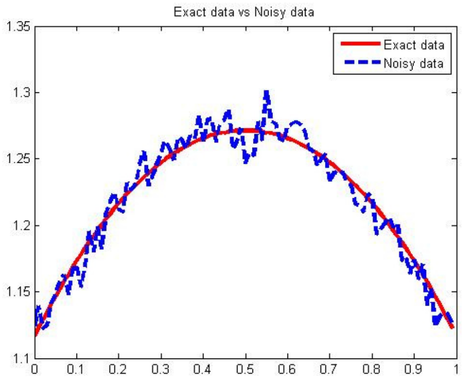

Figure 11: Exact data and noisy data when $\delta = 0.01$

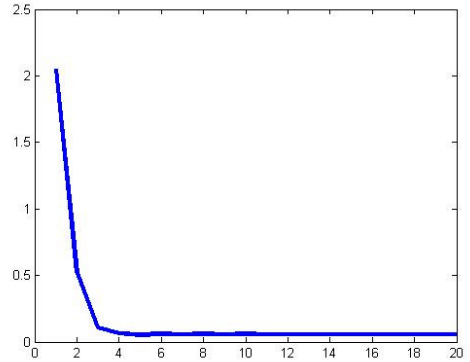

Figure 12: Error behivor when $\delta = 0.01$

6. Conclusion

In this paper, we examined the potential of the modified Lavrentiev method under a new non-linearity condition imposed on the operator. Under this assumption, we demonstrated that the scheme converges to the exact solution as $\delta \to 0$. Additionally, given appropriate conditions on $u_0 - u^\dagger$, we derived the convergence rate of the scheme. Lastly, all theoretical results were validated through numerical examples.

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