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Solvability of k-fractional Hilfer integral equations via Darbo's fixed point theorem

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ABSTRACT: In this article we are extended the Darbo's fixed point theorem using $\mathcal{U}$-class mapping. Using the Darbo type theorem, we give a solvability result for a k-fractional Hilfer integral equation along with an appropriate illustration.

Key Words: Measure of noncompactness, k-fractional Hilfer integral, fixed point theorem.

Contents

1. Introduction

Kuratowski's measure of noncompactness (MNC), introduced in 1930 [20], has been extensively studied in various mathematical contexts. Different research papers have explored the application of this measure in different areas. Metwali et al. defined a novel MNC in variable exponent Lebesgue spaces, extending classical Lebesgue spaces and enabling the study of various equations [23]. Khokhar et al. proposed the concept of $\mathcal{A}$-condensing operators by utilizing the MNC, focusing on best proximity points and fractional differential equations [19]. Telli et al. utilized the Kuratowski MNC in studying boundary-value problems for fractional differential equations with variable order and delays, showcasing the applicability of the measure in stability criteria [31].

Caponetti et al. [4] analyzed the Kuratowski measure of noncompactness(MNC) in spaces of vector-valued functions, leading to new criteria for compactness based on quantitative features. The MNC is crucial in various fixed point theorems, which are instrumental in exploring solutions to different equations. For instance, the MNC helps in identifying solutions for nonlinear fractional integral equations within Banach spaces, as evidenced by Golshan [14], Deb et al. [11], Metwali [23], and Gabeleh et al. [13]. Their research demonstrates the use of MNC in Lebesgue spaces and sequence spaces to confirm the existence of solutions via fixed point theorems. Moreover, combining the MNC with other criteria, like the Hausdorff measure, enhances the effectiveness of fixed point theorems in analyzing the solvability of integral equations, highlighting the synergy between fixed point theorems and the MNC.

G. Darbo [7] extended Schauder's fixed point theorem by integrating Kuratowski's MNC into his framework. The MNC has proven vital in broadening Darbo's theorem across various mathematical contexts. Hammad et al. [15] introduced a new fixed point theorem that expands Darbo's theorem with the MNC, while Taoudi [30] generalized Darbo's principle using a monotone MNC with specific properties. Additionally, Salem et al. [28] applied the MNC within Banach spaces to address the existence of solutions for fractional Sturm–Liouville operators using fixed point theorems, including Darbo's. Collectively, these studies underscore the MNC's importance in extending Darbo's fixed point theorem and solving diverse mathematical problems.

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The Darbo fixed point theorem, a fundamental mathematical concept, has been extensively studied and generalized. It establishes the existence of fixed points for certain mappings in various spaces. Extensions and generalizations of Darbo's theorem include proofs of solution existence and stability for fractional integral equations [5,25], new generalizations with relaxed assumptions, and applications in Banach spaces with monotone MNC and convex mappings [26,30]. Additionally, new contraction types, such as the F-Darbo contraction, have been proposed, offering broader results and applications for solving integral equations [18].

J. Banaś [3] introduced the MNC in 1980, and its study in Banach spaces has since become a key research area. These measures play a crucial role in determining the existence of solutions to integral boundary value problems [22], studying the representation of an MNC and their applications in Banach spaces [6], exploring interpolation of an MNC of polynomials on Banach spaces [21], examining solvability conditions for fractional integral equations in Banach spaces using the theory of an MNC [8], and applying the MNC in characterizing classes of compact operators, solving integral equations, and establishing the existence of optimal solutions for systems of integro-differentials in Banach spaces [13]. These research efforts collectively contribute to a deeper understanding of MNCs and their diverse applications in various mathematical contexts.

Inspired and motivated by [9,10,16,24,27,29], in the context of an MNC. In this paper, we are generalized well-known theorems, namely, Darbo's fixed point theorem. the theorems was generalized with the help of $\mathcal{U}$-class mapping. Herein, first we presented a solvability result of a k-fractional Hilfer integral equation along with a suitable example using the Darbo type theorem.

Let $(E, \| \,.\, \|)$ be a real Banach space and $B_r(\theta) = \{\zeta \in E : \| \zeta - \theta \| \leq r\}$. If $W(\neq \emptyset) \subseteq E$. Then

- $\bar{W}$ = represents the closure of W,
- $ConvW$ = represents the convex closure of W,
- $\mathfrak{B}_E$ = represents the set of all bounded and non-empty subsets of E
- $\mathfrak{C}_E$ = represents the family of all relatively compact sets
- $\mathbb{R}_+ = [0, \infty)$.

The MNC is defined as [3].

Definition 1.1 A MNC is a function $\hat{\wp} : \mathfrak{B}_E \to [0, \infty)$ which fulfills the following axioms:

(i) For all $W \in \mathfrak{B}_E$, we have $\hat{\wp}(W) = 0$ gives W is relatively compact.

(ii) $\ker \hat{\wp} = \{W \in \mathfrak{B}_E : \hat{\wp}(W) = 0\} \neq \emptyset$ and $\ker \hat{\wp} \subset \mathfrak{C}_E$.

(iii) $W \subseteq W_1 \implies \hat{\wp}(W) \leq \hat{\wp}(W_1)$.

(iv) $\hat{\wp}(\bar{W}) = \hat{\wp}(W)$.

(v) $\hat{\wp}(ConvW) = \hat{\wp}(W)$.

(vi) $\hat{\wp}\left(\hat{\lambda}W + \left(1 - \hat{\lambda}\right)W_1\right) \leq \hat{\lambda}\hat{\wp}(W) + \left(1 - \hat{\lambda}\right)\hat{\wp}(W_1)$ for $\hat{\lambda} \in [0, 1]$.

(vii) if $W_{\hat{k}} \in \mathfrak{B}_E$, $W_{\hat{k}} = \bar{W}_{\hat{k}}$, $W_{\hat{k}+1} \subset W_{\hat{k}}$, where $\hat{k} = 1, 2, 3, \ldots$ and $\lim\limits_{\hat{k}\to\infty} \hat{\wp}\left(W_{\hat{k}}\right) = 0$ then $\bigcap_{\hat{k}=1}^{\infty} W_{\hat{k}} \neq \emptyset$.

Here $ker\hat{\wp}$ represents the *kernel of measure* $\hat{\wp}$. Since $\hat{\wp}(W_\infty) \leq \hat{\wp}(W_{\hat{k}})$ and $\hat{\wp}(W_\infty) = 0$. So, $W_\infty = \bigcap_{\hat{k}=1}^{\infty} W_{\hat{k}} \in ker\hat{\wp}$.

Now we recall some definitions and theorems which are important in the context of present work:

Theorem 1.1 *(Schauder [1]) Assume V is a non-empty, bounded, closed, and convex subset(NBCCS) of a Banach Space E. Then a continuous and compact function $\mathcal{A} : V \to V$ has at least one fixed point.*

Theorem 1.2 *(Darbo [7]) Assume V is a NBCCS of a Banach Space E. If a constant $\lambda \in [0,1)$ exist for a continuous function $\mathcal{A}: V \to V$ such that*

$$\hat{\wp}(\mathcal{A}W) \leq \lambda \hat{\wp}(W),\ W \subseteq V.$$

Then $\mathcal{A}$ must have a fixed point.

The following are some of the related concepts which will be require to construct an extended Darbo's fixed point theorem(DFPT).

Definition 1.2 A $\mathcal{U}$- class function is a continuous function $f: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, if it fulfills the following conditions:
(1) $f(\hat{\hbar}_1, \hat{\hbar}_2) \leq \hat{\hbar}_1 + \hat{\hbar}_2$,
(2) $f(\hat{\hbar}_1, \hat{\hbar}_2) = \hat{\hbar}_1$ implies that either $\hat{\hbar}_1 = 0$ or $\hat{\hbar}_2 = 0$.
Also, $f(0,0) = 0$.
For example,
(1) $f(\hat{\hbar}_1, \hat{\hbar}_2) = \hat{\hbar}_1 + \hat{\hbar}_2$,
(2) $f(\hat{\hbar}_1, \hat{\hbar}_2) = \rho\hat{\hbar}_1,\ 0 < \rho < 1$.

Definition 1.3 [17] Consider that Ψ is the collection of altering distance functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ fulfilling all the conditions:
(1) $\psi(w_1) = 0 \iff w_1 = 0$.
(2) ψ is a continuous and increasing.

Definition 1.4 [2] An ultra-altering distance function is a continuous functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(\varpi) > 0$, $\varphi(0) \geq 0$ and $\varpi > 0$.

2. Fixed point result

Theorem 2.1 *Assume that $\mathcal{W}$ is a NBCCS of a Banach space E. Moreover if, $\mathcal{A}: \mathcal{W} \to \mathcal{W}$ is continuous mapping with*

$$\psi[\hat{\wp}(\mathcal{A}W)] \leq f[\psi(\hat{\wp}(W)), \varphi(\hat{\wp}(W))] - \varphi(\hat{\wp}(W)), \tag{2.1}$$

where $W \subset \mathcal{W}$, $\hat{\wp}$ is an arbitrary MNC, $f \in \mathcal{U}$, $\psi \in \Psi$ and φ is an ultra-altering distance function. Then $\mathcal{W}$ contains at least one fixed point of $\mathcal{A}$.

Proof: In order to obtain the main result, we first build a nested sequence and then we will apply the concept of the MNC. Construct a sequence $\{\mathcal{W}_n\}_{n=1}^{\infty}$ with $\mathcal{W}_1 = \mathcal{W}$ and $\mathcal{W}_{n+1} = Conv(\mathcal{A}\mathcal{W}_n)$ for $n \in \mathbb{N}$. Also $\mathcal{A}\mathcal{W}_1 = \mathcal{A}\mathcal{W} \subseteq \mathcal{W} = \mathcal{W}_1$, $\mathcal{W}_2 = Conv(\mathcal{A}\mathcal{W}_1) \subseteq \mathcal{W} = \mathcal{W}_1$. Proceeding in the same way we get $\mathcal{W}_1 \supseteq \mathcal{W}_2 \supseteq \mathcal{W}_3 \supseteq \ldots \supseteq \mathcal{W}_n \supseteq \mathcal{W}_{n+1} \supseteq \ldots$.

If $n_0 \in \mathbb{N}$ exists which fulfill the condition $f[\psi(\hat{\wp}(\mathcal{W}_{n_0})), \varphi(\hat{\wp}(\mathcal{W}_{n_0}))] = 0$. Then $\hat{\wp}(\mathcal{W}_{n_0}) = 0$, so $\mathcal{W}_{n_0}$ is a compact set. Then by virtue of Schauder's theorem, $\mathcal{A}$ must have a fixed point in $\mathcal{W}$

Consider $f[\psi(\hat{\wp}(\mathcal{W}_n)), \varphi(\hat{\wp}(\mathcal{W}_n))] > 0,\ n \in \mathbb{N}$.

Then for $n \in \mathbb{N}$ we have,

$$\begin{aligned}
&\psi(\hat{\wp}(\mathcal{W}_{n+1})) \\
&= \psi[\hat{\wp}(Conv\mathcal{A}\mathcal{W}_n)] \\
&= \psi[\hat{\wp}(\mathcal{A}\mathcal{W}_n)] \\
&\leq f[\psi(\hat{\wp}(\mathcal{W}_n)), \varphi(\hat{\wp}(\mathcal{W}_n))] - \varphi(\hat{\wp}(\mathcal{W}_n)) \\
&\leq \psi(\hat{\wp}(\mathcal{W}_n)).
\end{aligned}$$

As the mapping ψ is a non-decreasing, we get

$$\hat{\wp}(\mathcal{W}_{n+1}) \leq \hat{\wp}(\mathcal{W}_n).$$

Then, $\{\hat{\wp}(\mathcal{W}_n)\}$ is a decreasing and bounded below sequence.
Its follows converges to $k = \inf\{\mathcal{W}_n\}$.
If possible let $k > 0$. By using (2.3), we get

$$\psi(\hat{\wp}(\mathcal{W}_{n+1})) = \psi[\hat{\wp}(Conv\mathcal{A}\mathcal{W}_n)] \leq f[\psi(\hat{\wp}(\mathcal{W}_n)), \varphi(\hat{\wp}(\mathcal{W}_n))] - \varphi(\hat{\wp}(\mathcal{W}_n)).$$

As $n \to \infty$, we obtain

$$\psi(k) \leq f[\psi(k), \varphi(k)] - \varphi(k) \leq \psi(k).$$

Its follows that

$$f[\psi(k), \varphi(k)] = \psi(k).$$

Using by (2) of Definition 1.2, we get

$$\psi(k) = 0 \; or \; \varphi(k) = 0.$$

Using by (1) of Definition 1.3, we obtain,

$$k = 0$$
$$\implies \lim_{n\to\infty} \hat{\wp}(\mathcal{W}_n) = 0.$$

Since $\mathcal{W}_n \supseteq \mathcal{W}_{n+1}$, by virtue of Definition 1.1, $\mathcal{W}_\infty = \bigcap_{n=1}^{\infty} \mathcal{W}_n$ must be non-empty, closed and convex subset of $\mathcal{W}$ and under $\mathcal{A}$, $\mathcal{W}_\infty$ is invariant. Therefore if we apply Theorem 1.1 we can conclude that $\mathcal{A}$ must have a fixed point in $\mathcal{W}$. Which concludes the proof. □

Theorem 2.2 *Assume that $\mathcal{W}$ is a NBCCS of a Banach space E. Also a continuous mapping $\mathcal{A} : \mathcal{W} \to \mathcal{W}$ satisfies the condition*

$$\psi[\hat{\wp}(\mathcal{A}W)] \leq \psi(\hat{\wp}(W))\,\beta[\psi(\hat{\wp}(W))] - \varphi(\hat{\wp}(W)), \tag{2.2}$$

where $\hat{\wp}$ is an arbitrary MNC, $W \subset \mathcal{W}$, $\psi \in \Psi$ and φ is an ultra-altering distance function. Also, let $\beta : \mathbb{R}_+ \to [0, 1)$ be a continuous function. Then, $\mathcal{A}$ has at least one fixed point in $\mathcal{W}$.

Proof: Theorem 2.2 can be proved by putting $f(w_1, w_2) = w_1\beta(w_1)$ in Theorem 2.1. □

Theorem 2.3 *Assume that $\mathcal{W}$ is a NBCCS of a Banach space E. Also let $\beta : \mathbb{R}_+ \to [0, 1)$ be a continuous function and if a continuous mapping $\mathcal{A} : \mathcal{W} \to \mathcal{W}$ satisfies the condition*

$$\hat{\wp}(\mathcal{A}W) \leq \hat{\wp}(W)\,\beta[\hat{\wp}(W)] - \varphi(\hat{\wp}(W)), \tag{2.3}$$

where $W \subset \mathcal{W}$, $\hat{\wp}$ is an arbitrary MNC and φ is an ultra-altering distance function. Then in $\mathcal{W}$, $\mathcal{A}$ has at least one fixed point.

Proof: Theorem 2.3 can be obtain by putting $\psi(w_1) = w_1$ in Theorem 2.2. □

Corollary 2.1 *Assume that $\mathcal{W}$ is a NBCCS of a Banach space E. Also $\beta : \mathbb{R}_+ \to [0, 1)$ be a continuous mapping and let a continuous mapping $\mathcal{A} : \mathcal{W} \to \mathcal{W}$ fulfills the conditions*

$$\hat{\wp}(\mathcal{A}W) \leq [\beta\{\hat{\wp}(W)\} - 1]\hat{\wp}(W), \tag{2.4}$$

where $\hat{\wp}$ is an arbitrary MNC and $W \subset \mathcal{W}$. Then, $\mathcal{W}$ contains at least one fixed point of $\mathcal{A}$.

Proof: Corollary 2.1 can be proved by putting $\varphi(w_1) = w_1$ in Theorem 2.3. □

Corollary 2.2 *Assume that $\mathcal{W}$ is a NBCCS of a Banach space E. Also let a continuous mapping $\mathcal{A} : \mathcal{W} \to \mathcal{W}$ satisfies the condition*

$$\hat{\wp}(\mathcal{A}W) \leq h\hat{\wp}(W), \tag{2.5}$$

where $W \subset \mathcal{W}$, $\hat{\wp}$ is an arbitrary MNC. Then, $\mathcal{A}$ has at least one fixed point in $\mathcal{W}$.

Proof: In Theorem 2.3 by substituting $\beta(w_1) = \frac{1}{k}$ and $h = \frac{1-k}{k}$, we obtain the Darbo's theorem. □

3. Measure of noncompactness on $C(I)$

Let us consider the real continuous function space $E = C(I)$ on I, where $I = [0, T]$ equipped with the norm

$$\| \Delta \| = \sup \{|\Delta(w_1)| : w_1 \in I\}, \ \Delta \in E.$$

Consider a bounded subset $W(\neq \emptyset)$ of E. The modulus of the continuity of Δ denoted by $\hat{\wp}(\Delta, \varepsilon)$ for $\Delta \in W$ and $\varepsilon > 0$ is define as

$$\hat{\wp}(\Delta, \varepsilon) = \sup \{|\Delta(w_1) - \Delta(w_2)| : w_1, w_2 \in I, |w_1 - w_2| \leq \varepsilon\}.$$

Further, let us set

$$\hat{\wp}(W, \varepsilon) = \sup \{\hat{\wp}(\Delta, \varepsilon) : \Delta \in W\}; \ \hat{\wp}_0(W) = \lim_{\varepsilon \to 0} \hat{\wp}(W, \varepsilon).$$

Then the function $\hat{\wp}_0$ is generally known to be a MNC in E, with $\Theta(W) = \frac{1}{2}\hat{\wp}_0(W)$ (see [3]) as the Hausdorff MNC Θ.

4. Solvability of k-fractional integral equations

In this present section of the paper, we will demonstrate how we can apply our conclusions in regard to the existence result of a fractional integral equation in Banach space.
Assume $k \in \mathbb{N}$, $\alpha \in \mathbb{C} : \mathbb{R}(\alpha) \in (n-1, n]$, $n \in \mathbb{N}$, where $\mathbb{R}(\cdot)$ represents the real part of a complex number, the k-fractional Hilfer integral is defined by [12]

$$I_k^\alpha[f(\varrho)] = \frac{1}{k\Gamma_k(\alpha)} \int_0^\varrho (\varrho - \hat{\xi})^{\frac{\alpha}{k}-1} f(\hat{\xi}) d\hat{\xi}; \ \ \varrho \in [0, T].$$

In this study, we will be considering a functional integral equation:

$$L = \Delta(\varrho, H(\varrho, L(\varrho)), (I_k^\alpha L)(\varrho)), \tag{4.1}$$

where $0 < \alpha, k < 1$, and $\vartheta \in I = [0, T]$.
Let define

$$B_{r_0} = \{L \in X : \| L \| \leq r_0\}.$$

Assume that

(A) $H : I \times \mathbb{R} \to \mathbb{R}$, $\Delta : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions and constants $c_i \geq 0$; $i = 1, 2, 3$ exists satisfying the conditions

$$\left|\Delta(\varrho, H, I_1) - \Delta(\varrho, \bar{H}, \bar{I_1})\right| \leq c_1 \left|H - \bar{H}\right| + c_2 \left|I_1 - \bar{I_1}\right|, \ \varrho \in I, \ I_1, \bar{I_1} \in \mathbb{R}$$

and

$$\left|H(\varrho, L_1) - H(\varrho, \bar{L_1})\right| \leq c_3 \left|L_1 - \bar{L_1}\right|, \ \varrho \in I, \ L_1, \bar{L_1} \in \mathbb{R}.$$

(B) A constant $r_0 > 0$ exists which satisfies the conditions

$$\bar{\Delta} = \sup \left\{|\Delta(\varrho, H, I_1)| : \varrho \in I, H \in [-\bar{H}, \bar{H}], I_1 \in [-\bar{I}, \bar{I}]\right\} \leq r_0,$$

and $c_1 c_3 < 1$. Here,

$$\bar{H} = \sup \{|H(\varrho, L(\varrho))| : \varrho \in I, L(\varrho) \in [-r_0, r_0]\}$$

and

$$\bar{I} = \sup \{|(I_k^\alpha L)(\varrho)| : \varrho \in I, L(\varrho) \in [-r_0, r_0]\}.$$

(C) $\max |\Delta(\varrho, 0, 0)| = M$ and $H(\varrho, 0) = 0$ forall $\varrho \in I$.

(D) A positive solution r_0 exists such that

$$c_1 c_3 r_0 + M + \frac{c_2 r_0}{\alpha \Gamma_k(\alpha)} T^{\alpha/k} \leq r_0.$$

Theorem 4.1 *If the assumptions (A)-(D) holds, then the equation (4.1) has at least one solution in* $E = C(I)$.

Proof: Let $\mathcal{D} : E \to E$ be an operator defined by:

$$(\mathcal{D}L)(\varrho) = \Delta(\varrho, H(\varrho, L(\varrho)), (I_k^\alpha L)(\varrho)).$$

1^{st} **Step:** We show that $\mathcal{D}$ maps B_{r_0} into B_{r_0}. Let $L \in B_{r_0}$, We have

$$\begin{aligned}
&|(\mathcal{D}L)(\varrho)| \\
&\leq |\Delta(\varrho, H(\varrho, L(\varrho)), (I_k^\alpha L)(\varrho)) - \Delta(\varrho, 0, 0)| + |\Delta(\varrho, 0, 0)| \\
&\leq c_1 |H(\varrho, L(\varrho))| + c_2 |(I_k^\alpha L)(\varrho)| + |\Delta(\varrho, 0, 0)| \\
&\leq c_1 c_3 |L(\varrho)| + c_2 |(I_k^\alpha L)(\varrho)| + M.
\end{aligned}$$

Also,

$$\begin{aligned}
|(I_k^\alpha L)(\varrho)| &= \left| \frac{1}{k\Gamma_k(\alpha)} \int_0^\varrho (\varrho - \hat{\xi})^{\frac{\alpha}{k}-1} L(\hat{\xi}) d\hat{\xi} \right| \\
&\leq \frac{1}{k\Gamma_k(\alpha)} \int_0^\varrho (\varrho - \hat{\xi})^{\frac{\alpha}{k}-1} \left| L(\hat{\xi}) \right| d\hat{\xi} \\
&\leq \frac{r_0}{k\Gamma_k(\alpha)} \int_0^\varrho (\varrho - \hat{\xi})^{\frac{\alpha}{k}-1} d\hat{\xi} \\
&\leq \frac{r_0}{\alpha\Gamma_k(\alpha)} T^{\alpha/k}.
\end{aligned}$$

Hence, $\parallel L \parallel < r_0$ gives

$$\parallel \mathcal{D}L \parallel \leq c_1 c_3 r_0 + M + \frac{c_2 r_0}{\alpha\Gamma_k(\alpha)} T^{\alpha/k} \leq r_0.$$

It follows from the assumption (D) that $\mathcal{D}$ maps B_{r_0} into B_{r_0}.
2^{nd} **Step:** Next, we claim that $\mathcal{D}$ is continuous on B_{r_0}. Fix $\delta > 0$ and let $L, \bar{L} \in B_{r_0}$ such that $\parallel L - \bar{L} \parallel < \delta$. For all $x \in I$, we now obtain

$$\begin{aligned}
&\left|(\mathcal{D}L)(\varrho) - (\mathcal{D}\bar{L})(\varrho)\right| \\
&= \left|\Delta(\varrho, H(\varrho, L(\varrho)), (I_k^\alpha L)(\varrho)) - \Delta(\varrho, H(\varrho, \bar{L}(\varrho)), (I_k^\alpha \bar{L})(\varrho))\right| \\
&\leq c_1 \left|H(\varrho, L(\varrho)) - H(\varrho, \bar{L}(\varrho))\right| + c_2 \left|(I_k^\alpha L)(\varrho) - (I_k^\alpha \bar{L})(\varrho)\right| \\
&\leq c_1 c_3 \left|L(\varrho) - \bar{L}(\varrho)\right| + c_2 \left|(I_k^\alpha L)(\varrho) - (I_k^\alpha \bar{L})(\varrho)\right| \\
&\leq c_1 c_3 \parallel L - \bar{L} \parallel + c_2 \left|(I_k^\alpha L)(\varrho) - (I_k^\alpha \bar{L})(\varrho)\right|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left|(I_k^\alpha L)(\varrho) - (I_k^\alpha \bar{L})(\varrho)\right| \\
&= \left| \frac{1}{k\Gamma_k(\alpha)} \int_0^\varrho (\varrho - \hat{\xi})^{\frac{\alpha}{k}-1} (L(\hat{\xi}) - \bar{L}(\hat{\xi})) d\hat{\xi} \right| \\
&\leq \frac{1}{k\Gamma_k(\alpha)} \int_0^\varrho (\varrho - \hat{\xi})^{\frac{\alpha}{k}-1} \left| L(\hat{\xi}) - \bar{L}(\hat{\xi}) \right| d\hat{\xi} \\
&< \frac{\delta}{k\Gamma_k(\alpha)} \int_0^\varrho (\varrho - \hat{\xi})^{\frac{\alpha}{k}-1} d\hat{\xi} \\
&< \frac{\delta}{\alpha\Gamma_k(\alpha)} T^{\alpha/k}.
\end{aligned}$$

Hence, $\parallel L - \bar{L} \parallel < \delta$ gives

$$\left|(\mathcal{D}L)(\varrho) - (\mathcal{D}\bar{L})(\varrho)\right| < c_1 c_3 \delta + c_2 \frac{\delta}{\alpha\Gamma_k(\alpha)} T^{\alpha/k}.$$

As $\delta \to 0$, we get

$$\left|(\mathcal{D}L)(\vartheta) - (\mathcal{D}\bar{L})(\varrho)\right| \to 0,$$

which indicates that $\mathcal{D}$ is continuous on B_{r_0}.
3^{rd} **Step:** Here, we estimate $\mathcal{D}$ is relation to $\hat{\wp}_0$. Consider that $\varpi(\neq \emptyset) \subseteq B_{r_0}$. Let $\delta > 0$ be arbitrary and choose $L \in \varpi$ with $\varrho_1, \varrho_2 \in I$ such that $|\varrho_2 - \varrho_1| \leq \delta$ and $\varrho_2 \geq \varrho_1$.
Now,

$$\begin{aligned}
&|(\mathcal{D}L)(\varrho_2) - (\mathcal{D}L)(\varrho_1)| \\
&= |\Delta(\varrho_2, H(\varrho_2, L(\varrho_2)), (I_k^\alpha L)(\varrho_2)) - \Delta(\varrho_1, H(\varrho_1, L(\varrho_1)), (I_k^\alpha L)(\varrho_1))| \\
&\leq |\Delta(\varrho_2, H(\varrho_2, L(\varrho_2)), (I_k^\alpha L)(\varrho_2)) - \Delta(\varrho_2, H(\varrho_2, L(\varrho_2)), (I_k^\alpha L)(\varrho_1))| \\
&+ |\Delta(\varrho_2, H(\varrho_2, L(\varrho_2)), (I_k^\alpha L)(\varrho_1)) - \Delta(\varrho_2, H(\varrho_1, L(\varrho_1)), (I_k^\alpha L)(\varrho_1))| \\
&+ |\Delta(\varrho_2, H(\varrho_1, L(\varrho_1)), (I_k^\alpha L)(\varrho_1)) - \Delta(\varrho_1, H(\varrho_1, L(\varrho_1)), (I_k^\alpha L)(\varrho_1))| \\
&\leq c_2 |(I_k^\alpha L)(\varrho_2) - (I_k^\alpha L)(\varrho_1)| + c_1 |H(\varrho_2, L(\varrho_2)) - H(\varrho_1, L(\varrho_1))| + \beta_\Delta(I, \delta) \\
&\leq c_2 |(I_k^\alpha L)(\varrho_2) - (I_k^\alpha L)(\varrho_1)| + c_1 c_3 |L(\varrho_2) - L(\varrho_1)| + \beta_\Delta(I, \delta),
\end{aligned}$$

where

$$\beta_\Delta(I, \delta) = \sup \left\{ \begin{array}{c} |\Delta(\varrho_2, H, I_1) - \Delta(\varrho_1, H, I_1)| : |\varrho_2 - \varrho_1| \leq \delta; \varrho_1, \varrho_2 \in I; \\ H \in [-\bar{H}, \bar{H}], I_1 \in [-\bar{I}, \bar{I}] \end{array} \right\}.$$

Also,

$$\begin{aligned}
&|(I_k^\alpha L)(\varrho_2) - (I_k^\alpha L)(\varrho_1)| \\
&= \left| \frac{1}{k\Gamma_k(\alpha)} \int_0^{\varrho_2} (\varrho_2 - \hat{\xi})^{\frac{\alpha}{k}-1} L(\xi) d\hat{\xi} - \frac{1}{k\Gamma_k(\alpha)} \int_0^{\varrho_1} (\varrho_1 - \hat{\xi})^{\frac{\alpha}{k}-1} L(\hat{\xi}) d\hat{\xi} \right| \\
&\leq \frac{1}{k\Gamma_k(\alpha)} \left| \int_0^{\varrho_2} (\varrho_2 - \hat{\xi})^{\frac{\alpha}{k}-1} L(\hat{\xi}) d\hat{\xi} - \int_0^{\varrho_1} (\varrho_1 - \hat{\xi})^{\frac{\alpha}{k}-1} L(\hat{\xi}) d\hat{\xi} \right| \\
&\leq \frac{1}{k\Gamma_k(\alpha)} \left| \int_0^{\varrho_2} (\varrho_2 - \hat{\xi})^{\frac{\alpha}{k}-1} L(\hat{\xi}) d\hat{\xi} - \int_0^{\varrho_1} (\varrho_2 - \hat{\xi})^{\frac{\alpha}{k}-1} L(\hat{\xi}) d\hat{\xi} \right| \\
&+ \frac{1}{k\Gamma_k(\alpha)} \left| \int_0^{\vartheta_1} (\varrho_2 - \hat{\xi})^{\frac{\alpha}{k}-1} L(\hat{\xi}) d\hat{\xi} - \int_0^{\varrho_1} (\varrho_1 - \hat{\xi})^{\frac{\alpha}{k}-1} L(\hat{\xi}) d\hat{\xi} \right| \\
&\leq \frac{\| L \|}{k\Gamma_k(\alpha)} \left| \int_{\varrho_1}^{\vartheta_2} (\varrho_2 - \hat{\xi})^{\frac{\alpha}{k}-1} d\hat{\xi} + \int_0^{\varrho_1} (\varrho_2 - \hat{\xi})^{\frac{\alpha}{k}-1} d\hat{\xi} - \int_0^{\varrho_1} (\varrho_1 - \hat{\xi})^{\frac{\alpha}{k}-1} d\hat{\xi} \right| \\
&\leq \frac{\| L \|}{k\Gamma_k(\alpha)} \left| (\varrho_2 - \varrho_1)^{\frac{\alpha}{k}} + \varrho_2^{\frac{\alpha}{k}} - (\varrho_2 - \varrho_1)^{\frac{\alpha}{k}} - \varrho_1^{\frac{\alpha}{k}} \right|.
\end{aligned}$$

As $\delta \to 0$, then $\varrho_2 \to \varrho_1$ and so,

$$|(I_k^\alpha L)(\varrho_2) - (I_k^\alpha L)(\varrho_1)| \to 0.$$

Hence,

$$|(\mathcal{D}L)(\varrho_2) - (\mathcal{D}L)(\varrho_1)| \leq c_2 |(I_k^\alpha L)(\varrho_2) - (I_k^\alpha L)(\varrho_1)| + c_1 c_3 |L(\varrho_2) - L(\varrho_1)| + \beta_\Delta(I, \delta).$$

Which yields,

$$\hat{\wp}(\mathcal{D}L, \delta) \leq c_1 c_3 |L(\varrho_2) - L(\varrho_1)| + \beta_\Delta(I, \delta).$$

It is derived from the uniform continuity of Δ on $I \times [-\bar{H}, \bar{H}] \times [-\bar{I}, \bar{I}]$ that $\lim_{\delta \to 0} \beta_\Delta(I, \delta) \to 0$, as $\delta \to 0$. Setting $\sup_{L \in \varpi}$ as well as $\delta \to 0$, we obtain

$$\hat{\wp}_0(\mathcal{D}\varpi) \leq c_1 c_3 \hat{\wp}_0(\varpi).$$

By Corollary 2.2, we can state that $\mathcal{D}$ possess a fixed point in $\Delta \subseteq B_{r_0}$. This means that in the space E the Equation (4.1) has a solution. □

Example 4.1 Consider the following fractional integral equation:

$$L(\varrho) = \frac{L(\varrho)}{15+\varrho^2} + \frac{I_{\frac{1}{5}}^{\frac{2}{5}} L(\varrho)}{25} + g(\varrho) \tag{4.2}$$

for $\varrho \in [0,2] = I$.

Here,

$$g(\varrho) = k^2 e^{-\frac{\alpha}{k}\varrho} \frac{9+e^{\frac{\alpha}{k}\varrho}}{10+e^{\frac{\alpha}{k}\varrho}} = \frac{1}{25} e^{-2\varrho} \frac{9+e^{2\varrho}}{10+e^{2\varrho}}$$

and

$$I_{\frac{1}{5}}^{\frac{2}{5}} L(\varrho) = \frac{5}{\Gamma_{1/5}(2/5)} \int_0^{\varrho} (\varrho-\xi)\, L(\xi) d\xi.$$

Also,

$$\Delta(\varrho, H, I_1) = \frac{1}{25} e^{-2\varrho} \frac{9+e^{2\varrho}}{10+e^{2\varrho}} + H + \frac{I_1}{25}$$

and

$$H(\varrho, L) = \frac{L}{15+\varrho^2}.$$

Obviously, both Δ and H are continuous and satisfies the conditions:

$$|H(\varrho, L_1) - H(\varrho, L_2)| \leq \frac{|L_1 - L_2|}{15}$$

and

$$\left|\Delta(\varrho, H, I_1) - \Delta(\varrho, \bar{H}, \bar{I}_1)\right| \leq \left|H - \bar{H}\right| + \frac{1}{25}\left|I_1 - \bar{I}_1\right|,$$

respectively. Therefore

$$c_1 = 1,\ c_2 = \frac{1}{25},\ c_3 = \frac{1}{15} \quad and \quad c_1 c_3 = \frac{1}{15} < 1.$$

If $\|L\| \leq r_0$, then

$$\bar{H} = \frac{r_0}{15}$$

and

$$\bar{I}_1 = \frac{10 r_0}{\Gamma_{1/5}(2/5)}.$$

Further,

$$|\Delta(\varrho, H, I_1)| \leq \frac{r_0}{15} + \frac{1}{25}.\frac{10 r_0}{\Gamma_{1/5}(2/5)} + \frac{2}{55} \leq r_0.$$

If we choose $r_0 = 2$, then

$$\bar{H} = \frac{2}{15}$$

and

$$\bar{I}_1 = \frac{20}{\Gamma_{1/5}(2/5)}.$$

which gives

$$\bar{\Delta} = 0.667355 \leq 2.$$

However, also for $r_0 = 2$, assumption (D) is satisfied.
Consequently, we have achieved all of the assumptions of Theorem 4.1 from (A) to (D). Therefore from Theorem 4.1, we can say that the equation (4.2) has a solution in $E = C(I)$.

Example 4.2 Consider the following fractional integral equation:

$$L(\varrho) = \frac{L(\varrho)}{5+\varrho^4} + \frac{I_{\frac{1}{3}}^{\frac{2}{3}} L(\varrho)}{8} + g(\varrho) \tag{4.3}$$

for $\varrho \in [0,3] = I$.

Here,

$$g(\varrho) = k^2 e^{-\frac{\alpha}{k}\varrho} \frac{3+\varrho^{\frac{\alpha}{k}}}{15+\varrho^{\frac{\alpha}{k}}} = \frac{1}{9} e^{-2\varrho} \frac{3+\varrho^2}{15+\varrho^2}$$

and

$$I_{\frac{1}{3}}^{\frac{2}{3}} L(\varrho) = \frac{3}{\Gamma_{1/3}(2/3)} \int_0^{\varrho} (\varrho - \xi) L(\xi) d\xi.$$

Also,

$$\Delta(\varrho, H, I_1) = \frac{1}{9} e^{-2\varrho} \frac{3+\varrho^2}{15+\varrho^2} + H + \frac{I_1}{8}$$

and

$$H(\varrho, L) = \frac{L}{5+\varrho^4}.$$

Obviously, both Δ and H are continuous and satisfies the conditions:

$$|H(\varrho, L_1) - H(\varrho, L_2)| \le \frac{|L_1 - L_2|}{5}$$

and

$$\left|\Delta(\varrho, H, I_1) - \Delta(\varrho, \bar{H}, \bar{I}_1)\right| \le \left|H - \bar{H}\right| + \frac{1}{8}\left|I_1 - \bar{I}_1\right|,$$

respectively. Therefore

$$c_1 = 1,\ c_2 = \frac{1}{8},\ c_3 = \frac{1}{5} \quad and \quad c_1 c_3 = \frac{1}{5} < 1.$$

If $\|L\| \le r_0$, then

$$\bar{H} = \frac{r_0}{5}$$

and

$$\bar{I}_1 = \frac{27 r_0}{2\Gamma_{1/3}(2/3)}.$$

Further,

$$|\Delta(\varrho, H, I_1)| \le \frac{r_0}{5} + \frac{1}{8} . \frac{27 r_0}{2\Gamma_{1/3}(2/3)} + \frac{1}{45} \le r_0.$$

If we choose $r_0 = 3$, then

$$\bar{H} = \frac{2}{5}$$

and

$$\bar{I}_1 = \frac{81}{2\Gamma_{1/3}(2/3)}.$$

which gives

$$\bar{\Delta} = 2.52745 \le 3.$$

However, also for $r_0 = 3$, assumption (D) is satisfied.
Consequently, we have achieved all of the assumptions of Theorem 4.1 from (A) to (D). Therefore from Theorem 4.1, we can say that the equation (4.3) has a solution in $E = C(I)$.

5. Conclusion

Using generalized Darbo's fixed point theorem along with $\mathcal{U}$-class mappings, we can derive new results concerning fixed point theorems. This approach presents that Darbo's fixed point theorem is effective in establishing important fixed point results, particularly in proving the existence of solutions for a specific type of k-fractional Hilfer integral equation.

References

1. Agarwal, R. P., and O'regan, D., *Fixed point theory and applications,* Cambridge university press 141, (2001).
2. Ansari, A. H., *Note on $\varphi - \psi$-contractive type mappings and related fixed point,* In The 2nd regional conference on mathematics and applications, Payame Noor University 11, 377-380, (2014).
3. Banaś, J., and Goebel, K., *Measure of Noncompactness in Banach Spaces,* Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York 60, (1980).
4. Caponetti, D., Trombetta, A., and Trombetta, G., *Regular measures of noncompactness and Ascoli–Arzelà type compactness criteria in spaces of vector-valued functions,* Banach J. Math. Anal. 17(3), 48, (2023).
5. Chandra Deuri, B., V. Paunović, M., Das, A., and Parvaneh, V., *Solution of a fractional integral equation using the Darbo fixed point theorem,* J. Math. 2022(1), 8415616, (2022).
6. Chen, X., and Cheng, L., *Representation of measures of noncompactness and its applications related to an initial value problem in Banach spaces,* Science China Mathematics, 66(4), 745-776, (2023).
7. Darbo, G., *Punti uniti in trasformazioni a codominio non compatto,* Rendiconti del Seminario matematico della Università di Padova 24, 84-92, (1955).
8. Das, A., Hazarika, B., Parvaneh, V., and Mursaleen, M., *Solvability of generalized fractional order integral equations via measures of noncompactness,* Mathematical Sciences 15(3), 241-251, (2021).
9. Das, A., Hazarika, B., Arab, R., and Mursaleen, M., *Applications of a fixed point theorem to the existence of solutions to the nonlinear functional integral equations in two variables,* Rendiconti del Circolo Matematico di Palermo Series 2 68(1), 139-152, (2019).
10. Das, A., *Application of Measure of Noncompactness on Infinite System of Functional Integro-differential Equations with Integral Initial Conditions,* In Sequence Space Theory with Applications, Chapman and Hall/CRC, 45-62, (2022).
11. Deb, S., Jafari, H., Das, A., and Parvaneh, V., *New fixed point theorems via measure of noncompactness and its application on fractional integral equation involving an operator with iterative relations,* J. Inequa. Appl. 2023(1), 106, (2023).
12. Dorrego, G. A., and Cerutti, R. A., *The k-fractional Hilfer derivative,* Int. J. Math. Anal. 7(11), 543-550, (2013).
13. Gabeleh, M., Malkowsky, E., Mursaleen, M., and Rakočević, V., *A new survey of measures of noncompactness and their applications,* Axioms 11(6), 299, (2022).
14. Golshan, H. M., *On solution of fractional integral equation via measure of noncompactness and Petryshyn's fixed point theorem,* (2024). doi: 10.22541/au.170669940.03355969/v1
15. Hammad, H. A., Aydi, H., and De la Sen, M., *Solving a Nonlinear Fractional Integral Equation by Fixed Point Approaches Using Auxiliary Functions Under Measure of Noncompactness,* Inter. J. Anal. Appl. 22, 53-53, (2024).
16. Hazarika, B., Arab, R., and Mursaleen, M., *Applications of measure of noncompactness and operator type contraction for existence of solution of functional integral equations,* Complex Analysis and Operator Theory 13(8), 3837-3851, (2019).
17. Khan, M. S., Swaleh, M., and Sessa, S., *Fixed point theorems by altering distances between the points,* Bull. Aust. Math. Soc. 30(1), 1-9, (1984).
18. Karakaya, V., and Sekman, D., *A new type of contraction via measure of non-compactness with an application to Volterra integral equation,* Publications de l'Institut Mathematique 111(125), 111-121, (2022).
19. Khokhar, G. K., Patel, D. K., Patle, P. R., and Samei, M. E., *Optimum solution of $(k, \daleth)$-Hilfer FDEs by $\mathcal{A}$-condensing operators and the incorporated measure of noncompactness,* J. Inequa. Appl. 2024(1), 1-31, (2024).
20. Kuratowski, K., *Sur les espaces complets,* Fundamenta Mathematicae 1(15), 301-309, (1930).
21. Mastyło, M., and da Silva, E. B., *Measures of noncompactness of interpolated polynomials,* In Forum Mathematicum 35(2), 487-505, (2023).
22. Mesmouli, M. B., Hamza, A. E., and Rizk, D., *A Study of an IBVP of Fractional Differential Equations in Banach Space via the Measure of Noncompactness,* Fractal and Fractional 8(1), 30, (2023).
23. Metwali, M. M., *On measure of noncompactness in variable exponent Lebesgue spaces and applications to integral equations,* J. Inequa. Appl. 2023(1), 157, (2023).

24. Nashine, H. K., and Das, A., *Extension of Darbo's fixed point theorem via shifting distance functions and its application,* Nonlinear Analysis: Modelling and Control 27(2), 275-288, (2022).

25. Nikam, V., Shukla, A. K., Gopal, D., and Sumalai, P., *Some Darbo-type fixed-point theorems in the modular space and existence of solution for fractional ordered 2019-nCoV mathematical model,* Mathematical Methods in the Applied Sciences 47(18), 13923-13947, (2024).

26. Olszowy, L., and Zajac, T., *On Darbo-and Sadovskii-Type Fixed Point Theorems in Banach Spaces,* Symmetry 16(4), 392, (2024).

27. Rabbani, M., Das, A., Hazarika, B., and Arab, R., *Measure of noncompactness of a new space of tempered sequences and its application on fractional differential equations,* Chaos, Solitons & Fractals, 140, 110221, (2020).

28. Salem, A., Malaikah, H., and Kamel, E. S., *An Infinite System of Fractional Sturm–Liouville Operator with Measure of Noncompactness Technique in Banach Space,* Mathematics 11(6), 1444, (2023).

29. Srivastava, H. M., Das, A., Hazarika, B., and Mohiuddine, S., *Existence of solution for non-linear functional integral equations of two variables in Banach algebra,* Symmetry, 11(5), 674, (2019).

30. Taoudi, M. A., *On Darbo's fixed point principle,* Moroccan Journal of Pure and Applied Analysis 9(3), 304-310, (2023).

31. Telli, B., Souid, M. S., and Stamova, I., *Boundary-value problem for nonlinear fractional differential equations of variable order with finite delay via Kuratowski measure of noncompactness,* Axioms 12(1), 80, (2023).

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