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Non-Coercive Elliptic Problems with Measure Data in Musielak–Orlicz Spaces

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ABSTRACT: In this research, we investigate a class of nonlinear elliptic equations with measure data in Musielak–Orlicz spaces, under non-coercive growth conditions. Using the framework of renormalized solutions, we establish existence results by combining modular estimates and truncation techniques. No Δ_2-condition is assumed on the Musielak function, and the datum is assumed to belong to $L^1(\Omega) + W^{-1}E_{\overline{\varphi}}(\Omega)$. This work extends previous results to operators with nonstandard growth without coercivity.

Key Words: Renormalized solution, Musielak-Orlicz-Sobolev spaces, elliptic equations, lower order term.

Contents

1. Introduction

Nonlinear elliptic equations in Musielak–Orlicz spaces arise naturally in various applied contexts. These spaces are particularly suited to model physical phenomena with nonstandard behavior, such as non-Newtonian fluids whose viscosity depends on external factors like electric or magnetic fields. They are also used in image processing, for example in noise reduction and edge detection, and play an important role in the study of variational problems and partial differential equations involving low regularity data [11,17].

In the present paper, we deal with an existence result for a nonlinear elliptic problems associated to the following equation:

$$(\mathcal{P}) \begin{cases} A(u) - \operatorname{div}(\Phi(u)) + g(x,u,\nabla u) = f - \operatorname{div} F \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N (N \geq 2)$ and $A(u) = -\operatorname{div} a(x,u,\nabla u)$ is a Leray-Lions operator defined on $A : D(A) \subset W_0^1 L_\varphi(\Omega) \longrightarrow W^{-1} L_{\overline{\varphi}}(\Omega)$ where φ and $\overline{\varphi}$ are two complementary Musielak-Orlicz functions. The lower order term Φ is a continuous function on $\mathbb{R}$. The function $g(x,u,\nabla u)$ is a non linear lower order term with natural growth with respect to ∇u, satisfying the sign condition and the source term $f \in L^1(\Omega)$ and $F \in (E_{\overline{\varphi}}(\Omega))^N$.

The notion of renormalized solutions, originally formulated by DiPerna and Lions in [13] for the Boltzmann equation, has been successfully adapted to nonlinear elliptic problems. In [10], Boccardo et al. applied this concept to equations with right-hand sides in the dual space $W^{-1,p'}(\Omega)$, where the nonlinearity depends only on x and u. This approach was later extended by Rakotoson in [21] to cases where the data belong to $L^1(\Omega)$, and subsequently by Dal Maso et al. in [12] to encompass general measure data.

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In the context of Sobolev spaces with variable exponent, Bendahmane and Wittbold [6] addressed the existence and uniqueness of renormalized solutions for the nonlinear problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega)$ and the function $p(\cdot)$ is continuous on $\overline{\Omega}$ with values in $(1, +\infty)$.

In a different approach, Sanchón and Urbano [22] considered quasilinear equations involving general nonlinearities of the form

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

also with $f \in L^1(\Omega)$. They proved the existence and uniqueness of renormalized solutions and provided additional regularity properties.

On Orlicz-Sobolev spaces and variational problems, Benkirane and Bennouna studied in [9] the problem $(\mathcal{P})$ assuming that the nonlinearity g depends solely on x and u, under the additional assumption that the associated N-function satisfies the Δ_2-condition. This result was later generalized in [1] by Aharouch et al. by removing the Δ_2-assumption. When the function g also depends on ∇u, the problem $(\mathcal{P})$ was addressed in [2] by Benkirane et al. without imposing the Δ_2-condition on the N-function.

In the framework of Musielak–Orlicz spaces, the existence of solutions in the case $\Phi \equiv 0$ was first investigated by Oubeid, Benkirane, and Sidi El Vally in [20]. Later, Ait Khellou and Benkirane [3] studied problem $(\mathcal{P})$ in the case where the right-hand side belongs to $L^1(\Omega)$. A large number of papers was devoted to the study of the existence solutions of elliptic and parabolic problems under various assumptions and in different contexts for a review on classical results see [10,14,17,23].

The aim of this paper is to establish the existence of renormalized solutions to problem $(\mathcal{P})$ in Musielak–Orlicz spaces with nonstandard growth and non-coercive operators. Since classical weak formulations fail in the presence of measure data and lack of coercivity, we employ the framework of renormalized solutions combined with modular convergence and truncation techniques. Our results, obtained without assuming the Δ_2-condition, extend and generalize existing theories for elliptic problems with irregular data.

Specific examples of equations to which our result can be applied

$$-\operatorname{div}\left(\frac{\varphi(x, |\nabla u|)\nabla u}{|\nabla u|^2} + |u|^s u\right) + \varphi(x, |\nabla u|) = \mu \text{ in } \Omega,$$
$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u \log^\beta(1 + |\nabla u|) + |u|^s u\right) = \mu \,\text{in}\, \Omega,$$

where $p > 1, s > 0, \beta > 0$ and μ is a given Radon measure on Ω.

The paper is organized as follows. In Section 2, we recall some preliminaries and background material. Section 3 is devoted to several technical lemmas that will be instrumental in proving our main result. In Section 4, we state the basic assumptions, introduce the notion of renormalized solution, and present the main result. Finally, Section 5 is dedicated to the proof of the main theorem.

2. Preliminaries

2.1. Musielak-Orlicz function:

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions:

(a) $\varphi(x, \cdot)$ is an N-function for all $x \in \Omega$ (i.e. convex, strictly increasing, continuous, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$, for all $t > 0$, $\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0$ and $\lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty$),

(b) $\varphi(\cdot, t)$ is a measurable function.

The function φ is called a Musielak–Orlicz function.

For a Musielak-orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its non-negative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

The Musielak-orlicz function φ is said to satisfy the Δ_2-condition if for some $k > 0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x, 2t) \leq k\,\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{2.1}$$

When (2.1) holds only for $t \geq t_0 > 0$, then φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak-orlicz functions, we say that φ dominate γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x \in \Omega$

$$\gamma(x,t) \leq \varphi(x, ct) \text{ for all } t \geq t_0, \quad (\text{ resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant c we have

$$\lim_{t \longrightarrow 0} \left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0, \quad (\text{resp. } \lim_{t \longrightarrow \infty} \left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0).$$

Definition 2.1 *A Musielak function φ is called locally integrable on Ω if*

$$\int_E \varphi(x,t)dx = \int_\Omega \varphi\left(x, t\chi_E(x)\right) \mathrm{d}x < +\infty,$$

for all $t \geq 0$ and all measurable set $E \subset \Omega$ with $\operatorname{mes}(E) < +\infty$.

Remark 2.1 If $\gamma \prec\prec \varphi$ and γ is locally integrable on Ω, then $\forall c > 0$ there exists a nonnegative integrable function h such that

$$\gamma(x,t) \leq \varphi(x, ct) + h(x), \text{ for all } t \geq 0 \text{ and for a.e. } x \in \Omega. \tag{2.2}$$

Definition 2.2 *A Musielak function φ satisfies the log-Hölder continuity condition on Ω if there exists a constant $A > 0$ such that*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{log(\frac{1}{|x-y|})}\right)}$$

for all $t \geq 1$ and for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$.

Lemma 2.1 *[5]. Let Ω be a bounded open of $\mathbb{R}^N (N \geq 2)$ and let φ be a Musielak function satisfying the log-Hölder Continuity, then there exists an N-function M such that*

$$\varphi(x,t) \leq M(t), \textit{ for all } t \geq 1 \textit{ and for all } x \in \Omega.$$

Remark 2.2 The latter Lemma proves that the log-Hölder Continuity condition implies the local integrability.

2.2. Musielak-Orlicz space:

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, dx.$$

The set $K_\varphi(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable} \,/\, \rho_{\varphi,\Omega}(u) < \infty \right\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega)=\left\{u:\Omega\longrightarrow\mathbb{R}\text{ measurable }\Big/\rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big)<\infty,\ \text{for some }\lambda>0\right\}.$$

For a Musielak-Orlicz function φ we put: $\overline{\varphi}(x,s)=\sup_{t\geq 0}\{st-\varphi(x,t)\}$,
$\overline{\varphi}$ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to the variable s.

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega}=\inf\left\{\lambda>0/\int_\Omega\varphi\Big(x,\frac{|u(x)|}{\lambda}\Big)\,dx\leq 1\right\},$$

which is called the Luxemburg norm and the so–called Orlicz norm by:

$$|||u|||_{\varphi,\Omega}=\sup_{\|v\|_{\overline{\varphi}}\leq 1}\int_\Omega|u(x)v(x)|\,dx,$$

where $\overline{\varphi}$ is the Musielak Orlicz function complementary to φ. These two norms are equivalent [19].

We will also use the space $E_\varphi(\Omega)$ defined by

$$E_\varphi(\Omega)=\left\{u:\Omega\longrightarrow\mathbb{R}\text{ measurable }\Big/\rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big)<\infty,\ \text{for all }\lambda>0\right\}.$$

Remark 2.3 [5] The set E_φ is a closed subset of L_φ.

Theorem 2.1 *[5] Let Ω be a bounded open of $\mathbb{R}^N(N\geq 2)$ and let φ be a Musielak function satisfying the log-Hölder Continuity condition. Then $(E_\varphi(\Omega))'$ is isomorphic to $L_{\overline{\varphi}}(\Omega)$.*

We say that sequence of functions $u_n\in L_\varphi(\Omega)$ is modular convergent to $u\in L_\varphi(\Omega)$ if there exists a constant $\lambda>0$ such that

$$\lim_{n\to\infty}\rho_{\varphi,\Omega}\Big(\frac{u_n-u}{\lambda}\Big)=0.$$

For any fixed non-negative integer m we define

$$W^mL_\varphi(\Omega)=\Big\{u\in L_\varphi(\Omega):\forall|\alpha|\leq m,\ D^\alpha u\in L_\varphi(\Omega)\Big\}.$$

and

$$W^mE_\varphi(\Omega)=\Big\{u\in E_\varphi(\Omega):\forall|\alpha|\leq m,\ D^\alpha u\in E_\varphi(\Omega)\Big\}.$$

where $\alpha=(\alpha_1,...,\alpha_n)$ with non-negative integers α_i, $|\alpha|=|\alpha_1|+...+|\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^mL_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.

Let

$$\overline{\rho}_{\varphi,\Omega}(u)=\sum_{|\alpha|\leq m}\rho_{\varphi,\Omega}\Big(D^\alpha u\Big)\text{ and }\|u\|^m_{\varphi,\Omega}=\inf\Big\{\lambda>0:\overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big)\leq 1\Big\}$$

for $u\in W^mL_\varphi(\Omega)$, these functionals are a convex modular and a norm on $W^mL_\varphi(\Omega)$, respectively, and the pair $\Big(W^mL_\varphi(\Omega),\|\|^m_{\varphi,\Omega}\Big)$ is a Banach space if φ satisfies the following condition [19]:

$$\text{there exist a constant } c_0>0 \text{ such that } \inf_{x\in\Omega}\varphi(x,1)\geq c_0. \tag{2.3}$$

The space $W^mL_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m}L_\varphi(\Omega)=\Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi,\Pi E_{\overline{\varphi}})$ closed.

The space $W^m_0L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi,\Pi E_{\overline{\varphi}})$ closure of $\mathcal{D}(\Omega)$ in $W^mL_\varphi(\Omega)$. and the space $W^m_0E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^mL_\varphi(\Omega)$.

Let $W^m_0L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi,\Pi E_{\overline{\varphi}})$ closure of $\mathcal{D}(\Omega)$ in $W^mL_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m}L_{\overline{\varphi}}(\Omega)=\Big\{f\in D'(\Omega);\ f=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^{\alpha}f_{\alpha}\text{ with }f_{\alpha}\in L_{\overline{\varphi}}(\Omega)\Big\}.$$

and

$$W^{-m}E_{\overline{\varphi}}(\Omega)=\Big\{f\in D'(\Omega);\ f=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^{\alpha}f_{\alpha}\text{ with }f_{\alpha}\in E_{\overline{\varphi}}(\Omega)\Big\}.$$

We say that a sequence of functions $u_n\in W^mL_{\varphi}(\Omega)$ is modular convergent to $u\in W^mL_{\varphi}(\Omega)$ if there exists a constant $k>0$ such that

$$\lim_{n\to\infty}\overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n-u}{k}\Big)=0.$$

For φ and its complementary function $\overline{\varphi}$, the following inequality is called the Young's inequality [19]:

$$ts\le\varphi(x,t)+\overline{\varphi}(x,s),\quad\forall t,s\ge0,x\in\Omega. \tag{2.4}$$

This inequality implies that

$$|||u|||_{\varphi,\Omega}\le\rho_{\varphi,\Omega}(u)+1. \tag{2.5}$$

In $L_{\varphi}(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega}\le\rho_{\varphi,\Omega}(u)\text{ if }\|u\|_{\varphi,\Omega}>1. \tag{2.6}$$

$$\|u\|_{\varphi,\Omega}\ge\rho_{\varphi,\Omega}(u)\text{ if }\|u\|_{\varphi,\Omega}\le1. \tag{2.7}$$

For two complementary Musielak Orlicz functions φ and $\overline{\varphi}$, let $u\in L_{\varphi}(\Omega)$ and $v\in L_{\overline{\varphi}}(\Omega)$, then we have the Hölder inequality [19]

$$\Big|\int_{\Omega}u(x)v(x)\,dx\Big|\le\|u\|_{\varphi,\Omega}|||v|||_{\overline{\varphi},\Omega}. \tag{2.8}$$

3. Auxiliary Results

This subsection is devoted to some auxiliary lemmas and key inequalities used later in the prove of our results.

Lemma 3.1 *[5] Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ $(N\ge2)$ and let φ be a Musielak function satisfying the log-Hölder continuity such that*

$$\overline{\varphi}(x,1)\le c_1\ \textit{a.e in }\Omega\textit{ for some }c_1>0. \tag{3.1}$$

Then $\mathcal{D}(\Omega)$ is dense in $L_{\varphi}(\Omega)$ and in $W_0^1L_{\varphi}(\Omega)$ for the modular convergence.

Remark 3.1 Note that if $\displaystyle\lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t}=\infty$, then (3.1) holds.

Example 3.1 Let $p\in\mathcal{P}(\Omega)$ a bounded variable exponent on Ω, such that there exist a constant $A>0$ such that for all points $x,y\in\Omega$ with $|x-y|<\frac{1}{2}$, we have the inequality

$$|p(x)-p(y)|\le\frac{A}{\log\Big(\frac{1}{|x-y|}\Big)}$$

We can verify that the Musielak function defined by $\varphi(x,t)=t^{p(x)}\log(1+t)$, satisfies the conditions of Lemma 3.1.

Consequently, the action of a distribution S in $W^{-1}L_{\overline{\varphi}}(\Omega)$ on an element u of $W_0^1L_{\varphi}(\Omega)$ is well defined. It will be denoted by $<S,u>$.

The following lemma gives the modular Poincaré's inequality in Musielak-Orlicz spaces.

Lemma 3.2 *[5] Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ $(N \geq 2)$ and let φ be a Musielak function satisfying the conditions of lemma 3.1. Then there exist positive constants β, η and λ depending only on Ω and φ such that*

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \beta + \eta \int_\Omega \varphi(x, \lambda|\nabla u(x)|)dx \quad \forall u \in W_0^1L_\varphi(\Omega). \tag{3.2}$$

Corollary 3.1 *[5] (Poincaré Inequality) Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N(N \geq 2)$ and let φ be a Musielak function satisfying the same conditions of Lemma 3.2 Then there exists a constant $C > 0$ such that*

$$\|v\|_\varphi \leq C\|\nabla v\|_\varphi \quad \forall v \in W_0^1L_\varphi(\Omega).$$

Lemma 3.3 *[8] Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak–Orlicz function and let $u \in W_0^1L_\varphi(\Omega)$. Then $F(u) \in W_0^1L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e \text{ in } \{x \in \Omega : u(x) \in D\} \\ 0 & a.e \text{ in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 3.4 *Let $u_n, u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \to u$ for $\sigma(L_\varphi(\Omega), L_{\overline{\varphi}}(\Omega))$.*

Proof. Let $\lambda > 0$ be such that $\int_\Omega \varphi(x, \frac{u_n - u}{\lambda})dx \to 0$. Thus, for a subsequence, $u_n \to u$ a.e. in Ω. Take $v \in L_{\overline{\varphi}}(\Omega)$. Multiplying v by a suitable constant, we can assume $\lambda v \in L_{\overline{\varphi}}(\Omega)$. By young's inequality,

$$|(u_n - u)v| \leq \varphi(x, \frac{u_n - u}{\lambda}) + \overline{\varphi}(x, \lambda v),$$

which implies, by Vitali's theorem, that $\int_\Omega |(u_n - u)v|dx \to 0$.

Definition 3.1 *Let Ω be an open subset of $\mathbb{R}^N$. We say that Ω has the segment property if there exist a locally finite open covering $\{O_i\}$ of the boundary $\partial\Omega$ of Ω and corresponding vectors $\{y_i\}$ such that if $x \in \Omega \cap O_i$ for some i, then $x + ty_i \in \Omega$ for $0 < t < 1$.*

Lemma 3.5 *[7] Suppose that Ω satisfies the segment property and let $u \in W_0^1L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1L_\varphi(\Omega).$$

Furthermore, if $u \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $||u_n||_\infty \leq (N + 1)||u||_\infty$.

Lemma 3.6 *[9] Let Ω be an open bounded subset of $\mathbb{R}^N$ satisfying the segment property. If $u \in (W_0^1L_\varphi(\Omega))^N$ then*

$$\int_\Omega div\, u\, dx = 0.$$

Lemma 3.7 *Let $(f_n), f \in L^1(\Omega)$ such that*
i) $f_n \geq 0$ a.e in Ω,
ii) $f_n \longrightarrow f$ a.e in Ω,
iii) $\int_\Omega f_n(x)dx \longrightarrow \int_\Omega f(x)dx$,
then $f_n \longrightarrow f$ strongly in $L^1(\Omega)$.

Lemma 3.8 *(The Nemytskii Operator) Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and $\overline{\varphi}$ be two Musielak-Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|f(x,s)| \leq c(x) + k_1 \overline{\varphi}_x^{-1} \varphi(x, k_2|s|). \tag{3.3}$$

where k_1 and k_2 are real positives constants and $c(.) \in E_{\overline{\varphi}}(\Omega)$.
Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is continuous from

$$\left(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2})\right)^p = \prod \left\{ u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2} \right\}.$$

into $(L_{\overline{\varphi}}(\Omega))^q$ for the modular convergence.
Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \overline{\varphi}$ then N_f is strongly continuous from $\left(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2})\right)^p$ to $(E_\gamma(\Omega))^q$.

4. Assumptions and Main Result

Throughout the paper, Ω will be a bounded Lipschitz subset of $\mathbb{R}^N$ $N \geq 2$, and let φ and γ two Musielak–Orlicz functions such that φ satisfies the conditions of Lemma 3.2 and $\gamma \prec\prec \varphi$.

Let $A : D(A) \subset W_0^1 L_\varphi(\Omega) \longrightarrow W^{-1} L_{\overline{\varphi}}(\Omega)$ be a mapping given by

$$A(u) = -\text{ div } a(x, u, \nabla u),$$

where $\overline{\varphi}$ is the Musielak–Orlicz function complementary to φ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$:

$$|a(x,s,\xi)| \leq k_1 \left(c(x) + \overline{\varphi}_x^{-1} \gamma(x, k_2|s|) + \overline{\varphi}_x^{-1} \varphi(x, k_3|\xi|) \right), \tag{4.1}$$

$$\left(a(x,s,\xi) - a(x,s,\xi') \right)(\xi - \xi') > 0, \tag{4.2}$$

$$a(x,s,\xi).\xi \geq \alpha \varphi(x, |\xi|), \tag{4.3}$$

where $c(.)$ belongs to $E_{\overline{\varphi}}(\Omega)$, $c(.) \geq 0$ and $\alpha, k_i \in \mathbb{R}_+^*$. for $i = 1, 2, 3$.

Furthermore, let $g(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ be a Caratheodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, satisfying the following conditions

$$|g(x,s,\xi)| \leq b(|s|)(d(x) + \varphi(x, |\xi|)) \tag{4.4}$$

$$g(x,s,\xi)s \geq 0, \tag{4.5}$$

where $b : \mathbb{R} \to \mathbb{R}^+$is a continuous and increasing function while d is a given nonnegative function in $L^1(\Omega)$.

The right–hand side of $(\mathcal{P})$ and $\Phi : \mathbb{R} \to \mathbb{R}^N$, are assumed to satisfy

$$\Phi \in \mathcal{C}^0 \left(\mathbb{R}, \mathbb{R}^N \right), \tag{4.6}$$

$$f \in L^1(\Omega) \text{ and } F \in (E_{\overline{\varphi}}(\Omega))^N. \tag{4.7}$$

Note that no growth hypothesis is assumed on the function Φ, which implies that the term $-\text{div}(\Phi(u))$ may be meaningless, even as a distribution.

Let us define the truncation $T_k : \mathbb{R} \to \mathbb{R}$ at height $k > 0$ by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\frac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

Definition 4.1 *A measurable function $u:\Omega\to\mathbb{R}$ is called renormalized solution of $(\mathcal{P})$ if $T_k(u)\in W_0^1L_\varphi(\Omega), a\left(x,T_k(u),\nabla T_k(u)\right)\in\left(L_{\overline{\varphi}}(\Omega)\right)^N$,*

$$\lim_{m\to+\infty}\int_{\{x\in\Omega: m\leq|u(x)|\leq m+1\}} a(x,u,\nabla u)\nabla u\,dx=0,$$

and

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u)h(u)-\operatorname{div}(\Phi(u)h(u))+h'(u)\Phi(u)\nabla u\\ +g(x,u,\nabla u)h(u)=fh(u)-\operatorname{div}(Fh(u))+h'(u)F\nabla u \text{ in } \mathcal{D}'(\Omega),\\ \text{for every } h\in C_c^1(\mathbb{R}).\end{cases}\tag{4.8}$$

The aim of this paper is to prove the following existence result:

Theorem 4.1 *Suppose that assumptions (4.1)–(4.7) are fulfilled. Then, problem $(\mathcal{P})$ has at least one renormalized solution.*

5. Proof of the Main Result

Step 1: Approximate problem.

For $n\in\mathbb{N}^*$, let f_n be regular functions which strongly converge to f in $L^1(\Omega)$ such that $||f_n||_1\leq c$ for some constant c and Φ_n is a Lipschitz continuous bounded function from $\mathbb{R}$ into $\mathbb{R}^N$ and set $\Phi_n(s)=\Phi(T_n(s))$ and $g_n(x,s,\xi)=\frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$.

Consider the approximate problem:

$$(\mathcal{P}_n)\begin{cases} u_n\in W_0^1L_\varphi(\Omega),\\ -\operatorname{div} a\left(x,u_n,\nabla u_n\right)-\operatorname{div}\Phi_n\left(u_n\right)+g_n\left(x,u_n,\nabla u_n\right)=f_n-\operatorname{div}F \text{ in } \mathcal{D}'(\Omega).\end{cases}$$

For fixed $n>0$, it's obvious to observe that $g_n(x,s,\xi)\xi\geq 0$, $|g_n(x,s,\xi)|\leq|g(x,s,\xi)|$ and $|g_n(x,s,\xi)|\leq n$, Since g_n is bounded for any fixed n, as a consequence, proving of a weak solution $u_n\in W_0^1L_\varphi(\Omega)$ of $(\mathcal{P}_n)$ is an easy task (see e.g. [7, Theorem 8], [15, Proposition 1]).

Step 2 : A priori estimates

Taking u_n as test function in $(\mathcal{P}_n)$, we get

$$\begin{aligned}&\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx+\int_\Omega\Phi_n\left(u_n\right)\cdot\nabla u_n dx\\ &\quad+\int_\Omega g_n\left(x,u_n,\nabla u_n\right)u_n dx=\int_\Omega f_n u_n dx+\int_\Omega F.\nabla u_n dx.\end{aligned}\tag{5.1}$$

The Liptschitz character of Φ_n, Stokes formula together with the boundary condition $u_n=0$ on $\partial\Omega$, make it possible to obtain

$$\int_\Omega\Phi_n(u)\cdot\nabla u_n dx=0.\tag{5.2}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega F\cdot\nabla u_n dx&=\int_\Omega\frac{2}{\alpha}F\cdot\frac{\alpha}{2}\nabla u_n dx\\ &\leq\int_\Omega\overline{\varphi}\left(x,\frac{2}{\alpha}|F|\right)dx+\frac{\alpha}{2}\int_\Omega\varphi\left(x,|\nabla u_n|\right)dx.\end{aligned}\tag{5.3}$$

Since $g_n\left(x,u_n,\nabla u_n\right)u_n\geq 0$, we obtain from (5.1)

$$\left|\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx\right|\leq C_1+\frac{\alpha}{2}\int_\Omega\varphi\left(x,|\nabla T_k\left(u_n\right)|\right)dx.\tag{5.4}$$

Thanks to (4.3), we have

$$\int_\Omega \varphi\left(x, |\nabla T_k(u_n)|\right) dx \le c_1 + c_2 k. \tag{5.5}$$

On the other hand we have

$$\int_\Omega g_n\left(x, u_n, \nabla u_n\right) u_n dx \le C_3. \tag{5.6}$$

Now, choosing $v = (1/\lambda)\left|T_k(u_n)\right|$ in (3.2) we obtain

$$\int_\Omega \varphi\left(x, \frac{1}{\lambda}\left|T_k(u_n)\right|\right) \mathrm{d}x \le \beta + \eta \int_\Omega \varphi\left(x, |\nabla T_k(u_n)|\right) \mathrm{d}x \le c_3 + c_4 k, \tag{5.7}$$

then

$$\begin{aligned} \operatorname{meas}\{|u_n| > k\} &\le \frac{1}{\inf\limits_{x\in\Omega} \varphi\left(x, \frac{k}{\lambda}\right)} \int_{\{|u_n|>k\}} \varphi\left(x, \frac{k}{\lambda}\right) dx \\ &\le \frac{1}{\inf\limits_{x\in\Omega} \varphi\left(x, \frac{k}{\lambda}\right)} \int_\Omega \varphi\left(x, \frac{1}{\lambda}\left|T_k(u_n)\right|\right) dx \\ &\le \frac{c_3 + c_4 k}{\inf\limits_{x\in\Omega} \varphi\left(x, \frac{k}{\lambda}\right)} \qquad \forall n, \forall k > 0, \end{aligned} \tag{5.8}$$

which implies, for any $\nu > 0$,

$$\operatorname{meas}\{|u_n - u_m| > \nu\} \le \operatorname{meas}\{|u_n| > k\} + \operatorname{meas}\{|u_m| > k\} + \operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > \nu\}$$

and so that

$$\operatorname{meas}\{|u_n - u_m| > \nu\} \le \frac{2\left(c_3 + c_4 k\right)}{\inf\limits_{x\in\Omega} \varphi\left(x, \frac{k}{\lambda}\right)} + \operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > \nu\}. \tag{5.9}$$

From (5.5), we deduce that $T_k(u_n)$ is bounded in $W_0^1 L_\varphi(\Omega)$ and we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon > 0$, by using (5.9) and the fact that $\frac{2(c_3+c_4k)}{\inf_{x\in\Omega}\varphi(x,\frac{k}{\lambda})} \to 0$ as $k \to +\infty$ there exists $k(\varepsilon) > 0$ such that

$$\operatorname{meas}\{|u_n - u_m| > \nu\} \le \varepsilon, \qquad \text{for all } n, m \ge n_0(k(\varepsilon), \nu).$$

This proves that (u_n) is a Cauchy sequence in measure in Ω, thus, u_n converges almost everywhere to some measurable function u. Finally, for all $k > 0$, we have for a subsequence

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma\left(\Pi L_\varphi, \Pi E_{\overline{\varphi}}\right) \\ T_k(u_n) \to T_k(u) & \text{strongly in } E_\varphi(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{5.10}$$

Step 3: Boundedness of $\left(a(x, T_k(u_n), \nabla T_k(u_n))\right)_n$

Let $\vartheta \in \left(E_\varphi(\Omega)\right)^N$ such that $\|\vartheta\|_{\varphi,\Omega} = 1$. Thanks to (4.2), we can write,

$$\int_\Omega \left[a\left(x, T_k(u_n), \nabla T_k(u_n)\right) - a\left(x, T_k(u_n), \frac{\vartheta}{k_3}\right)\right]\left[\nabla T_k(u_n) - \frac{\vartheta}{k_3}\right] dx \ge 0,$$

hence

$$\begin{aligned}
&\int_{\Omega} \frac{1}{k_3} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \vartheta \mathrm{d} x \\
&\quad \leq \int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) \mathrm{d} x \\
&-\int_{\Omega} a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\left(\nabla T_k\left(u_n\right)-\frac{\vartheta}{k_3}\right) \mathrm{d} x \\
&\leq k C_1+C_2-\int_{\Omega} a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right) \nabla T_k\left(u_n\right) \mathrm{d} x \\
&+\frac{1}{k_3} \int_{\Omega} a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right) \vartheta \mathrm{d} x .
\end{aligned}$$

By using Young's inequality in the last two terms of the last side and (5.5) we get

$$\begin{aligned}
&\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \vartheta \mathrm{d} x \\
&\leq\left(k C_1+C_2\right) k_3+3 k_1\left(1+k_3\right) \int_{\Omega} \overline{\varphi}\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3 k_1}\right) \mathrm{d} x \\
&+3 k_1 k_3 \int_{\Omega} \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right) \mathrm{d} x+3 k_1 \int_{\Omega} \varphi(x,|\vartheta|) \mathrm{d} x \\
&\leq\left(k C_1+C_2\right) k_3+3 k_1 k_3\left(k C_1+C_2\right)+3 k_1 \\
&+3 k_1\left(1+k_3\right) \int_{\Omega} \overline{\varphi}\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3 k_1}\right) \mathrm{d} x .
\end{aligned}$$

From (4.1) and the convexity of $\overline{\varphi}$, it follows that

$$\overline{\varphi}\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3 k_1}\right) \leq \frac{1}{3}\left(\overline{\varphi}(x, d(x))+\gamma\left(x, k_2\left|T_k\left(u_n\right)\right|\right)+\varphi(x,|\vartheta|)\right) .$$

By Remark 2.1, there exists a function $h \in L^1(\Omega)$ satisfying $\gamma\left(x, k_2\left|T_k\left(u_n\right)\right|\right) \leq \gamma\left(x, k_2 k\right) \leq \varphi(x, 1)+h(x)$. Integrating over Ω then yields

$$\begin{aligned}
&\int_{\Omega} \overline{\varphi}\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{v}{k_3}\right)\right|}{3 k_1}\right) \mathrm{d} x \\
&\leq \frac{1}{3}\left(\int_{\Omega} \overline{\varphi}(x, c(x)) \mathrm{d} x+\int_{\Omega} h(x) \mathrm{d} x\right. \\
&\left.+\int_{\Omega} \varphi(x, 1) d x+\int_{\Omega} \varphi(x,|\vartheta|) \mathrm{d} x\right) \leq c_k^{\prime},
\end{aligned}$$

where $c_k^{\prime}$ is a constant depending on k. Thus,

$$\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \vartheta d x \leq c_k^{\prime}, \quad \forall \vartheta \in\left(E_{\varphi}(\Omega)\right)^N \quad \text { with }\|\vartheta\|_{\varphi, \Omega}=1,$$

and thus $\left\|a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\right\|_{\overline{\varphi}, \Omega} \leq c_k^{\prime}$, which implies that,

$$\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\right)_n \quad \text { is bounded in } L_{\overline{\varphi}}(\Omega)^N . \tag{5.11}$$

Step 4: Renormalization identity for the approximate solutions

By testing the approximate problem $(\mathcal{P}_n)$ with the function $\theta_m(r) = T_{m+1}(r) - T_m(r)$ for $m \geq 1$, we obtain

$$\begin{aligned} &\int_\Omega a\left(x, u_n, \nabla u_n\right) \nabla \theta_m\left(u_n\right) dx + \int_\Omega \Phi_n\left(u_n\right) \nabla \theta_m\left(u_n\right) dx \\ &\quad + \int_\Omega g_n\left(x, u_n, \nabla u_n\right) \theta_m\left(u_n\right) dx = \int_\Omega f_n \theta_m\left(u_n\right) dx + \int_\Omega F . \nabla \theta_m\left(u_n\right) dx. \end{aligned} \tag{5.12}$$

Let us consider the functions

$$\begin{aligned} \phi(t) &= \Phi_n(t)\, \chi_{\{s\in\mathbb{R}: m\leq |s| \leq m+1\}}(t), \\ \tilde{\phi}(t) &= \int_0^t \phi(\tau)\, d\tau. \end{aligned}$$

By Lemma 3.3, it follows that $\tilde{\phi}(u_n) \in \left(W_0^1 L_\varphi(\Omega)\right)^N$. Then, applying Lemma 3.6, we obtain

$$\begin{aligned} \int_\Omega \Phi_n(u_n)\, \nabla \theta_m(u_n)\, dx &= \int_\Omega \Phi_n(u_n)\, \chi_{\{s\in\mathbb{R}: m\leq |s| \leq m+1\}}(u_n)\, \nabla u_n\, dx \\ &= \int_\Omega \phi(u_n)\, \nabla u_n\, dx = \int_\Omega \operatorname{div}(\tilde{\phi}(u_n))\, dx = 0. \end{aligned}$$

Using the sign condition (4.5) we have $g_n\left(x, u_n, \nabla u_n\right) \theta_m\left(u_n\right) \geq 0$ a.e. in Ω, and knowing that $\nabla \theta_m\left(u_n\right) = \nabla u_n \chi_{\{m\leq|u_n|\leq m+1\}}$ a.e. in Ω, we get

$$\int_{\{m\leq|u_n|\leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f_n, \theta_m(u_n)\rangle + \int_{\{m\leq|u_n|\leq m+1\}} F \nabla u_n dx.$$

By Holder's inequality and (5.5) we have

$$\int_{\{m\leq|u_n|\leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f_n, \theta_m(u_n)\rangle + C_4 \int_{\{m\leq|u_n|\leq m+1\}} \overline{\varphi}(x, |F|) dx.$$

It is straightforward to verify that

$$\left\|\nabla \theta_m(u_n)\right\|_{\varphi,\Omega} \leq \left\|\nabla u_n\right\|_{\varphi,\Omega}.$$

Then, by applying (5.5) and (5.10), we deduce that

$$\theta_m(u_n) \rightharpoonup \theta_m(u) \quad \text{weakly in } W_0^1 L_\varphi(\Omega) \text{ with respect to } \sigma\left(\Pi L_\varphi(\Omega), \Pi E_\varphi(\Omega)\right).$$

As a consequence, we obtain the estimate

$$\lim_{n\to\infty} \int_{\{m\leq|u_n|\leq m+1\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n\, dx \leq \langle f, \theta_m(u)\rangle .$$

Moreover, since $\theta_m(u) \rightharpoonup 0$ weakly in $W_0^1 L_\varphi(\Omega)$ with respect to $\sigma\left(\Pi L_\varphi(\Omega), \Pi E_\varphi(\Omega)\right)$, it follows that

$$\lim_{m\to\infty} \lim_{n\to\infty} \int_{\{m\leq|u_n|\leq m+1\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n\, dx \leq \lim_{m\to\infty} \langle f, \theta_m(u)\rangle = 0.$$

Finally, by invoking (4.3), we conclude that

$$\lim_{m\to\infty} \lim_{n\to\infty} \int_{\{m\leq|u_n|\leq m+1\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n\, dx = 0. \tag{5.13}$$

Step 5: Almost everywhere convergence of the gradients

In this subsubsection we pose $\phi(s)=se^{\lambda s^2}$ with $\lambda=\left(\frac{b(k)}{2\alpha}\right)^2$. One can easily verify that for all $s\in\mathbb{R}$

$$\phi'(s)-\frac{b(k)}{\alpha}|\phi(s)|\geq\frac{1}{2}. \tag{5.14}$$

For $m\geq k$, we define the function $\rho_m(s)$ by

$$\rho_m(s)=\begin{cases}1 & \text{if} \quad |s|\leq m\\ m+1-|s| & \text{if} \quad m\leq|s|\leq m+1\\ 0 & \text{if} \quad |s|\geq m+1.\end{cases}$$

Let $\{v_j\}_j\subset D(\Omega)$ such that $v_j\to u$ in $W_0^1L_\varphi(\Omega)$ for the modular convergence and a.e. in Ω. And let us define the following functions

$$\theta_n^j=T_k(u_n)-T_k(v_j),\ \theta^j=T_k(u)-T_k(v_j) \text{ and } z_{n,m}^j=\phi(\theta_n^j)\rho_m(u_n).$$

Testing the problem $(\mathcal{P}_n)$ with the function $z_{n,m}^j$, we get

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla z_{n,m}^j dx+\int_\Omega \Phi_n(u_n)\nabla\phi\big(T_k(u_n)-T_k(v_j)\big)\rho_m(u_n)dx\\&+\int_{\{m\leq|u_n|\leq m+1\}}\Phi_n(u_n)\nabla u_n\rho_m'(u_n)\phi\big(T_k(u_n)-T_k(v_j)\big)dx\\&+\int_\Omega g_n(x,u_n,\nabla u_n)z_{n,m}^j dx=\int_\Omega f_n z_{n,m}^j dx+\int_\Omega F\nabla z_{n,m}^j dx.\end{aligned} \tag{5.15}$$

Denote by $\epsilon_i(n,j)$, $i=0,1,2,...$, various sequences of real numbers which tend to 0 when n and $\longrightarrow\infty$, i.e.

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon_i(n,j)=0.$$

Thanks to (5.5) and (5.10), we have $z_{n,m}^j\to\phi\left(\theta^j\right)\rho_m(u)$ weakly in $W_0^1L_\varphi(\Omega)$ as $n\to\infty$ for $\sigma\left(\Pi L_\varphi,\Pi E_{\overline{\varphi}}\right)$, then

$$\int_\Omega f_n z_{n,m}^j dx\to\int_\Omega f\phi\left(\theta^j\right)\rho_m(u)dx \text{ as } n\to\infty,$$

using the modular convergence of v_j, we get $\theta^j\to 0$ as $j\to\infty$, so that

$$\int_\Omega f_n z_{n,m}^j dx=\varepsilon_0(n,j).$$

Also, we have

$$\lim_{n\to+\infty}\int_\Omega F.\nabla z_{n,m}^j dx=\int_\Omega F.\nabla\theta^j\phi'\left(\theta^j\right)\rho_m(u)dx+\int_\Omega F.\nabla u\phi\left(\theta^j\right)\rho_m'(u)dx,$$

so that, by Lebesgue's theorem one has

$$\lim_{j\to+\infty}\int_\Omega F.\nabla u\phi\left(\theta^j\right)\rho_m'(u)dx=0.$$

Assume that there exists $\lambda>0$ such that $\varphi\left(x,\frac{|\nabla v_j-\nabla u|}{\lambda}\right)$ converges strongly to zero in $L^1(\Omega)$ as $j\to+\infty$, and that $\varphi\left(x,\frac{|\nabla u|}{\lambda}\right)$ belongs to $L^1(\Omega)$. Under these conditions, the convexity of the Musielak function φ allows us to conclude that

$$\begin{aligned}&\varphi\left(x,\frac{\left|\nabla T_k\left(v_j\right)\phi'\left(\theta^j\right)\rho_m(u)-\nabla T_k(u)\rho_m(u)\right|}{4\lambda\phi'(2k)}\right)\\&\leq\frac{1}{4}\varphi\left(x,\frac{|\nabla v_j-\nabla u|}{\lambda}\right)+\frac{1}{4}\left(1+\frac{1}{\phi'(2k)}\right)\varphi\left(x,\frac{|\nabla u|}{\lambda}\right),\end{aligned}$$

then, using modular convergence of $\{\nabla v_j\}$ in $L_\varphi(\Omega)^N$ and Vitali's theorem, yields

$$\nabla T_k\left(v_j\right) \phi'\left(\theta^j\right) \rho_m(u) \to \nabla T_k(u) \rho_m(u) \quad \text{in} \quad \left(L_\varphi(\Omega)\right)^N, \text{ as } j \to +\infty,$$

for the modular convergence, and then

$$\lim_{j\to+\infty} \int_\Omega F.\nabla T_k(u) \phi'\left(\theta^j\right) \rho_m(u) dx = \int_\Omega F.\nabla T_k(u) \rho_m(u) dx,$$

we have proved that

$$\int_\Omega F.\nabla z^j_{n,m} dx = \epsilon_1(n,j).$$

It is straightforward to observe that, due to the modular convergence of the sequence $\{v_j\}_j$, one obtains

$$\lim_{j\to+\infty} \lim_{n\to+\infty} \int_{\{m\le|u_n|\le m+1\}} \Phi_n(u_n) \nabla u_n \rho'_m(u_n) \phi\big(T_k(u_n) - T_k(v_j)\big) dx = 0.$$

As for the third term on the left-hand side of (5.15), we can express it as

$$\begin{aligned}&\int_\Omega \Phi_n(u_n) \nabla \phi\big(T_k(u_n) - T_k(v_j)\big) \rho_m(u_n) dx\\ &= \int_\Omega \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta^j_n) \rho_m(u_n) dx - \int_\Omega \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta^j_n) \rho_m(u_n) dx.\end{aligned}$$

Firstly, we have

$$\lim_{j\to+\infty} \lim_{n\to+\infty} \int_\Omega \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta^j_n) \rho_m(u_n) dx = 0.$$

According to (5.10), it holds that

$$\Phi_n(u_n)\, \phi'\left(\theta^j_n\right) \rho_m(u_n) \to \Phi(u)\, \phi'\left(\theta^j\right) \rho_m(u),$$

almost every where in Ω as $n \to +\infty$. In addition, it can be shown that

$$\|\Phi_n(u_n) \phi'(\theta^j_n) \rho_m(u_n)\|_{\overline{\varphi}} \le \overline{\varphi}(x, c_m \phi'(2k))|\Omega| + 1,$$

where $c_m = \max_{|t|\le m+1} \Phi(t)$. Applying [[18], Theorem 14.6] we get

$$\lim_{n\to+\infty} \int_\Omega \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta^j_n) \rho_m(u_n) dx = \int_\Omega \Phi(u) \nabla T_k(v_j) \phi'(\theta^j) \rho_m(u) dx.$$

Using the modular convergence of the sequence $\{v_j\}_j$, it follows that

$$\lim_{j\to+\infty} \lim_{n\to+\infty} \int_\Omega \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta^j_n) \rho_m(u_n) dx = \int_\Omega \Phi(u) \nabla T_k(u) \rho_m(u) dx.$$

Then, thanks to Lemma 3.6 we obtain

$$\int_\Omega \Phi(u) \nabla T_k(u) \rho_m(u) dx = 0.$$

Therefore, we write

$$\int_\Omega \Phi_n(u_n) \nabla \phi\big(T_k(u_n) - T_k(v_j)\big) \rho_m(u_n) dx = \epsilon_2(n,j).$$

Since $g_n(x,u_n,\nabla u_n)\, z^j_{n,m} \ge 0$ on the set $\{|u_n| > k\}$, and $\rho_m(u_n) = 1$ on $\{|u_n| \le k\}$, identity (5.15) yields

$$\int_\Omega a(x,u_n,\nabla u_n) \nabla z^j_{n,m}\, dx + \int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\, \phi(\theta^j_n)\, dx \le \epsilon_3(n,j). \tag{5.16}$$

We now proceed to estimate the first term on the left-hand side of (5.15) by rewriting it as

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla z^j_{n,m}dx &= \int_\Omega a(x,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla T_k(v_j))\phi'(\theta^j_n)\rho_m(u_n)dx\\ &+\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\phi(\theta^j_n)\rho'_m(u_n)dx\\ &=\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(v_j))\phi'(\theta^j_n)dx\\ &-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\phi'(\theta^j_n)\rho_m(u_n)dx\\ &+\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\phi(\theta^j_n)\rho'_m(u_n)dx,\end{aligned}$$

and then

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla z^j_{n,m}dx\\ &=\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j)\Big)\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j\Big)\phi'(\theta^j_n)dx\\ &+\int_\Omega a\big(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j\big)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j\big)\phi'(\theta^j_n)dx\\ &-\int_{\Omega\setminus\Omega^s_j} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\phi'(\theta^j_n)dx\\ &-\int_{|[u_n]>k)} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\phi'(\theta^j_n)\rho_m(u_n)dx\\ &+\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\phi(\theta^j_n)\rho'_m(u_n)dx,\end{aligned}\tag{5.17}$$

where by χ^s_j, $s>0$, we denote the characteristic function of the subset
$\Omega^s_j=\{x\in\Omega:|\nabla T_k(v_j)|\le s\}$.

By fixing m and s, we will pass to the limit in n and in j in the second, third, fourth and fifth term on the right hand side of (5.17). For the second term, we have

$$\begin{aligned}&\int_\Omega a\big(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j\big)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j\big)\phi'(\theta^j_n)dx\\ &\to\int_\Omega a\big(x,T_k(u),\nabla T_k(v_j)\chi^s_j\big)\big(\nabla T_k(u)-\nabla T_k(v_j)\chi^s_j\big)\phi'(\theta^j)dx,\end{aligned}$$

as $n\to+\infty$. According to Lemma 3.8, we have

$$a\big(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j\big)\,\phi'\big(\theta^j_n\big)\to a\big(x,T_k(u),\nabla T_k(v_j)\chi^s_j\big)\,\phi'\big(\theta^j\big)$$

strongly in $(E_{\overline{\varphi}}(\Omega))^N$ as $n\to\infty$. Moreover, from (5.5),

$$\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$$

weakly in $(L_\varphi(\Omega))^N$. Let us denote by χ^s the characteristic function of the set $\Omega^s=\{x\in\Omega:|\nabla T_k(u)|\le s\}$.

Since $\nabla T_k(v_j)\chi^s_j\to\nabla T_k(u)\chi^s$ strongly in $(E_\varphi(\Omega))^N$ as $j\to+\infty$, it follows that

$$\int_\Omega a\big(x,T_k(u),\nabla T_k(v_j)\chi^s_j\big)\cdot\big(\nabla T_k(u)-\nabla T_k(v_j)\chi^s_j\big)\,\phi'\big(\theta^j\big)\,dx\to 0,$$

as $j \to \infty$. Consequently, we conclude that

$$\int_\Omega a\big(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s\big) \cdot \big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)\, \phi'\big(\theta_n^j\big)\, dx = \epsilon_4(n,j). \tag{5.18}$$

We now turn to the estimation of the third term in (5.17). From (4.3), it is clear that $a(x,s,0)=0$ for almost every $x \in \Omega$ and all $s \in \mathbb{R}$. Consequently, by (5.11), the sequence $\big(a(x, T_k(u_n), \nabla T_k(u_n))\big)_n$ is bounded in $(L_{\overline{\varphi}}(\Omega))^N$ for every $k \geq 0$.

Thus, up to a subsequence (still indexed by n), there exists a function $l_k \in (L_{\overline{\varphi}}(\Omega))^N$ such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup l_k \quad \text{weakly in } (L_{\overline{\varphi}}(\Omega))^N \text{ with respect to } \sigma(\Pi L_{\overline{\varphi}}, \Pi E_{\varphi}). \tag{5.19}$$

Moreover, since $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_j^s} \in (E_{\overline{\varphi}}(\Omega))^N$, we deduce that

$$\int_{\Omega\setminus\Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(v_j)\, \phi'\big(\theta_n^j\big)\, dx \to \int_{\Omega\setminus\Omega_j^s} l_k \cdot \nabla T_k(v_j)\, \phi'\big(\theta^j\big)\, dx,$$

as $n \to +\infty$. The modular convergence of the sequence $\{v_j\}$ then implies that

$$-\int_{\Omega\setminus\Omega_j^s} l_k \nabla T_k(v_j) \phi'(\theta^j) dx \to -\int_{\Omega\setminus\Omega^s} l_k \nabla T_k(u) dx,$$

as $j \to +\infty$. This, proves

$$-\int_{\Omega\setminus\Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) dx = -\int_{\Omega\setminus\Omega^s} l_k \nabla T_k(u) dx + \epsilon_5(n,j). \tag{5.20}$$

For the fourth term, we remark that $\rho_m(u_n) = 0$ on the subset $\{|u_n| \geq m+1\}$, then we obtain

$$\begin{aligned}
&-\int_{\{|u_n|>k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j)) \phi'(\theta_n^j) \rho_m(u_n) dx \\
&= -\int_{\{|u_n|>k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \rho_m(u_n) dx.
\end{aligned}$$

Since

$$\begin{aligned}
&-\int_{\{|u_n|>k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \rho_m(u_n) dx \\
&= -\int_{\{|u|>k\}} l_{m+1} \nabla T_k(u) \rho_m(u) dx + \epsilon_5(n,j),
\end{aligned}$$

observing that $\nabla T_k(u) = 0$ on the subset $\{|u| > k\}$, one has

$$-\int_{\{|u_n|>k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \rho_m(u_n) dx = \epsilon_6(n,j). \tag{5.21}$$

For the last term of (5.17) we obtain

$$\begin{aligned}
&[\Big| \int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \rho_m'(u_n) dx \Big| \\
&= \Big| \int_{\{m \leq |u_p| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \rho_m'(u_n) dx \Big| \\
&\leq \phi(2k) \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx.
\end{aligned}$$

To estimate the last term of the previous inequality, we use $(T_1(u_n - T_m(u_n)) \in W_0^1 L_\varphi(\Omega))$ as test function in $(\mathcal{P}_n)$, to get

$$\begin{aligned}
&\int_{\{m\leq |u_n|\leq m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx + \int_{\{m\leq |u_n|\leq m+1\}} \Phi_n(u_n)\nabla u_n dx \\
&+ \int_{\{|u_n|\geq m\}} g_n(x,u_n,\nabla u_n) T_1(u_n - T_m(u_n))dx = \langle f_n, T_1(u_n - T_m(u_n))\rangle \\
&+ \int_{\{m\leq |u_n|\leq m+1\}} F\nabla u_n dx.
\end{aligned}$$

Then, applying Lemma 3.6, we obtain

$$\int_{\{m\leq |u_n|\leq m+1\}} \Phi_n(u_n)\nabla u_n dx = 0.$$

Observing that $g_n(x,u_n,\nabla u_n)\,T_1(u_n - T_m(u_n)) \geq 0$ on the set $\{|u_n| \geq m\}$, and invoking Young's inequality, we derive the estimate

$$\begin{aligned}
\int_{\{m\leq |u_n|\leq m+1\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\, dx &\leq \langle f_n, T_1(u_n - T_m(u_n))\rangle \\
&\quad + \int_{\{m\leq |u_n|\leq m+1\}} \overline{\varphi}(x,|F|)\, dx.
\end{aligned}$$

Consequently, we conclude that

$$\begin{aligned}
&\Big|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \phi(\theta_n^j)\rho_m'(u_n)dx\Big| \\
&\quad \leq 2\phi(2k)\Big(\int_{\{m\leq |u_n|\}} |f_n|dx + \int_{\{m\leq |u_n|\leq m+1\}} \overline{\varphi}(x,|F|)dx\Big).
\end{aligned} \tag{5.22}$$

Combining the results from (5.18), (5.20), (5.21), and (5.22), we arrive at

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla z_{n,m}^j dx \\
&\quad \geq \int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big) \\
&\quad \times (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)\phi'(\theta_n^j)dx \\
&\quad - \alpha\phi(2k)\Big(\int_{\{m\leq |u_n|\}} |f_n|dx + \int_{\{m\leq |u_n|\leq m+1\}} \overline{\varphi}(x,|F|)dx\Big) \\
&\quad - \int_{\Omega\setminus\Omega^s} l_k\cdot\nabla T_k(u)dx + \epsilon_7(n,j).
\end{aligned} \tag{5.23}$$

We now focus on the second term on the left-hand side of (5.16). It holds that

$$\begin{aligned}
&\Big|\int_{\{|u_p|\leq k\}} g_n(x,u_n,\nabla u_n)\phi(\theta_n^j)dx\Big| \\
&= \Big|\int_{\{|u_n|\leq k\}} g_n(x,T_k(u_n),\nabla T_k(u_n))\phi(\theta_n^j)dx\Big| \\
&\leq b(k)\int_\Omega M(|\nabla T_k(u_n)|)|\phi(\theta_n^j)|dx + b(k)\int_\Omega d(x)|\phi(\theta_n^j)|dx \\
&\leq \frac{b(k)}{\alpha}\int_\Omega a_n(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\phi(\theta_n^j)|dx + \epsilon_8(n,j).
\end{aligned}$$

Then

$$\begin{aligned}
&\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\phi(\theta_n^j)dx\Big|\\
&\le \frac{b(k)}{\alpha}\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big)\\
&\times\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\Big)|\phi(\theta_n^j)|dx\\
&+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\big)|\phi(\theta_n^j)|dx\\
&+\frac{b(k)}{\alpha}\int_\Omega a_n(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s|\phi(\theta_n^j)|dx+\epsilon_9(n,j).
\end{aligned}\tag{5.24}$$

We proceed as above to get

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\big)|\phi(\theta_n^j)|dx=\epsilon_9(n,j)$$

and

$$\frac{b(k)}{\alpha}\int_\Omega a_n(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s|\phi(\theta_n^j)|dx=\epsilon_{10}(n,j).$$

Hence, we have

$$\begin{aligned}
&\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\phi(\theta_n^j)dx\Big|\\
&\le \frac{b(k)}{\alpha}\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big)\\
&\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\Big)|\phi(\theta_n^j)|dx+\epsilon_{11}(n,j).
\end{aligned}\tag{5.25}$$

From (5.16), (5.23) and (5.25), we get

$$\begin{aligned}
&\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\big)\\
&\Big(\phi'(\theta_n^j)-\tfrac{b(k)}{\alpha}|\phi(\theta_n^j)|\Big)dx\\
&\le\int_{\Omega\setminus\Omega^s} l_k\nabla T_k(u)dx+\alpha\phi(2k)\Big(\int_{\{m\le|u_n|\}}|f_n|dx+\int_{\{m\le|u_n|\le m+1\}}\overline{\varphi}(x,|F|)dx\Big)\\
&+\epsilon_{12}(n,j).
\end{aligned}$$

By (5.14), we have

$$\begin{aligned}
&\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\big)dx\\
&\le 2\int_{\Omega\setminus\Omega^s} l_k\nabla T_k(u)dx+4\alpha\phi(2k)\Big(\int_{\{m\le|u_n|\}}|f_n|dx+\int_{\{m\le|u_n|\le m+1\}}\overline{\varphi}(x,|F|)dx\Big)\\
&+\epsilon_{12}(n,j).
\end{aligned}\tag{5.26}$$

On the other hand, we can write

$$\begin{aligned}
&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s))(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\,dx\\
&= \int_\Omega \left(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\right)\left(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right)dx\\
&+ \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\left(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\right)dx\\
&- \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi^s)(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\,dx\\
&+ \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\left(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right)dx.
\end{aligned}$$

We will first let $n \to \infty$, followed by $j \to \infty$, in the last three terms on the right-hand side of the above identity. Proceeding analogously to the arguments in (5.17) and (5.24), we deduce that

$$\begin{aligned}
&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\left(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\right)dx = \epsilon_{13}(n,j),\\
&\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi^s)(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\,dx = \epsilon_{14}(n,j),\\
&\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\left(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right)dx = \epsilon_{15}(n,j).
\end{aligned} \tag{5.27}$$

So that

$$\begin{aligned}
&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s))(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\,dx\\
&= \int_\Omega \left(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\right)\left(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right)dx\\
&+ \epsilon_{16}(n,j).
\end{aligned} \tag{5.28}$$

Let $r \le s$. Making use of (4.2), (5.26), and (5.28), we can express

$$\begin{aligned}
0 &\le \int_{\Omega^r} (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))\,dx\\
&\le \int_{\Omega^s} (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))\,dx\\
&= \int_{\Omega^s} (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s))(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\,dx\\
&\le \int_{\Omega} (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s))(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\,dx\\
&= \int_\Omega \left(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\right)\left(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right)dx\\
&+ \epsilon_{15}(n,j)\\
&\le 2\int_{\Omega\setminus\Omega^s} l_k \nabla T_k(u)\,dx + 2\alpha\phi(2k)\left(\int_{\{m\le|u_n|\}} |f_n|\,dx + \int_{\{m\le|u_n|\le m+1\}} \overline{\varphi}(x,|F|)\,dx\right)\\
&+ \epsilon_{17}(n,j).
\end{aligned}$$

By passing to the limit in n and then in j one has,

$$\begin{aligned}
0 &\le \limsup_{n\to+\infty} \int_{\Omega^r} \left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right) - a\left(x,T_k\left(u_n\right),\nabla T_k(u)\right)\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\right)dx\\
&\le 2\int_{\Omega\setminus\Omega^s} l_k \nabla T_k(u)dx + 4\alpha\phi(2k)\left(\int_{\{m\le|u_n|\}} |f|dx + \int_{\{m\le|u_n|\le m+1\}} \overline{\varphi}(x,|F|)dx\right).
\end{aligned}$$

Letting $s \to +\infty$ and then $m \to +\infty$, taking into account that $l_k \nabla T_k(u) \in L^1(\Omega), f \in L^1(\Omega), |F| \in (E_{\overline{\varphi}}(\Omega))^N, |\Omega \backslash \Omega^s| \to 0$, and $|\{m \leq |u| \leq m+1\}| \to 0$, one has

$$\int_{\Omega^r} \left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(x, T_k\left(u_n\right), \nabla T_k(u)\right)\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\right) dx$$

tends to 0 as $n \to +\infty$. As in [16], we deduce that there exists a subsequence of $\{u_n\}$ still indexed by n such that

$$\nabla u_n \to \nabla u \text{ a. e. in } \Omega. \tag{5.29}$$

Thus, by taking account that (5.11) and (5.10) we can apply [[18], Theorem 14.6] to obtain

$$a(x, u, \nabla u) \in (L_{\overline{\varphi}}(\Omega))^N$$

and

$$a\left(x, u_n, \nabla u_n\right)) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_{\overline{\varphi}}(\Omega))^N \text{ for } \sigma\left(\Pi L_{\overline{\varphi}}, \Pi E_{\varphi}\right). \tag{5.30}$$

Step 6: Modular convergence of the truncations.

From inequality (5.26), we obtain

$$\begin{aligned}
&\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) dx \\
&\leq \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(v_j\right) \chi_j^s dx \\
&\quad + \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k\left(v_j\right) \chi_j^s\right) dx \\
&\quad + 2\alpha\phi(2k)\left(\int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{\varphi}(x, |F|) dx\right) \\
&\quad + 2\int_{\Omega \backslash \Omega^s} a\left(x, T_k(u), \nabla T_k(u)\right) \nabla T_k(u) dx + \epsilon_{12}(n, j),
\end{aligned}$$

which implies, by using (5.27), that

$$\begin{aligned}
&\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) dx \\
&\leq \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(v_j\right) \chi_j^s dx \\
&\quad + 2\alpha\phi(2k)\left(\int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{\varphi}(x, |F|) dx\right) \\
&\quad + 2\int_{\Omega \backslash \Omega^s} a\left(x, T_k(u), \nabla T_k(u)\right) \nabla T_k(u) dx + \epsilon_{18}(n, j).
\end{aligned}$$

The passage to the limitto the limit in n on both sides of this inequality and using (5.30) implies that

$$\begin{aligned}
&\limsup_{n \to +\infty} \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) dx \\
&\leq \int_\Omega a\left(x, T_k(u), \nabla T_k(u)\right) \nabla T_k\left(v_j\right) \chi_j^s dx \\
&\quad + 2\alpha\phi(2k)\left(\int_{\{m \leq |u|\}} |f| dx + \int_{\{m \leq |u| \leq m+1\}} \overline{\varphi}(x, |F|) dx\right) \\
&\quad + 2\int_{\Omega \backslash \Omega^s} a\left(x, T_k(u), \nabla T_k(u)\right) \nabla T_k(u) dx,
\end{aligned}$$

and by passing to the limit in j we obtain

$$\begin{aligned}&\limsup_{n\to+\infty}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dx\\&\leq\int_\Omega a\left(x,T_k(u),\nabla T_k(u)\right)\nabla T_k(u)\chi^s dx\\&\quad+2\alpha\phi(2k)\left(\int_{\{m\leq|u|\}}|f|dx+\int_{\{m\leq|u|\leq m+1\}}\overline{\varphi}(x,|F|)dx\right)\\&\quad+2\int_{\Omega\setminus\Omega^s}a\left(x,T_k(u),\nabla T_k(u)\right)\nabla T_k(u)dx.\end{aligned}$$

Letting s and then $m\to+\infty$, one has

$$\limsup_{n\to+\infty}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dx\leq\int_\Omega a\left(x,T_k(u),\nabla T_k(u)\right)\nabla T_k(u)dx.$$

Now, thinks to (4.3), (5.5), (5.29) and applying Fatou's lemma, we have

$$\int_\Omega a\left(x,T_k(u),\nabla T_k(u)\right)\nabla T_k(u)dx\leq\liminf_{n\to\infty}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dx.$$

It follows that

$$\lim_{n\to+\infty}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dx=\int_\Omega a\left(x,T_k(u),\nabla T_k(u)\right)\nabla T_k(u)dx.$$

By Lemma 3.7 we conclude that for every $k>0$

$$a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)\to a\left(x,T_k(u),\nabla T_k(u)\right)\nabla T_k(u)\text{ strongly in }L^1(\Omega).\tag{5.31}$$

The convexity of the Musielak function φ and (4.3) allow us to have

$$\begin{aligned}&\varphi\left(x,\frac{\left|\nabla T_k\left(u_n\right)-\nabla T_k(u)\right|}{2}\right)\\&\leq\frac{1}{2\alpha}a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)+\frac{1}{2\alpha}a\left(x,T_k(u),\nabla T_k(u)\right)\nabla T_k(u),\end{aligned}$$

so, by Vitali's theorem one has

$$\lim_{|E|\to0}\sup_n\int_E\varphi\left(x,\frac{\left|\nabla T_k\left(u_n\right)-\nabla T_k(u)\right|}{2}\right)dx=0.$$

Consequently, for every $k>0$

$$T_k\left(u_n\right)\to T_k(u)\text{ in }W_0^1L_\varphi(\Omega)\text{ for the modular convergence.}\tag{5.32}$$

Step 7: Equi-integrability of the non-linearities.

We shall prove that $g_n\left(x,u_n,\nabla u_n\right)\to g(x,u,\nabla u)$ strongly in $L^1(\Omega)$ by using Vitali's theorem. Since $g_n\left(x,u_n,\nabla u_n\right)\to g(x,u,\nabla u)$ a.e in Ω, by (5.29), it suffices to prove that $g_n\left(x,u_n,\nabla u_n\right)$ are uniformly equi-integrable in Ω.

Let E be measurable subset of Ω and let $m>0$. Using (4.3) and (4.4) we can write

$$\begin{aligned}&\int_E\left|g_n\left(x,u_n,\nabla u_n\right)\right|dx\\&=\int_{E\cap\{|u_n|\leq m\}}\left|g_n\left(x,u_n,\nabla u_n\right)\right|dx+\int_{E\cap\{|u_n|>m\}}\left|g_n\left(x,u_n,\nabla u_n\right)\right|dx\\&\leq b(m)\int_E d(x)dx+b(m)\int_E a\left(x,T_m\left(u_n\right),\nabla T_m\left(u_n\right)\right)\nabla T_m\left(u_n\right)dx\\&\quad+\frac{1}{m}\int_\Omega g_n\left(x,u_n,\nabla u_n\right)u_n dx.\end{aligned}$$

By (4.4) and (5.6) it follows that

$$0 \le \int_\Omega g_n\left(x,u_n,\nabla u_n\right) u_n dx \le c,$$

so

$$0 \le \frac{1}{m}\int_\Omega g_n\left(x,u_n,\nabla u_n\right) u_n dx \le \frac{c}{m},$$

then

$$\lim_{m\to+\infty}\frac{1}{m}\int_\Omega g_n\left(x,u_n,\nabla u_n\right) u_n dx = 0.$$

Thanks to (5.31), the sequence

$$\left\{a\left(x,T_m\left(u_n\right),\nabla T_m\left(u_n\right)\right)\nabla T_m\left(u_n\right)\right\}_n \text{ is equi-integrable.}$$

This fact allows us to get

$$\lim_{|E|\to 0}\sup_n \int_E a\left(x,T_m\left(u_n\right),\nabla T_m\left(u_n\right)\right)\cdot\nabla T_m\left(u_n\right)dx = 0,$$

which shows that $g_n\left(x,u_n,\nabla u_n\right)$ is equi-integrable. Thus, Vitali's theorem implies that $g(x,u,\nabla u) \in L^1(\Omega)$ and

$$g_n\left(x,u_n,\nabla u_n\right) \to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega). \tag{5.33}$$

Step 8: Renormalization identity for the solutions.

In this subsubsection, we aim to prove the following identity:

$$\lim_{m\to\infty}\int_{\{m\le|u|\le m+1\}} a(x,u,\nabla u)\cdot\nabla u\,dx = 0. \tag{5.34}$$

For each fixed $m \ge 0$, we observe that

$$\begin{aligned}
&\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx\\
&=\int_\Omega a(x,u_n,\nabla u_n)\cdot(\nabla T_{m+1}(u_n)-\nabla T_m(u_n))\,dx\\
&=\int_\Omega a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla T_{m+1}(u_n)\,dx\\
&\quad-\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\cdot\nabla T_m(u_n)\,dx.
\end{aligned}$$

Thanks to the convergence result established in (5.31), and taking the limit as $n\to\infty$ for fixed m, we obtain:

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx\\
&=\int_\Omega a(x,T_{m+1}(u),\nabla T_{m+1}(u))\cdot\nabla T_{m+1}(u)\,dx\\
&\quad-\int_\Omega a(x,T_m(u),\nabla T_m(u))\cdot\nabla T_m(u)\,dx\\
&=\int_\Omega a(x,u,\nabla u)\cdot(\nabla T_{m+1}(u)-\nabla T_m(u))\,dx\\
&=\int_{\{m\le|u|\le m+1\}} a(x,u,\nabla u)\cdot\nabla u\,dx.
\end{aligned}$$

Finally, by applying the result of (5.13) and letting $m\to\infty$, we conclude that identity (5.34) holds.

Step 9: Passing to the limit.

Finally, in this step thanks to (5.31) and Lemma 3.7, one hasa

$$a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n \longrightarrow a(x, u, \nabla u) \cdot \nabla u \text{ strongly in } L^1(\Omega). \tag{5.35}$$

Let $h \in \mathcal{C}_c^1(\mathbb{R})$ and $\theta \in \mathcal{D}(\Omega)$. Inserting $h\left(u_n\right) \theta$ as test function in $\left(\mathcal{P}_n\right)$, we get

$$\begin{aligned} & \int_{\Omega} a\left(x, u_n, \nabla u_n\right) \nabla u_n h^{\prime}\left(u_n\right) \theta d x+\int_{\Omega} a\left(x, u_n, \nabla u_n\right) \nabla \theta h\left(u_n\right) d x \\ & \quad+\int_{\Omega} \Phi_n\left(u_n\right) \nabla\left(h\left(u_n\right) \theta\right) d x+\int_{\Omega} g_n\left(x, u_n, \nabla u_n\right) h\left(u_n\right) \theta d x \\ & \quad=\left\langle f_n, h\left(u_n\right) \theta\right\rangle+\int_{\Omega} F \nabla\left(h\left(u_n\right) \theta\right) d x . \end{aligned} \tag{5.36}$$

We now pass to the limit as $n \rightarrow+\infty$ in each term of the equality (5.36).
Since both h and $h^{\prime}$ have compact support in $\mathbb{R}$, there exists a real number $\nu>0$ such that

$$\operatorname{supp}(h) \subset[-\nu, \nu] \quad \text{and} \quad \operatorname{supp}\left(h^{\prime}\right) \subset[-\nu, \nu].$$

Therefore, for any $n>\nu$, we have

$$\Phi_n(t) h(t)=\Phi\left(T_\nu(t)\right) h(t), \quad \text{and} \quad \Phi_n(t) h^{\prime}(t)=\Phi\left(T_\nu(t)\right) h^{\prime}(t).$$

Moreover, the functions Φh and $\Phi h^{\prime}$ belong to $\left(\mathcal{C}^0(\mathbb{R}) \cap L^{\infty}(\mathbb{R})\right)^N$.
Since $u_n \in W_0^1 L_{\varphi}(\Omega)$, there exists two positive constants η_1, η_2 such that

$$\int_{\Omega} \varphi\left(x, \frac{\left|\nabla u_n\right|}{\eta_1}\right) d x \leq \eta_2 .$$

Let τ be a positive constant such that $\left\|h\left(u_n\right)|\nabla \theta|\right\|_{\infty} \leq \tau$ and $\left\|h^{\prime}\left(u_n\right) \theta\right\|_{\infty} \leq \tau$. For η large enough, we have

$$\begin{aligned} \int_{\Omega} \varphi\left(x, \frac{\left|\nabla\left(h\left(u_n\right) \theta\right)\right|}{\eta}\right) d x & \leq \int_{\Omega} \varphi\left(x, \frac{\left|h\left(u_n\right) \nabla \theta\right|+\left|h^{\prime}\left(u_n\right) \theta\right|\left|\nabla u_n\right|}{\eta}\right) d x \\ & \leq \int_{\Omega} \varphi\left(x, \frac{\tau+\frac{\tau \eta_1\left|\nabla u_n\right|}{\eta_1}}{\eta}\right) d x \\ & \leq \int_{\Omega} \varphi\left(x, \frac{\tau}{\eta}\right) d x+\frac{\tau \eta_1}{\eta} \int_{\Omega} \varphi \quad\left(x, \frac{\left|\nabla u_n\right|}{\eta_1}\right) d x \\ & \leq \int_{\Omega} \varphi(x, 1) d x+\frac{\tau \eta_1 \eta_2}{\eta} \leq C, \end{aligned}$$

which implies that $h\left(u_n\right) \theta$ is bounded in $W_0^1 L_{\varphi}(\Omega)$ and then we deduce that

$$h\left(u_n\right) \theta \rightharpoonup h(u) \theta \text{ weakly in } W_0^1 L_{\varphi}(\Omega) \text{ for } \sigma\left(\Pi L_{\varphi}, \Pi E_{\bar{\varphi}}\right). \tag{5.37}$$

Which give

$$\left\langle f, h\left(u_n\right) \varphi\right\rangle \rightarrow\langle f, h(u) \varphi\rangle.$$

Let E be a measurable subset of Ω. we pose $c_\nu=\max_{|t| \leq \nu} \Phi(t)$. And denoting by $\|v\|_{\varphi, \Omega}$ the Orlicz norm of a function $v \in L_{\varphi}(\Omega)$. We thinking to the strengthened Hölder inequality with both Orlicz and Luxemburg norms, we have

$$\begin{aligned} & \left\|\Phi\left(T_\nu\left(u_n\right)\right) \chi_E\right\|_{\bar{\varphi}, \Omega}=\sup_{\|v\|_{\varphi, \Omega} \leq 1}\left|\int_E \Phi\left(T_\nu\left(u_n\right)\right) v d x\right| \\ & \leq c_\nu \sup_{\|v\|_{\varphi, \Omega} \leq 1}\left\|\chi_E\right\|_{\bar{\varphi}, \Omega}\|v\|_{\varphi, \Omega} \\ & \leq c_\nu|E| \varphi^{-1}\left(x, \frac{1}{|E|}\right) . \end{aligned}$$

Consequently,

$$\lim_{|E|\to 0} \sup_n \left\| \Phi\left(T_\nu\left(u_n\right)\right) \chi_E \right\|_{(\overline{\varphi},\Omega)} = 0.$$

Then, in light of (5.10) and by applying Lemma 11.2 from [18], we deduce that

$$\Phi\left(T_\nu\left(u_n\right)\right) \to \Phi\left(T_\nu(u)\right) \quad \text{strongly in } \left(E_{\overline{\varphi}}(\Omega)\right)^N.$$

This, together with (5.37), enables us to pass to the limit in the third term of (5.36), obtaining

$$\int_\Omega \Phi\left(T_\nu\left(u_n\right)\right) \cdot \nabla\left(h\left(u_n\right)\theta\right)\, dx \to \int_\Omega \Phi\left(T_\nu(u)\right) \cdot \nabla(h(u)\theta)\, dx.$$

Observe that

$$\left| a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n\, h'\left(u_n\right)\theta \right| \leq c'\, a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n,$$

so that, by virtue of (5.35) and using Vitali's convergence theorem, we conclude

$$\int_\Omega a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n\, h'\left(u_n\right)\theta\, dx \to \int_\Omega a(x, u, \nabla u) \cdot \nabla u\, h'(u)\theta\, dx.$$

Concerning the second term in (5.36), a similar argument yields

$$h\left(u_n\right)\nabla\theta \to h(u)\nabla\theta \quad \text{strongly in } \left(E_\varphi(\Omega)\right)^N,$$

and from (5.30) we deduce

$$\int_\Omega a\left(x, u_n, \nabla u_n\right) \cdot \nabla\theta\, h\left(u_n\right)\, dx \to \int_\Omega a(x, u, \nabla u) \cdot \nabla\theta\, h(u)\, dx,$$

as well as

$$\int_\Omega F \cdot \nabla\theta\, h\left(u_n\right)\, dx \to \int_\Omega F \cdot \nabla\theta\, h(u)\, dx.$$

Since $h(u_n)\theta \to h(u)\theta$ weakly-* in $L^\infty(\Omega)$ (with respect to $\sigma^*(L^\infty, L^1)$), we can use (5.33) to pass to the limit in the fourth term of (5.36) and obtain

$$\int_\Omega g_n(x, u_n, \nabla u_n)\, h(u_n)\theta\, dx \to \int_\Omega g(x, u, \nabla u)\, h(u)\theta\, dx.$$

Combining all these limits, we finally pass to the limit in each term of (5.36), yielding

$$\begin{aligned} &\int_\Omega a(x, u, \nabla u) \cdot \left[h'(u)\theta\nabla u + h(u)\nabla\theta\right] dx + \int_\Omega \Phi(u)h'(u)\theta \cdot \nabla u\, dx \\ &\quad + \int_\Omega \Phi(u)h(u) \cdot \nabla\theta\, dx + \int_\Omega g(x, u, \nabla u)h(u)\theta\, dx \\ &= \int_\Omega f h(u)\theta\, dx + \int_\Omega F \cdot \left[h'(u)\theta\nabla u + h(u)\nabla\theta\right] dx, \end{aligned}$$

for every $h \in \mathcal{C}_c^1(\mathbb{R})$ and every $\theta \in \mathcal{D}(\Omega)$, which establishes Theorem 4.1.

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