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Applications of Haar Wavelet Transform in Thermal Mitigation, Bacterial Growth Simulation, Urban Demographic Dynamics and Vehicle Suspension Dynamics [*]

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ABSTRACT: In the field of signal and image processing, the Haar wavelet transform plays a vital role because of its optimal and productive computational properties. In this paper, the Haar wavelet transform method was implemented to find a solution of differential equations that are used in daily life. The study provides a comprehensive analysis of the Haar wavelet transform that helps both applied and theoretical research in the field of science and technology.

Keywords: Discrete wavelet transform, Haar wavelet, differential equations, wavelet analysis.

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1. Introduction

Over the last two decades wavelet theory contributes in the field of science and technology to solve various differential equations. Wavelet analysis is an effective tool used in image, signal processing and numerical analysis [6]. Wavelet transform methods are useful to find solutions of differential and integral equations having highly localized in scale and position [2]. Fourier transform is an effective resource to analyze the stationary signal while wavelet transform is suitable for both stationary as well as non-stationary signals [10,1]. The French word 'ondelette' means small wave.'onde' translated into English word 'wave' namely 'wavelet' [10]. The oldest and more user-friendly wavelet is 'Haar wavelet' that coined by Alfred Haar which represents a pulse pair rectangular function.

2. Preliminaries

The wavelet transform is one of the mathematical transforms that are used to decompose signal into the time and frequency information. Based on time-frequency localization, computational requirement and applications, the wavelet transform is divided into two main types namely continuous wavelet transform, and discrete wavelet transform. Furthermore, it is also classified as stationary wavelet transform, wavelet packet transform, and complex wavelet transform.

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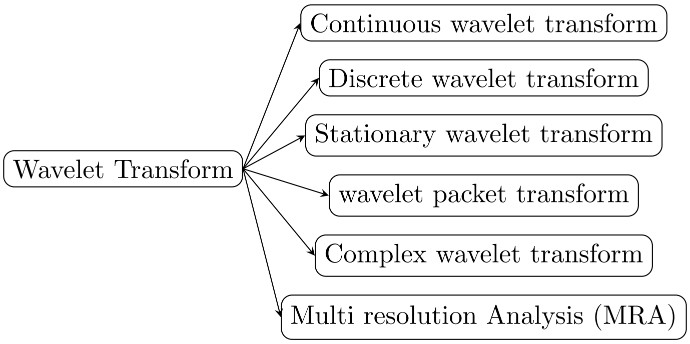

Definition 2.1 *Discrete wavelet transform: The discrete wavelet transform (DWT) for a signal $f(t)$ is defined as [11]:*

$$\mathcal{W}_f\left(\frac{1}{2^j}, \frac{2}{2^j}\right) = \frac{1}{2^{\frac{j}{2}}} \int_{-\infty}^{\infty} f(t)\psi(2^j t - k)\, dt$$

where $j, k \in \mathbb{Z}$.

Definition 2.2 *Continuous wavelet transform: The continuous wavelet transform (CWT) for a signal $f(t)$ is defined as [11]:*

$$\mathcal{W}_f(a,b) = \int_{-\infty}^{\infty} f(t)\psi^*_{a,b}(t)\, dt$$

where $\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$ and a is the scaled parameter while b is the translation parameter and ψ^* is the complex conjugate of the wavelet function.

Theorem 2.1 ***Properties of Continuous Wavelet Transform*** *[9] Let $f, g \in L^2(\mathbb{R})$, then*

1. *Linearity Property:*

$$\mathcal{W}_{a_1 f(t)+a_2 g(t)} = a_1\mathcal{W}_{f(t)} + a_2\mathcal{W}_{g(t)}$$

2. *Translation Property:*

$$\mathcal{W}_{f(t-k)}(a,b) = \mathcal{W}_f(a, b-k)$$

3. *Symmetry Property:*

$$\mathcal{W}_{f(-t)}(a,b) = \mathcal{W}_f(a,-b)$$

4. *Dilation Property:*

$$\mathcal{W}_{f(at)}(a,b) = \frac{1}{\sqrt{|a|}}\mathcal{W}_f\left(\frac{a}{|a|}, \frac{b}{a}\right)$$

5. *Anti-linearity:*

$$\mathcal{W}_{\overline{f(t)}}(a,b) = \overline{\mathcal{W}_f(a,b)}.$$

Proof: To prove the linearity Property, let $f(t)$ and $g(t)$ be the two functions with constants a_1 and a_2, the linear combination $a_1 f(t) + a_2 g(t)$ of The continuous wavelet transform is given by:

$$\begin{aligned}
\mathcal{W}_{a_1 f + a_2 g}(a,b) &= \int_{-\infty}^{\infty} [a_1 f(t) + a_2 g(t)] \cdot \psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}} \\
&= a_1 \int_{-\infty}^{\infty} f(t)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}} + a_2 \int_{-\infty}^{\infty} g(t)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}} \\
&= a_1\mathcal{W}_f(a,b) + a_2\mathcal{W}_g(a,b)
\end{aligned}$$

To prove the translation property, let $g(t) = f(t-k)$. The continuous wavelet transform of $g(t)$ is given by:

$$\mathcal{W}_g(a,b) = \int_{-\infty}^{\infty} g(t)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}}$$

$$\begin{aligned}\mathcal{W}_g(a,b) &= \int_{-\infty}^{\infty} g(t)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}}\\ &= \int_{-\infty}^{\infty} f(t-k)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}}\end{aligned}$$

Let $u = t-k$,$du = dt$,$t = u+k$,

$$\begin{aligned}\mathcal{W}_g(a,b) &= \int_{-\infty}^{\infty} f(u)\cdot\psi^*\left(\frac{u+k-b}{a}\right)\frac{du}{\sqrt{|a|}}\\ &= \int_{-\infty}^{\infty} f(u)\cdot\psi^*\left(\frac{u-(b-k)}{a}\right)\frac{du}{\sqrt{|a|}}\\ &= \mathcal{W}_f(a,b-k).\end{aligned}$$

To prove the symmetry Property, the continuous wavelet transform (CWT) of a function $f(t)$ is given by:

$$\mathcal{W}_f(a,b) = \int_{-\infty}^{\infty} f(t)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}}$$

Let $g(t) = f(-t)$,CWT of $g(t)$,

$$\begin{aligned}\mathcal{W}_g(a,b) &= \int_{-\infty}^{\infty} g(t)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}}\\ &= \int_{-\infty}^{\infty} f(-t)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}}\end{aligned}$$

Substitute $v = -t$, $dv = -dt$ and assume that $\psi(t)$ is symmetric

$$\begin{aligned}\mathcal{W}_g(a,b) &= \int_{\infty}^{-\infty} f(v)\cdot\psi^*\left(\frac{-v-b}{a}\right)\frac{-dv}{\sqrt{|a|}}\\ &= \int_{-\infty}^{\infty} f(v)\cdot\psi^*\left(\frac{v+b}{a}\right)\frac{dv}{\sqrt{|a|}}\\ &= \int_{-\infty}^{\infty} f(v)\cdot\psi^*\left(\frac{v-(-b)}{a}\right)\frac{dv}{\sqrt{|a|}}\\ &= \mathcal{W}_f(a,-b).\end{aligned}$$

To prove the dilation Property, consider the continuous wavelet transform (CWT) of a function $f(t)$ is given by,

$$\mathcal{W}_f(a,b) = \int_{-\infty}^{\infty} f(t)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}}$$

Now we find CWT of $g(t) = f(at)$

$$\begin{aligned}\mathcal{W}_g(a,b) &= \int_{-\infty}^{\infty} g(t)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}}\\ &= \int_{-\infty}^{\infty} f(at)\frac{1}{\sqrt{|a|}}\psi^*\left(\frac{t-b}{a}\right)dt\end{aligned}$$

Substitute $at = u$, $adt = du$

$$\begin{aligned}
\mathcal{W}_g(a,b) &= \int_{-\infty}^{\infty} f(u)\frac{1}{\sqrt{|a|}}\psi^*\left(\frac{\frac{u}{a}-b}{a}\right)\frac{du}{a} \\
&= \frac{1}{\sqrt{|a|}|a|}\int_{-\infty}^{\infty} f(u)\psi^*\left(\frac{u-ba}{a^2}\right)du \\
&= \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty} f(u)\psi^*_{\frac{1}{a},\frac{b}{a}}(u)\,du \\
&= \frac{1}{\sqrt{|a|}}W_f\left(\frac{1}{a},\frac{b}{a}\right) \\
&= \frac{1}{\sqrt{|a|}}W_f\left(\frac{a}{|a|},\frac{b}{a}\right).
\end{aligned}$$

To prove the anti-linearity, the CWT of $f(t)$ is given by:

$$\mathcal{W}_f(a,b) = \int_{-\infty}^{\infty} f(t)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}}$$

The CWT of $\overline{f(t)}$is given by

$$\mathcal{W}_{\overline{f(t)}}(a,b) = \int_{-\infty}^{\infty} \overline{f(t)}\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}}$$

Taking conjugate of $W_f(a,b)$,

$$\begin{aligned}
\overline{\mathcal{W}_f(a,b)} &= \overline{\int_{-\infty}^{\infty} f(t)\cdot\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}}} \\
&= \int_{-\infty}^{\infty} \overline{f(t)}\cdot\overline{\psi^*\left(\frac{t-b}{a}\right)}\frac{dt}{\sqrt{|a|}}
\end{aligned}$$

But $\overline{\psi^*\left(\frac{t-b}{a}\right)} = \psi\left(\frac{t-b}{a}\right)$
Hence

$$\overline{\mathcal{W}_f(a,b)} = \int_{-\infty}^{\infty} \overline{f(t)}\cdot\psi\left(\frac{t-b}{a}\right)\frac{dt}{\sqrt{|a|}} = \mathcal{W}_{\overline{f(t)}}(a,b).$$

$\square$

Definition 2.3 *The Haar wavelet function $\psi(t)$ [3,5] is defined as follows:*

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \textit{otherwise} \end{cases}$$

The translated and scaled Haar wavelet function $\psi^*\left(\frac{t-b}{a}\right)$ is given by:

$$\psi^*\left(\frac{t-b}{a}\right) = \begin{cases} 1, & b \le t < b + \frac{1}{2}a \\ -1, & b + \frac{1}{2}a \le t < b + a \\ 0, & \text{otherwise} \end{cases}$$

The haar wavelet has compact support and $\int_{-\infty}^{\infty}\psi(t)dt = 0$.
Now, we find the Haar wavelet transform of some functions:

1. Haar wavelet transform of $g(t) = t^2 + t + 1$

   $\mathcal{W}_g(a,b) = \int_{-\infty}^{\infty} g(t)\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{a}$
   $= \int_{-\infty}^{\infty} (1+t+t^2)\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{a}$
   $= \frac{1}{a}\left[\int_b^{b+0.5a} 1\,dt + \int_b^{b+0.5a} t\,dt + \int_b^{b+0.5a} t^2\,dt - \int_b^{b+0.5a} 1\,dt - \int_b^{b+0.5a} t\,dt - \int_b^{b+0.5a} t^2\,dt\right]$
   $= \frac{1}{a}\left[0.5a + \frac{ab+0.25a^2}{2} + \frac{1.5b^2a+0.75ba^2+0.125a^3}{3} - 0.5a - \frac{ab+0.75a^2}{2} - \frac{1.5b^2a+2.25ba^2+0.875a^3}{3}\right]$
   $= \frac{1}{a}\left[\left(0.5a + \frac{ab+0.25a^2}{2} + \frac{1.5b^2a+0.75ba^2+0.125a^3}{3}\right) + \left(-0.5a - \frac{ab+0.75a^2}{2} - \frac{1.5b^2a+2.25ba^2+0.875a^3}{3}\right)\right]$
   $= -\frac{1}{a}\left[\frac{0.50a^2}{2} + \frac{1.5ba^2}{3} + \frac{0.75a^3}{3}\right]$

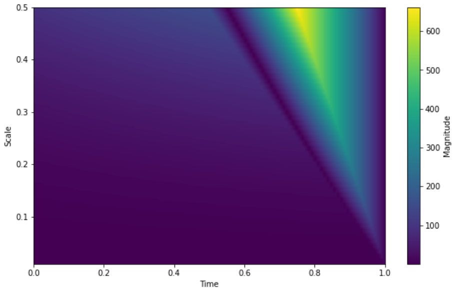

Figure 1: Haar wavelet transform of $g(t) = t^2 + t + 1$.

2. Haar wavelet transform of $g(t) = cos(mt)$

$$
\begin{aligned}
\mathcal{W}_{g(t)}(a,b) &= \int_{-\infty}^{\infty} \cos(mt)\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{a} \\
&= \frac{1}{a}\left[\int_b^{b+0.5a} \cos(mt)\,dt - \int_{b+0.5a}^{b+a} \cos(mt)\,dt\right] \\
&= \frac{1}{a}\left[\frac{\sin(m(b+0.5a)) - \sin(mb)}{m} - \frac{\sin(m(b+a)) - \sin(m(b+0.5a))}{m}\right] \\
&= \frac{2\sin(m(b+0.5a)) - \sin(mb) - \sin(m(b+a))}{am}.
\end{aligned}
$$

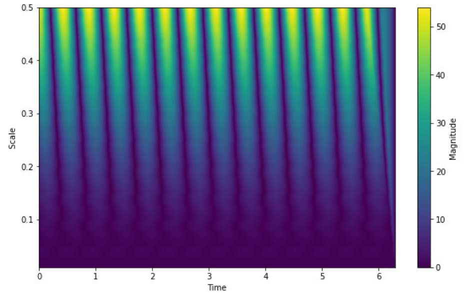

Figure 2: Haar wavelet transform of $g(t) = cos(mt)$, $m = 7$.

3. Haar wavelet transform of $g(t) = e^t$

$$\begin{aligned}\mathcal{W}_{g(t)}(a,b) &= \int_{-\infty}^{\infty} e^t \psi^*\left(\frac{t-b}{a}\right)\frac{dt}{a} \\ &= \int_{b}^{b+0.5a} e^t\,dt - \int_{b+0.5a}^{b+a} e^t\,dt. \\ &= (e^{b+0.5a} - e^b) - (e^{b+a} - e^{b+0.5a}) \\ &= 2e^{b+0.5a} - e^b - e^{b+a}.\end{aligned}$$

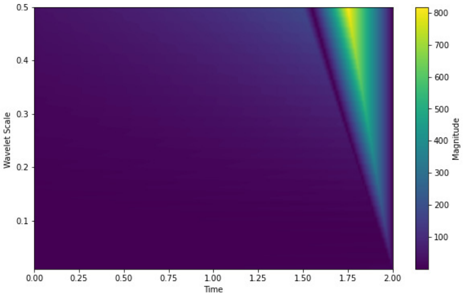

Figure 3: Haar wavelet transform of $g(t) = e^t$.

4. Haar wavelet transform of $f'(t)$, derivative of $f(t)$ By the definition of CWT

$$\mathcal{W}_f(a,b) = \int_{-\infty}^{\infty} f(t)\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{a}$$

The CWT of $f'(t)$ is:

$$\mathcal{W}_{f'}(a,b) = \int_{-\infty}^{\infty} f'(t)\psi^*\left(\frac{t-b}{a}\right)\frac{dt}{a}$$

By using integration by part rule,

$$\mathcal{W}_{f'}(a,b) = \left.\frac{f(t)\psi^*\left(\frac{t-b}{a}\right)}{a}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\frac{\psi'^*\left(\frac{t-b}{a}\right)}{a^2}\,dt$$

Assuming that at infinity both $\psi^*\left(\frac{t-b}{a}\right)$ and $f(t)$ the boundary term vanishes Therefore,we get

$$\mathcal{W}_{f'}(a,b) = -\frac{1}{a^2}\int_{-\infty}^{\infty} f(t)\frac{\psi'^*\left(\frac{t-b}{a}\right)}{a}\,dt$$

But as haar wavelet function,

$$\psi(t) = \begin{cases} 1, & 0 \le t < 0.5 \\ -1, & 0.5 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

The derivative, $\psi'(t) = \delta(t) - \delta(t-0.5)$, where $\delta(t)$ is the Dirac delta function. Therefore,

$$\psi'^*\left(\frac{t-b}{a}\right) = \frac{1}{a}\left(\delta\left(\frac{t-b}{a}\right) - \delta\left(\frac{t-b-0.5a}{a}\right)\right)$$

Hence,

$$\mathcal{W}_{f'}(a,b) = -\frac{1}{a^2}\int_{-\infty}^{\infty} f(t)\frac{1}{a}\left(\delta\left(\frac{t-b}{a}\right) - \delta\left(\frac{t-b-0.5a}{a}\right)\right) dt$$

By simplifying,

$$\mathcal{W}_{f'}(a,b) = -\frac{1}{a^3}\left[f(b) - f(b+0.5a)\right].$$

Definition 2.4 *The Haar wavelet family [14,7,15,8] is defined for $t \in [0,1)$ as follows:*

$$h_i(t) = 2^{\frac{j}{2}}\psi(2^j x - k) = \begin{cases} 1, & t \in [\rho_1, \rho_2) \\ -1, & t \in [\rho_2, \rho_3) \\ 0 & otherwise \end{cases}$$

where $\rho_1 = \frac{k}{m}, \rho_2 = \frac{k+\frac{1}{2}}{m}, \rho_3 = \frac{k+1}{m}$
where $k = 0, 1, 2, ..., m-1, m = 2^j, j = 0, 1, ..., J, i = m+k+1$
Here, k denotes the translation parameter, j indicates the dilation parameter or scaling (changing level) or level of wavelet and J describes the maximum level of resolution.

For least values $k = 0, m = 1$ we have $i = 2$ and the maximum value of i, is $i = 2M = 2^{J+1}$. In particular for $i = 1$, the corresponding scaling function also called the father wavelet for the family of Haar wavelets is given by

$$h_1(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

For $i = 2$, the function $h_2(t)$ is the function 'mother wavelet' for the family of the Haar wavelet family is defined as

$$h_2(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

For $i = 3$, the function $h_3(t)$ is defined as

$$h_3(t) = \begin{cases} 1, & 0 \le t < \frac{1}{4} \\ -1, & \frac{1}{4} \le t < \frac{2}{4} \\ 0 & \text{otherwise} \end{cases}$$

For $i = 4$, the function $h_4(t)$ is defined as

$$h_4(t) = \begin{cases} 1, & 0 \le t < \frac{1}{8} \\ -1, & \frac{1}{8} \le t < \frac{2}{8} \\ 0 & \text{otherwise} \end{cases}$$

Integration of Haar wavelets [15]:
We use the following notation in the Chen and Hsiao method(CHM).

$$\begin{aligned} P_{i,\alpha}(t) &= \int_A^t \int_A^t \dots \int_A^t h_i(\tau) d\tau^\alpha \\ &= \frac{1}{(\alpha-1)}\int_A^x (x-t)^{\alpha-1} h_i(\tau) d\tau \end{aligned}$$

where $\alpha = 1, 2, ...n, i = 1, 2, ..., 2M$
The above integrals can also be calculated analytically,

$$P_{i,\alpha}(t) = \begin{cases} 0, & t < \rho_1 \\ \frac{(t-\rho_1)^\alpha}{\alpha!}, & t \in [\rho_1, \rho_2) \\ \frac{(t-\rho_1)^\alpha - 2(t-\rho_2)^\alpha}{\alpha!}, & t \in [\rho_2, \rho_3) \\ \frac{(t-\rho_1)^\alpha - 2(t-\rho_2)^\alpha + (t-\rho_3)}{\alpha!}, & \text{otherwise} \end{cases}$$

In particular ,

$$p_{i,1}(t) = \int_0^t h_i(t)dt = \begin{cases} t - \rho_1, & t \in [\rho_1, \rho_2) \\ \rho_3 - t, & t \in [\rho_2, \rho_3) \\ 0 & \text{otherwise} \end{cases}$$

$$p_{i,v}(t) = \int_0^t p_{i,v-1}(t)dt, v = 2, 3, ...$$

$$p_{i,2}(t) = \int_0^t p_{i,1}(t)dt = \int_0^t \int_0^t [h_i(t)]dt^2 = \begin{cases} \frac{(t-\rho_1)^2}{2}, & t \in [\rho_1, \rho_2) \\ \frac{-(\rho_3-t)^2}{2} + \frac{1}{4m^2}, & t \in [\rho_2, \rho_3) \\ \frac{1}{4m^2} & t \in [\rho_3, 1) \\ 0, & \text{otherwise} \end{cases}$$

$$p_{i,3}(t) = \int_0^t p_{i,2}(t)dt = \int_0^t \int_0^t \int_0^t [h_i(t)]dt^3 = \begin{cases} \frac{(t-\rho_1)^3}{6}, & t \in [\rho_1, \rho_2) \\ \frac{-(\rho_3-t)^3}{6} + \frac{(t-\rho_2)}{4m^2}, & t \in [\rho_2, \rho_3) \\ \frac{(t-\rho_2)}{4m^2} & t \in [\rho_3, 1) \\ 0, & \text{otherwise} \end{cases}$$

Similarly, we can find $p_{i,v}(t), v = 4, 5, ...$
The integrals $p_{i,v}(t), v = 1, 2, 3, ...$ are used to find solutions of the differential equations [12].

Function Approximation [6,9,14]:
The square integrable function $z(t) \in L^2[1, 0]$ is a decomposed by haar series of an infinite number of terms.

$$z(t) = \sum_{i=1}^{\infty} a_i h_i(t)$$

where a_i are haar coefficients.If $z(t)$ is piece-wise constant in each sub- interval then $z(t)$ will be terminated finite M terms. i.e.

$$z(t) = \sum_{i=1}^{2M} a_i h_i(t)$$

3. Haar Wavelet Method

Let n^{th} order linear differential equation be

$$c_1 z^{(n)}(t) + c_2 z^{(n-1)}(t) + ... + c_n z(t) = g(t), t \in [A, B) \tag{3.1}$$

take $M = 2^{j+1}$,$\Delta = \frac{B-A}{M}$, use method [4] to find $z(t)$, we used following steps:

1. Let $z^{(n)}(t) = \sum_{i=1}^{2M} a_i h_i(t)$, where a_i are wavelet coefficients and h_i are haar wavelets.
2. Obtain the appropriate v order of $z(t)$ by integrating $z^{(n)}(t)$ with respect to t from 0 to t.
3. Replace the values of $z^{(v)}(t)$ and $z^{(n)}(t)$ in equation (1).
4. Solving the system of linear equations for the wavelet coefficients a_i, we get the solution $z(t)$.

4. Numerical Examples

Example 1 *City Population Model*

A town of population is currently $2,00,000$ and it grows with continuous annual rate of 5% Here, Current population, $P_0 = 2,00,000$ and growth rate, $r = 0.05$ with time t years.
The population growth model is given by differential equation:

$$\frac{dP}{dt} = rP$$

while solving by traditional methods with initial population is given by:

$$P(t) = 200000e^{0.05t}$$

Solution by Haar wavelet method

Let

$$\frac{dP}{dt} = P^{'}(t) = \sum_{i=1}^{2M} a_i h_i(t)$$

on integrating with limits 0 to t,

$$\begin{aligned} \int_0^t P^{'}(t)dt &= \int_0^t \sum_{i=1}^{2M} a_i h_i(t)dt \\ P(t) - P(0) &= \sum_{i=1}^{2M} a_i \int_0^t h_i(t)dt \\ P(t) &= P_0 + \sum_{i=1}^{2M} a_i p_{i,1}(t) \end{aligned}$$

Differential equation becomes,

$$\sum_{i=1}^{2M} a_i h_i(t) = r[P_0 + \sum_{i=1}^{2M} a_i p_{i,1}(t)]$$

$$\sum_{i=1}^{2M} a_i [h_i(t) - rp_{i,1}(t)] = rP_0$$

After solving the system of linear equations, we get wavelet coefficients a_i.
The computational work has done with the help of MATLAB R2024 software.

Table 1: City Population Model

t /32	**Exact solution**	**Haar solution**	**Absolute Error**
1	2003127.44267831	2003127.44267839	$8.00937414 \times 10^{-8}$
3	2009397.00702880	2009397.00702720	$1.60001218 \times 10^{-6}$
5	2015686.19441290	2015686.19441299	$9.01054591 \times 10^{-8}$
7	2021995.06624851	2021995.06624861	$9.98843461 \times 10^{-8}$
9	2028323.68414576	2028323.68414588	$1.19907781 \times 10^{-7}$
11	2034672.10990760	2034672.10990570	$1.90013088 \times 10^{-6}$
13	2041040.40553044	2041040.40553070	$2.60071829 \times 10^{-7}$
15	2047428.63320472	2047428.63320473	$1.00117177 \times 10^{-8}$
17	2053836.85531551	2053836.85531542	$8.98726285 \times 10^{-8}$
19	2060265.13444316	2060265.13444336	$2.00001523 \times 10^{-7}$
21	2066713.53336389	2066713.53336382	$6.98491931 \times 10^{-8}$
23	2073182.11505039	2073182.11505021	$1.79978088 \times 10^{-7}$
25	2079670.94267246	2079670.94267235	$1.10128894 \times 10^{-7}$
27	2086180.07959761	2086180.07959725	$3.59956175 \times 10^{-7}$
29	2092709.58939168	2092709.58939178	$1.00117177 \times 10^{-7}$
31	2099259.53581946	2099259.53581949	$2.98023224 \times 10^{-8}$

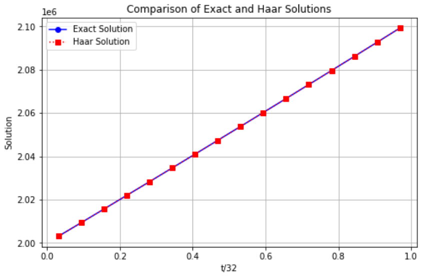

Figure 4: The exact and haar solution for City Population Model.

Example 2 *Bacterial Growth Model*

A bacterial culture initially contains 400 bacteria and grows at a continuous rate of 4% per hour.

The bacterial growth model is given by a differential equation:

$$\frac{dN}{dt} = rN, N(0) = N_0 = 400, r = 0.04$$

while solving by traditional methods with initial population is given by:
Number of Bacteria after t Hours

$$N(t) = 400e^{0.04t}$$

Solution by Haar wavelet method
Let us

$$\frac{dN}{dt} = N^{'}(t) = \sum_{i=1}^{2M} a_i h_i(t)$$

on integrating with limits 0 to t,

$$\begin{aligned}\int_0^t N^{'}(t)dt &= \int_0^t \sum_{i=1}^{2M} a_i h_i(t)dt \\ N(t) - N(0) &= \sum_{i=1}^{2M} a_i \int_0^t h_i(t)dt \\ N(t) &= N_0 + \sum_{i=1}^{2M} a_i p_{i,1}(t)\end{aligned}$$

The differential equation becomes

$$\sum_{i=1}^{2M} a_i h_i(t) = r[N_0 + \sum_{i=1}^{2M} a_i p_{i,1}(t)]$$

$$\sum_{i=1}^{2M} a_i [h_i(t) - r p_{i,1}(t)] = rN_0$$

After solving the system of linear equations, we get wavelet coefficients a_i. The computational work has been done with the help of MATLAB R2024 software.

Table 2: Bacterial Growth Model

t /32	**Exact solution**	**Haar solution**	**Absolute Error**
1	400.5003126	400.5003132	5.70×10^{-7}
3	401.502816	401.50281900	2.98×10^{-6}
5	402.5078288	402.50782980	1.00×10^{-6}
7	403.5153573	403.51535790	6.40×10^{-7}
9	404.5254077	404.52540970	2.01×10^{-6}
11	405.5379864	405.53798400	2.40×10^{-6}
13	406.5530997	406.55309980	7.00×10^{-8}
15	407.570754	407.57076800	1.40×10^{-5}
17	408.5909556	408.59095570	7.00×10^{-8}
19	409.6137109	409.61371090	3.00×10^{-8}
21	410.6390263	410.63902550	8.10×10^{-7}
23	411.6669082	411.66690770	5.00×10^{-7}
25	412.697363	412.69736900	6.00×10^{-6}
27	413.7303972	413.73039750	3.40×10^{-7}
29	414.7660171	414.76601810	9.60×10^{-7}
31	415.8042294	415.80422970	2.90×10^{-7}

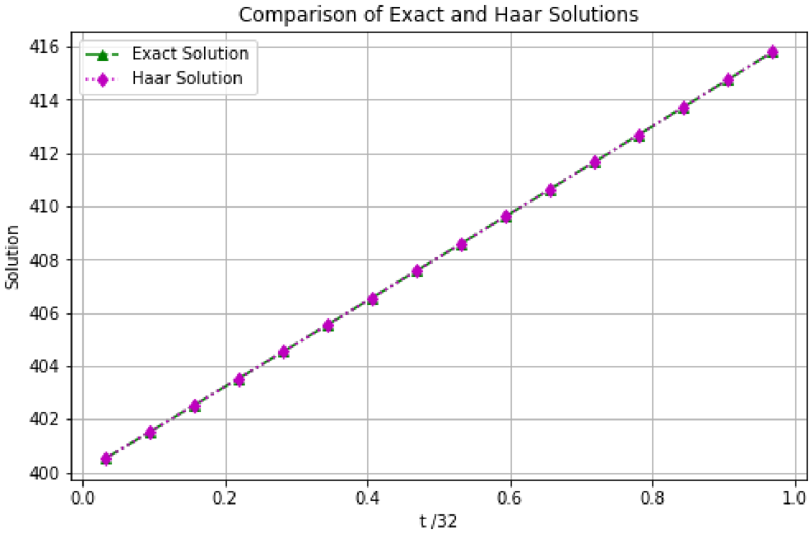

Figure 5: Bacterial Growth Model.

Example 3 *Newton's law of cooling*

A cup of coffee is initially at a temperature of 100 ° C. The ambient room temperature is 25° C. The coffee was cooled at a rate proportional to the difference between its temperature and the ambient temperature. The cooling constant is $k = 0.04$ per minute.

The bacterial growth model is given by differential equation:

$$\frac{dT}{dt} = -k(T - T_{env}), T(0) = T_0 = 100, T_{env} = 25, k = 0.04$$

while solving by traditional methods with initial population is given by:
Number of Bacteria after t Hours

$$T(t) = 25 + 75e^{-0.04t}$$

Solution by Haar wavelet method

Let

$$\frac{dT}{dt} = T^{'}(t) = \sum_{i=1}^{2M} a_i h_i(t)$$

on integrating with limits 0 to t,

$$\int_0^t T^{'}(t)dt = \int_0^t \sum_{i=1}^{2M} a_i h_i(t)dt$$

$$T(t) - T(0) = \sum_{i=1}^{2M} a_i \int_0^t h_i(t)dt$$

$$T(t) = T_0 + \sum_{i=1}^{2M} a_i p_{i,1}(t)$$

Differential equation becomes,

$$\sum_{i=1}^{2M} a_i h_i(t) = -k[T_0 + \sum_{i=1}^{2M} a_i p_{i,1}(t) - T_{env}]$$

$$\sum_{i=1}^{2M} a_i[h_i(t) + kp_{i,1}(t)] = -kT_0 + k.T_{env}$$

After solving the system of linear equations, we get wavelet coefficients a_i. The computational work has been done with the help of MATLAB R2024 software.

Table 3: Newtons law of cooling

t /32	Exact solution	Haar solution	Absolute Error
1	99.90630857	99.90630859	2.06564×10^{-8}
3	99.71927669	99.71927677	8.48122×10^{-8}
5	99.5327118	99.5327118	3.2454×10^{-9}
7	99.34661274	99.34661278	4.19873×10^{-8}
9	99.16097835	99.16097829	5.58425×10^{-8}
11	98.97580746	98.97580739	7.00283×10^{-8}
13	98.79109892	98.79109884	8.32515×10^{-8}
15	98.60685158	98.60685152	6.10833×10^{-8}
17	98.42306428	98.42306429	8.02299×10^{-8}
19	98.23973588	98.23973588	2.73852×10^{-9}
21	98.05686522	98.05686531	8.88664×10^{-8}
23	97.87445117	97.87445119	1.93488×10^{-8}
25	97.69249259	97.69249259	4.27418×10^{-9}
27	97.51098833	97.51098834	1.08843×10^{-8}
29	97.32993727	97.32993729	2.35813×10^{-8}
31	97.14933827	97.14933827	3.93493×10^{-9}

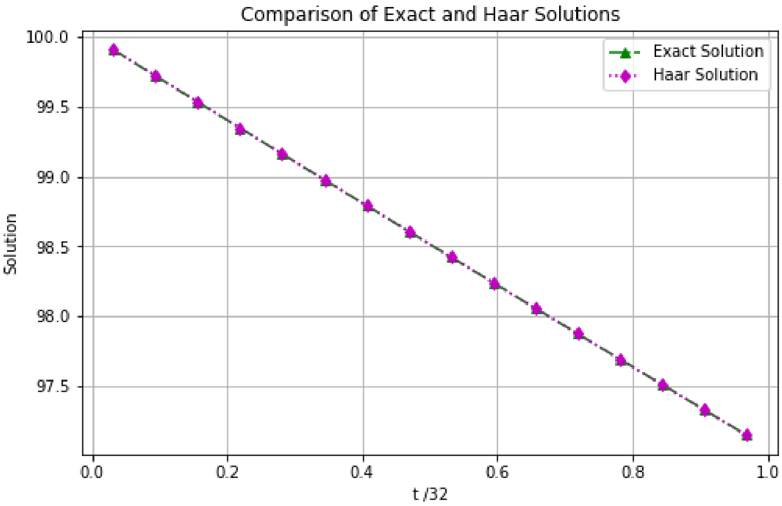

Figure 6: The exact and haar solution for Newtons law of cooling.

Example 4 *Vehicle Suspension System with Impulse Force*

A vehicle with a mass of 250 kg encounters a sudden bump on the road, causing an impulse force of 5000 N at $t = 0$. The vehicle suspension system has a damping coefficient of 1000 Ns/m and a spring stiffness of 15,000 N/m. Assume the vehicle starts at rest, with zero initial displacement and velocity.

Mass of vehicle body: $m = 250\,\mathrm{kg}$, Damping coefficient: $c = 1000\,\mathrm{Ns/m}$, Spring stiffness: $k = 15000\,\mathrm{N/m}$, Impulse force magnitude: $5000\,\mathrm{N}$, $\delta(t)$-Dirac delta function

Now,

$$250\frac{d^2y}{dt^2} + 1000\frac{dy}{dt} + 15000y = 5000\delta(t)$$

i.e.

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 60y = 20\delta(t)$$

damping ratio ζ and the damped natural frequency ω_d is calculated by

$$\zeta = \frac{c}{2\sqrt{mk}} \approx 0.258$$

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \approx 7.48\,\mathrm{rad/s}$$

Thus, the displacement of the vehicle will exhibit damped oscillatory motion with the general form:

$$y(t) = e^{-\zeta\omega_n t}(A\cos(\omega_d t) + B\sin(\omega_d t))$$

Using initial conditions, $y(0) = 0, y'(0) = 20$ we get the displacement of the vehicle as a function of time t is:

$$y(t) = e^{-2t} \cdot 2.67\sin(7.48t)\,\mathrm{meter}$$

Solution by Haar wavelet method

Here,

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 60y = 20\delta(t), y(0) = 0, y'(0) = 20$$

Let

$$y^{''}(t) = \sum_{i=1}^{2M} a_i h_i(t)$$

on integrating with limits 0 to t,

$$\int_0^t y^{''}(t)dt = \int_0^t \sum_{i=1}^{2M} a_i h_i(t)dt$$

$$y^{'}(t) - y'(0) = \sum_{i=1}^{2M} a_i \int_0^t h_i(t)dt$$

$$y^{'}(t) = 20 + \sum_{i=1}^{2M} a_i p_{i,1}(t)$$

Again on integrating with limits 0 to t

$$\int_0^t y^{'}(t)dt = \int_0^t \sum_{i=1}^{2M} a_i p_{i,1}(t)$$

$$y^(t) - y(0) = 20t + \sum_{i=1}^{2M} a_i \int_0^t p_{i,1}(t)dt$$

$$y(t) = 20t + \sum_{i=1}^{2M} a_i p_{i,2}(t)$$

Differential equation becomes,

$$\sum_{i=1}^{2M} a_i h_i(t) + 4[20 + \sum_{i=1}^{2M} a_i p_{i,1}(t)] + 60[20t + \sum_{i=1}^{2M} a_i p_{i,2}(t)] = 20\delta(t)$$

$$\sum_{i=1}^{2M} a_i[h_i(t) + 4p_{i,1}(t) + 60p_{i,2}(t)] = 20\delta(t) - 1200t - 80$$

After solving the system of linear equations, we get wavelet coefficients a_i. The computational work has been done with the help of MATLAB R2024 software.

Table 4: Vehicle Suspension System with Impulse Force

t /32	**Exact solution**	**Haar solution**	**Absolute Error**
1	0.618058113	0.618058121	8.20677×10^{-9}
3	1.719385669	1.719385670	1.11436×10^{-9}
5	2.449433592	2.449433586	5.85255×10^{-9}
7	2.652651881	2.652651883	1.88758×10^{-9}
9	2.286798556	2.286798547	9.11427×10^{-9}
11	1.431710803	1.431710853	4.96335×10^{-8}
13	0.271888014	0.27188803	1.13184×10^{-8}
15	-0.943314784	-0.943314786	1.78981×10^{-9}
17	-1.953295128	-1.953295125	3.19154×10^{-9}
19	-2.542088227	-2.542088231	4.01894×10^{-9}
21	-2.584544975	-2.584544978	2.75243×10^{-9}
23	-2.072929412	-2.072929417	5.18907×10^{-9}
25	-1.118283664	-1.118283669	4.85662×10^{-9}
27	0.073669178	0.07366917	7.85847×10^{-9}
29	1.246855752	1.246855754	1.63096×10^{-9}
31	2.149864406	2.149864499	9.2617×10^{-8}

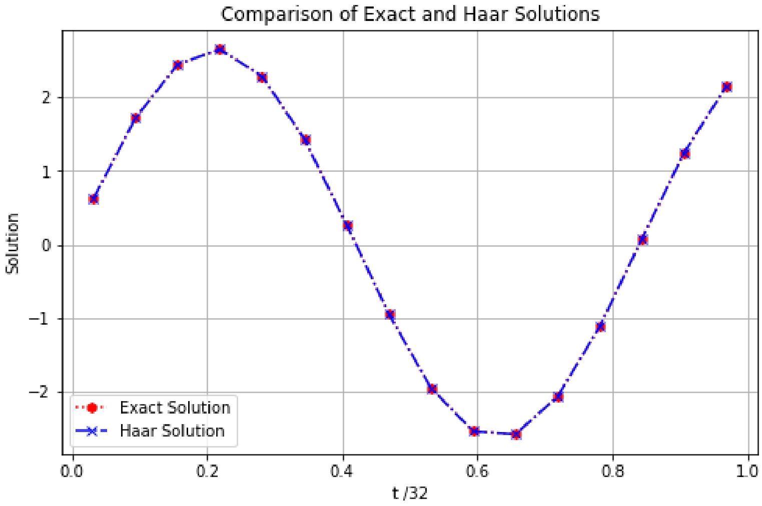

Figure 7: The exact and haar solution for Vehicle Suspension System with Impulse Force.

5. Conclusion

The Haar wavelet transform method was used to find solutions of dynamical differential equations. The absolute errors in the computations demonstrate the accuracy of the proposed method. Due to the sparsity of transform matrices and smaller wavelet coefficients, the method becomes more reliable, which helps in minimal computational cost. The study offers a clearer pathway to solving different types of differential equations in the Science and Engineering fields.

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