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Some Non-Trivial Trigonometric Identities From Theta Functions

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ABSTRACT: It is always exciting to find relations between quantities that are not immediately and obviously related. In this regard, connecting theta functions and continued fractions with trigonometric functions is fascinating. Though trigonometric identities can also be established by elementary methods or by the help of software, the fact that they can be obtained by theta functions and continued fractions is intriguing. In the present research article, we deduce twelve trigonometric identities, using theta function identities listed by Ramanujan along with modular relations of few continued fractions.

Key Words: Theta functions, modular equations, trigonometric identities.

Contents

1. Introduction

Throughout the article, let $q \in \mathbb{C}$ in such a way that $|q| < 1$. Then the q-shifted factorial or q-rising factorial [5] is customarily defined as

$$(\zeta; q)_m := \prod_{k=1}^{m} (1 - \zeta q^{k-1}), \quad \text{and} \quad (\zeta; q)_\infty := \prod_{k=0}^{\infty} (1 - \zeta q^k),$$

where $\zeta \in \mathbb{C}$ and $m \in \mathbb{N}$. By convention, $(\zeta; q)_0 := 1$. This is the q-analogue of the renowned Pochhammer symbol

$$(\zeta)_m = \begin{cases} \zeta(\zeta+1)\dots(\zeta+m-1) & ; m \in \mathbb{N} \\ 1 & ; m = 0, \end{cases}$$

in the sense that

$$\lim_{q \to 1} \frac{(q^\zeta; q)_m}{(1-q)^m} = (\zeta)_m. \tag{1.1}$$

We use the following notation for convenience:

$$(\zeta_1, \zeta_2, \dots, \zeta_m; q)_\infty = (\zeta_1; q)_\infty (\zeta_2; q)_\infty \dots (\zeta_m; q)_\infty.$$

Now,

$$f(\gamma, \delta) = \sum_{k=-\infty}^{\infty} \gamma^{k(k+1)/2} \delta^{k(k-1)/2}, \qquad |\gamma\delta| < 1$$

is Ramanujan's general theta function. The following is the Jacobi's triple product identity in terms of Ramanujan's general theta function [5, p. 35, Entry 19]:

$$f(\gamma, \delta) = (-\gamma, -\delta, \gamma\delta; \gamma\delta)_\infty.$$

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The following are the theta functions [5, p. 36, Entry 22 (i)-(iii)] arising from $f(\gamma,\delta)$:

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k-1)}{2}} = (q;q)_\infty, \tag{1.2}$$

$$\varphi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = \frac{(-q;q^2)^2_\infty}{(q^2;q^2)_\infty}, \tag{1.3}$$

and

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

Ramanujan recorded numerous wonderful continued fractions in the unorganized portions of his notebook. Some of the q-continued fractions of Ramanujan give interesting representations for certain q-products. Adiga et al. [1] established continued fraction representations of the similar kind for $\varphi(q)$ and $\psi(q)$ using Ramanujan's general continued fraction through induction. In the recent past, several mathematicians and physicists have investigated the theta functions, trigonometric functions, zeta functions, Mellin's transform, multiple zeta functions, L-series, multiple q-Bernoulli numbers, Bell and Dowling polynomials (see [14,17]), which serve whole different perspective to classical Ramanujan's theory of q-series. These numbers and functions are used in p-adic analysis and other areas of complex analysis and mathematical physics. The generalized hypergeometric function [16] ${}_{s+1}\Phi_s(z)$ is defined by

$${}_{s+1}\Phi_s\left[\begin{matrix} x_0, x_1, \ldots, x_s \\ y_1, y_2, \ldots, y_s \end{matrix} ; z\right] = \sum_{k=0}^{\infty} \frac{(x_0)_k (x_1)_k \ldots (x_s)_k}{(y_1)_k (y_2)_k \ldots (y_s)_k} \frac{z^k}{k!}, |z| < 1,$$

Ramanujan documented numerous theta function identities in his notebooks. These identities bear applications in numerous fields. The following are the identities listed by Ramanujan [6, pp. 140-142], which form the link connecting his theta functions with the trigonometric functions. As $q \to 1^-$, he stated the following identities:

$$\frac{f(-q^k,-q^l)}{f(-q^r,-q^s)} \sim \frac{\sin\left(\frac{k\pi}{m}\right)}{\sin\left(\frac{r\pi}{m}\right)}, \tag{1.4}$$

and

$$\frac{f(-q^k,-q^l)}{\varphi(-q^{m/2})} \sim \sin\left(\frac{k\pi}{m}\right),$$

where k, l, r and $s \in \mathbb{N}$, such that $k + l = m = r + s$. Since right hand side of (1.4) is a constant (devoid of q), employing the definition as 'asymptotic to', we deduce

$$\lim_{q\to 1^-} \frac{f(-q^k,-q^l)}{f(-q^r,-q^s)} = \frac{\sin\left(\frac{k\pi}{m}\right)}{\sin\left(\frac{r\pi}{m}\right)}. \tag{1.5}$$

Similarly, we obtain

$$\lim_{q\to 1^-} \frac{f(-q^k,-q^l)}{\varphi(-q^{m/2})} = \sin\left(\frac{k\pi}{m}\right). \tag{1.6}$$

These identities can be exploited to obtain various particular or generalized trigonometric sums using the known theta function identities of Ramanujan. The detailed history can be found in [7], where Berndt and Yeap gave the generalization of the trigonometric sums in various dimensions by obtaining some of the reciprocity theorems utilizing contour integration technique. Berndt and Zaharescu [8] generalized several non-trivial trigonometric sums of Liu [15]. For similar work, [4] and [12] can be referred. Vinay et al. [19]

obtained several trigonometric sums from theta function identities of Ramanujan. Yathirajsharma [20] constructed four generalized finite q-trigonometric sums inspired by Gosper's q-trigonometric identities. Berndt et al. [9] demonstrated the closed form evaluation of two categories of finite trigonometric sums, both consisting of sines only. Allouche and Zeilberger [2] demonstrated the direct elementary proofs of few trigonometric identities established by Harshitha et al. [12] and they also proved two conjectures from [12]. They also noted how the use of software like MAPLE provides such identities within fraction of seconds. However, it has been indeed fascinating to obtain trigonometric identities from Ramanujan's theta function identities. This makes us even wonder if any of Ramanujan's identities were influenced by the trigonometric identities at the first place!

In this article, we establish the following trigonometric sums, which follow from (1.5), (1.6), and theta function identities recorded in Section 2. Rather than proving the trigonometric identities using elementary methods or using software, we make use of Ramanujan's theta function identities as well as modular relations on few continued fractions. Some of the identities, like (1.7), can be proved using the elementary angle sum formula, but our interest lies in proofs using the suitable theta function identities, thus appreciating the strange yet fascinating relationship between Ramanujan's theta functions, continued fractions and trigonometric functions. However, these identities can be verified using MAPLE.

Theorem 1.1 *The below-mentioned trigonometric identities hold good.*

$$\sin\left(\frac{\pi}{10}\right)\sin\left(\frac{3\pi}{10}\right)=\frac{1}{4}. \tag{1.7}$$

$$\sin\left(\frac{3\pi}{10}\right)-\sin\left(\frac{\pi}{10}\right)=\frac{1}{2}. \tag{1.8}$$

$$\sin^2\left(\frac{\pi}{5}\right)+\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right)-\sin^2\left(\frac{2\pi}{5}\right)=0. \tag{1.9}$$

$$\sin\left(\frac{5\pi}{14}\right)-\sin\left(\frac{3\pi}{14}\right)+\sin\left(\frac{\pi}{14}\right)=\frac{1}{2}. \tag{1.10}$$

$$\frac{\sin\left(\frac{4\pi}{9}\right)}{\sin\left(\frac{2\pi}{9}\right)}+\frac{\sin\left(\frac{\pi}{9}\right)}{\sin\left(\frac{4\pi}{9}\right)}-\frac{\sin\left(\frac{2\pi}{9}\right)}{\sin\left(\frac{\pi}{9}\right)}=0. \tag{1.11}$$

$$\sin\left(\frac{9\pi}{22}\right)-\sin\left(\frac{7\pi}{22}\right)+\sin\left(\frac{5\pi}{22}\right)-\sin\left(\frac{3\pi}{22}\right)+\sin\left(\frac{\pi}{22}\right)=\frac{1}{2}. \tag{1.12}$$

$$\frac{\sin\left(\frac{\pi}{11}\right)}{\sin\left(\frac{5\pi}{11}\right)}+\frac{\sin\left(\frac{2\pi}{11}\right)}{\sin\left(\frac{\pi}{11}\right)}-\frac{\sin\left(\frac{3\pi}{11}\right)}{\sin\left(\frac{4\pi}{11}\right)}-\frac{\sin\left(\frac{4\pi}{11}\right)}{\sin\left(\frac{2\pi}{11}\right)}+\frac{\sin\left(\frac{5\pi}{11}\right)}{\sin\left(\frac{3\pi}{11}\right)}=1. \tag{1.13}$$

$$\sin^2\left(\frac{\pi}{8}\right)+2\sin\left(\frac{\pi}{8}\right)\sin\left(\frac{3\pi}{8}\right)-\sin^2\left(\frac{3\pi}{8}\right)=0. \tag{1.14}$$

$$\sin^2\left(\frac{\pi}{6}\right)+2\sin^3\left(\frac{\pi}{6}\right)-\sin\left(\frac{\pi}{6}\right)=0. \tag{1.15}$$

$$\sin\left(\frac{\pi}{6}\right)+2\sin^3\left(\frac{\pi}{6}\right)+4\sin^4\left(\frac{\pi}{6}\right)=1. \tag{1.16}$$

$$\sin^3\left(\frac{\pi}{10}\right)\sin^2\left(\frac{\pi}{5}\right)\sin\left(\frac{3\pi}{10}\right)-\sin^2\left(\frac{\pi}{10}\right)\sin^2\left(\frac{3\pi}{10}\right)\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right)$$
$$-\sin\left(\frac{\pi}{10}\right)\sin^2\left(\frac{2\pi}{5}\right)\sin^3\left(\frac{3\pi}{10}\right)+\sin^3\left(\frac{2\pi}{5}\right)\sin^3\left(\frac{\pi}{5}\right)=0. \tag{1.17}$$

$$\sin^2\left(\frac{\pi}{12}\right)-4\sin\left(\frac{\pi}{12}\right)\sin\left(\frac{5\pi}{12}\right)+\sin^2\left(\frac{5\pi}{12}\right)=0. \tag{1.18}$$

The article's subsequent content is categorized as follows. First, we state the theta function identities of Ramanujan and the modular relations on Ramanujan-Göllnitz-Gordon continued fraction $H(q)$, Ramanujan's cubic continued fraction $G(q)$, and the continued fractions $I(q)$ and $J(q)$, each of order ten and $U(q)$ of order twelve as Lemmas, in Section 2. Also, we document three results in Section 2 that are required to prove the main theorem. We demonstrate the proof of our main results in Section 3, that are stated above as Theorem 1.1.

2. Preliminaries

We document few results that are useful in proving trigonometric identities listed in the Theorem 1.1.

Lemma 2.1 ([5]) *Suppose* $\gamma\in(0,1)$, $z=\pi\dfrac{{}_2\Phi_1\left(\frac{1}{2},\frac{1}{2};1;1-\gamma\right)}{{}_2\Phi_1\left(\frac{1}{2},\frac{1}{2};1;\gamma\right)}$, *and* $q=e^{-z}$, *then*

$${}_2\Phi_1\left(\frac{1}{2},\frac{1}{2};1;\gamma\right)=\varphi^2(q).$$

If $0<\gamma$, $\delta<1$, and the equality

$$n\frac{{}_2\Phi_1\left(\frac{1}{2},\frac{1}{2};1;1-\gamma\right)}{{}_2\Phi_1\left(\frac{1}{2},\frac{1}{2};1;\gamma\right)}=\frac{{}_2\Phi_1\left(\frac{1}{2},\frac{1}{2};1;1-\delta\right)}{{}_2\Phi_1\left(\frac{1}{2},\frac{1}{2};1;\delta\right)}$$

holds, then any identity connecting γ and δ induced by the expression atop is known as a modular equation of n^{th} degree. It is said that δ is of degree n over γ.

Lemma 2.2 ([13]) *For every* $n\in\mathbb{N}$, *for any* $\gamma\in(0,1)$, *there exists* $\delta\in(0,1)$ *such that* δ *is of* n^{th} *degree over* γ.

Lemma 2.3 ([13]) *If* $\gamma\in(0,1)$, *then as* $q\to1^-$, $\gamma\to1^-$. *Consequently, if* δ *is of* n^{th} *degree over* γ, *then as* $q\to1^-$, $\delta\to1^-$.

We now document the theta function identities which are useful to prove our main theorem.

Lemma 2.4 *We have the below-mentioned theta function identities:*

$$\frac{\varphi^2(q)-\varphi^2(q^5)}{4q(q^{10};q^{10})^2_\infty}=(-q,-q^3,-q^7,-q^9;q^{10})_\infty. \tag{2.1}$$

$$\frac{\varphi(q^{1/5})-\varphi(q^5)}{2(q^{10};q^{10})_\infty}=q^{1/5}(-q^3,-q^7;q^{10})_\infty+q^{4/5}(-q,-q^9;q^{10})_\infty. \tag{2.2}$$

$$\frac{f(-q^{1/5})}{f(-q^5)}=\frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty}-q^{1/5}-q^{2/5}\frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}. \tag{2.3}$$

$$\frac{\varphi(q^{1/7})-\varphi(q^7)}{2(q^{14};q^{14})_\infty}=q^{1/7}(-q^5,-q^9;q^{14})_\infty+q^{4/7}(-q^3,-q^{11};q^{14})_\infty+q^{9/7}(-q,-q^{13};q^{14})_\infty. \tag{2.4}$$

$$\frac{(q^4,q^5;q^9)_\infty}{(q^2,q^7;q^9)_\infty}+q\frac{(q,q^8;q^9)_\infty}{(q^4,q^5;q^9)_\infty}=\frac{(q^2,q^7;q^9)_\infty}{(q,q^8;q^9)_\infty}. \tag{2.5}$$

$$\begin{aligned}\frac{\varphi(q^{1/11})-\varphi(q^{11})}{(q^{22};q^{22})_\infty}=&2q^{1/11}f(q^9,q^{13})+2q^{4/11}(-q^7,-q^{15};q^{22})_\infty+2q^{9/11}(-q^5,-q^{17};q^{22})_\infty\\&+2q^{16/11}(-q^3,-q^{19};q^{22})_\infty+2q^{25/11}(-q,-q^{21};q^{22})_\infty.\end{aligned} \tag{2.6}$$

$$\begin{aligned}\frac{f(-q^{1/11})}{f(-q^{11})}=&\frac{(q^4,q^7;q^{11})_\infty}{(q^2,q^9;q^{11})_\infty}-q^{1/11}\frac{(q^2,q^9;q^{11})_\infty}{(q,q^{10};q^{11})_\infty}+q^{5/11}-q^{2/11}\frac{(q^5,q^6;q^{11})_\infty}{(q^3,q^8;q^{11})_\infty}\\&+q^{7/11}\frac{(q^3,q^8;q^{11})_\infty}{(q^4,q^7;q^{11})_\infty}-q^{15/11}\frac{(q,q^{10};q^{11})_\infty}{(q^5,q^6;q^{11})_\infty}.\end{aligned} \tag{2.7}$$

The above identities can be found in [5, p. 262, p. 262, p. 270, p. 303, p. 349, p. 362, p. 363] respectively.

Lemma 2.5 ([11]) *For*

$$H(q):=q^{1/2}\frac{f(-q,-q^7)}{f(-q^3,-q^5)},$$

we have

$$H^2(q)=H(q^2)\frac{1-H(q^2)}{1+H(q^2)}. \tag{2.8}$$

Lemma 2.6 ([10]) *For*

$$G(q):=q^{1/3}\frac{f(-q,-q^5)}{f(-q^3,-q^3)},$$

we have the following:

$$G^2(q)+2G^2(q^2)G(q)-G(q^2)=0. \tag{2.9}$$

and

$$G^3(q)=G(q^3)\frac{1-G(q^3)+G^2(q^3)}{1+2G(q^3)+4G^2(q^3)}. \tag{2.10}$$

Lemma 2.7 ([3]) *For*

$$I(q):=q^{3/4}\frac{f(-q,-q^9)}{f(-q^4,-q^6)}\qquad \textit{and}\qquad J(q):=q^{1/4}\frac{f(-q^2,-q^8)}{f(-q^3,q^7)},$$

we have

$$I^4+\frac{1}{J^4}=\frac{J^2}{I^2}-2+3\frac{I^2}{J^2}. \tag{2.11}$$

Lemma 2.8 ([18]) *For*

$$U(q):=q\frac{f(-q,-q^{11})}{f(-q^5,-q^7)},$$

we have

$$U^2(q)+U(q^2)+2U(q)U(q^2)-U^2(q)U(q^2)+U^2(q^2)=0. \tag{2.12}$$

3. Proof of Main Theorem

For proving our main results, we need the following limits of theta functions. For $n \in \mathbb{N}$, from (1.2), we deduce

$$\lim_{q\to 1^-} \frac{f(-q)}{f(-q^n)} = \lim_{q\to 1^-} \frac{(q;q)_\infty}{(q^n;q^n)_\infty} = \lim_{q\to 1^-} \prod_{\substack{m=1,\\ m\neq nk,\\ k\in\mathbb{N}}}^{\infty} (1-q^m) = 0. \tag{3.1}$$

Also, from [12] we have

$$\lim_{q\to 1^-} \frac{\varphi^2(-q)}{\varphi^2(-q^n)} = 0. \tag{3.2}$$

Proof: [Proof of (1.7)] On switching q by $-q$ in (2.1), one can deduce

$$\frac{\varphi^2(-q)-\varphi^2(-q^5)}{-4q(q^{10};q^{10})^2_\infty} = (q,q^3,q^7,q^9;q^{10})_\infty.$$

On simplification, one can arrive at

$$\frac{\varphi^2(-q)}{\varphi^2(-q^5)} - 1 = -4q\frac{f(-q,-q^9)}{\varphi^2(-q^5)}\frac{f(-q^3,-q^7)}{\varphi^2(-q^5)}.$$

Taking the limit on both the sides as $q\to 1^-$, one can get

$$\lim_{q\to 1^-} \frac{\varphi^2(-q)}{\varphi^2(-q^5)} - 1 = \lim_{q\to 1^-}\left(-4q\frac{f(-q,-q^9)}{\varphi^2(-q^5)}\frac{f(-q^3,-q^7)}{\varphi^2(-q^5)}\right).$$

Upon using (1.6) and (3.2) on right and left hand sides of the above respectively, one can establish the result (1.7). □

Proof: [Proof of (1.8)] On switching q by $-q$ in (2.2), one can deduce

$$\frac{\varphi(-q^{1/5})-\varphi(-q^5)}{2(q^{10};q^{10})_\infty} = -q^{1/5}(q^3,q^7;q^{10})_\infty - q^{4/5}(q,q^9;q^{10})_\infty.$$

On simplification, one can arrive at

$$\frac{\varphi(-q^{1/5})}{\varphi(-q^5)} - 1 = -2q^{1/5}\frac{f(-q^3,-q^7)}{\varphi(-q^5)} + 2q^{4/5}\frac{f(-q,-q^9)}{\varphi(-q^5)}.$$

Taking the limit on both the sides as $q\to 1^-$, upon using (1.6) and (3.2), one can establish the result (1.8). □

Proof: [Proof of (1.9)] Upon letting $q\to 1^-$ in (2.3), using (3.1) and (1.5), one can establish the result. □

Proof: [Proof of (1.10)] On switching q by $-q$ in (2.4), one can deduce

$$\frac{\varphi(-q^{1/7})-\varphi(-q^7)}{2(q^{14};q^{14})_\infty} = -q^{1/7}(q^5,q^9;q^{14})_\infty + q^{4/7}(q^3,q^{11};q^{14})_\infty - q^{9/7}(q,q^{13};q^{14})_\infty.$$

On simplification, one can arrive at

$$\frac{\varphi(-q^{1/7})}{\varphi(-q^7)} - 1 = -2q^{1/7}\frac{f(-q^5,-q^9)}{\varphi(-q^7)} + 2q^{4/7}\frac{f(-q^3,-q^{11})}{\varphi(-q^7)} - 2q^{9/7}\frac{f(-q,-q^{13})}{\varphi(-q^7)}.$$

Taking the limit on both the sides as $q \to 1^-$, upon using (1.6) and (3.2), one can establish the result (1.10). ☐

Proof: [Proof of (1.11)] Upon letting $q \to 1^-$ in (2.5), using (1.5), one can establish the result. ☐

Proof: [Proof of (1.12)] On switching q by $-q$ in (2.6), one can deduce

$$\begin{aligned}\frac{\varphi(-q^{1/11})-\varphi(-q^{11})}{2(q^{22};q^{22})_\infty} =& - q^{1/11}(q^9,q^{13};q^{22})_\infty + q^{4/11}(q^7,q^{15};q^{22})_\infty - q^{9/11}(q^5,q^{17};q^{22})_\infty \\ &+ q^{16/11}(q^3,q^{19};q^{22})_\infty - q^{25/11}(q,q^{21};q^{22})_\infty.\end{aligned}$$

On simplification, one can arrive at

$$\begin{aligned}\frac{\varphi(-q^{1/11})}{\varphi(-q^{11})} - 1 =& - 2q^{1/11}\frac{f(-q^9,-q^{13})}{\varphi(-q^{11})} + 2q^{4/11}\frac{f(-q^7,-q^{15})}{\varphi(-q^{11})} - 2q^{9/11}\frac{f(-q^5,-q^{17})}{\varphi(-q^{11})} \\ &+ 2q^{16/11}\frac{f(-q^3,-q^{19})}{\varphi(-q^{11})} - 2q^{25/11}\frac{f(-q,-q^{21})}{\varphi(-q^{11})}.\end{aligned}$$

Taking the limit on both the sides as $q \to 1^-$, upon using (1.6) and (3.2), one can establish the result (1.12). ☐

Proof: [Proof of (1.13)] Upon letting $q \to 1^-$ in (2.7), using (3.1) and (1.5), one can establish the result. ☐

Proof: [Proof of (1.14)-(1.18)] Upon letting $q \to 1^-$ in (2.8)-(2.12), using (1.5), one can establish the results (1.14)-(1.18) respectively. ☐

4. Conclusion

In conclusion, Ramanujan's theta functions offer a profound connection to trigonometric functions, which is really captivating. Utilizing theta function identities of Ramanujan, we derived some trigonometric identities. We also established few more trigonometric identities using the known modular relations on Ramanujan-Göllnitz-Gordon continued fraction, Ramanujan's cubic continued fraction, and the continued fractions of order ten and twelve, all of which identities can be verified using MAPLE. We hope that many more trigonometric sums could be established using the known theta function identities in a similar fashion. As we continue to explore the depth of Ramanujan's theta functions, we may uncover further applications and connections to many other areas as well.

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