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Semi-Analytical Approach to Study the Nonlinear Oxygen Absorption Kinetics and Oxygen Diffusion in a Spherical Cell

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ABSTRACT: The phenomenon of oxygen diffusion in spherical cells, governed by Michaelis-Menten uptake kinetics, gives rise to a class of challenging singular boundary value problems. A semi-analytical method namely differential transform method is proposed to acquire analytical solution of aforesaid with nonlinear uptake kinetics. The proposed method is found to produce more trustworthy findings when the obtained results are compared with those available in the literature. We discuss the impact of the maximal reaction rate, Michaelis constant, and cell membrane permeability on the dimensionless oxygen concentration, supported by numerical and graphical results. The proposed method ability to handle the strongly nonlinear term make it a promising tool.

Keywords: Nonlinear singular BVP, differential transform method, error analysis, convergence analysis.

Contents

1. Introduction

Over the past decades, nonlinear singular boundary value problems (NSBVP) have attracted a lot of attention because they have a diverse variety of applications in science and technology. Only a few of the numerous physical phenomena that are simulated by NSBVPs include the diffusion of oxygen in a cell, electrohydrodynamic movement of a fluid in an isothermal gas sphere, thermal behaviour of a spherical cloud of gas, and thermal explosion. This aim of the manuscript is to proposed a reliable semi-analytical method for studying NSBVPs with Neumann and Robin boundary condition describing oxygen absorption kinetics and the oxygen diffusion in a spherical cell of the subsequent form:

$$\frac{d}{d\chi}\left(\Psi(\chi)\,\frac{d}{d\chi}c(\chi)\right) = \Psi(\chi)\,f\left(\chi, c(\chi)\right), \quad 0 < \chi \leq 1, \tag{1.1}$$

subjected to boundary conditions

$$\left[\frac{d}{d\chi}c(\chi)\right]_{\chi=0} = 0, \quad \left[\alpha\, c(\chi) + \beta\,\frac{d}{d\chi}c(\chi)\right]_{\chi=1} = a, \tag{1.2}$$

with $\Psi(\chi) = \chi^{\kappa}\,P(\chi)$, $\kappa > 0$ and $P(\chi)$ is a nonnegative function and $\alpha > 0$, $\beta \geq 0$, and a is a finite constant with the following assumptions:

- A1: $\Psi(0) = 0$, $\Psi(\chi) > 0$ for $\chi \in (0, 1]$, and $\Psi(\chi) \in C^1\left((0, 1]\right)$.

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- A2: $\Psi(\chi) = \chi^{\kappa} P(\chi)$ in $[0,1]$ and for some $\mu > 1$, the function $\frac{1}{\Psi(\chi)}$ is analytic in the $\{\chi : |\chi| < \mu\}$.
- A3: The function $f(\chi, c(\chi))$ is continuous and $\frac{\partial f}{\partial c} \geq 0$ exists and is continuous.

The reader can discover references [1,2] that discuss the uniqueness and existence of solutions to (1.1) and (1.2). Solving SBVPs is difficult and time-consuming because singular points occur. One of the real world phenomena described by SBVP is oxygen absorption kinetics and the oxygen diffusion in a spherical cell which has gain attention of many researchers. Lin originally proposed and solved a model of oxygen diffusion and nonlinear uptake in a sphere [3]. After that, McElwain re-examined and re-solved the same model [4]. The original draughts by Lin and McElwain [3,4] contain the full dimensional governing equation.

The governing equation for nonlinear oxygen diffusion and absorption kinetics in spherical cells, represented as a singular boundary value problem (SBVP), has been a subject of extensive research using both numerical and analytical approaches. Numerical methods, such as finite difference schemes [5], cubic and B-spline collocation approaches [6,7], and shooting techniques [8], have been widely employed to tackle these problems due to the general absence of exact analytical solutions. However, these methods suffer from significant limitations: they provide solutions only at discrete points, require substantial computational resources for acceptable accuracy, and often produce unsatisfactory approximations in the presence of strong nonlinearities and singularities. In response to these limitations, the past two decades have witnessed the development of numerous analytical and semi-analytical techniques designed specifically for singular boundary value problems. Methods such as the Adomian Decomposition Method (ADM) [9], the Modified ADM (MADM) [10], the Optimal Homotopy Analysis method (OHAM) [11], the Variational Iteration Method (VIM) [12], Maclaurin series [13], the Homotopy Perturbation Method (HPM) [14], the optimal HPM [15], the iterative based normal S-method [16], the DTM-padé approach [17] have been successfully applied to obtain continuous and differentiable solutions without the need for extensive discretization.

In addition, numerical approaches such as the local discontinuous Galerkin scheme [18], wavelet-based methods [19], and spectral Galerkin-collocation schemes [20] based on modified Chebyshev polynomials have been developed for these kinds of model problems. For a recent review, readers are referred to [21]. More recently, advanced spectral methods leveraging orthogonal polynomials—such as Legendre polynomials [22], shifted Chebyshev polynomials [23,24], generalized airfoil polynomials [25,26], Chelyshkov and Morgan-Voyce functions [27,28], and Genocchi-based approximation algorithms [29] have emerged as powerful tools, offering exponential convergence and high accuracy even in the presence of singularities

While the aforementioned analytical techniques have contributed significantly to solving singular boundary value problems (SBVPs), each possesses inherent limitations. The ADM struggles with the computational complexity of calculating Adomian polynomials for intricate nonlinearities and often fails to provide a convergent series solution over a wider domain for strongly nonlinear problems. Similarly, the VIM requires the often non-trivial determination of Lagrange multipliers via variational theory and can involve cumbersome, repetitive calculations of unnecessary terms, leading to computational inefficiency.

To circumvent these challenges, we propose the application of the Differential Transform Method (DTM) for solving nonlinear SBVPs. The DTM is a powerful semi-analytical technique that has been successfully employed across a wide range of linear and nonlinear problems, including reaction-convection-diffusion systems [30], magnetohydrodynamic flows in porous channels [31,32], and boundary eigenvalue problems [33]. More recently, DTM has been extended to fractional-order nonlinear models arising in quantum mechanics and field theory [34,35]. Its principal advantages include bypassing the need for linearization, perturbation techniques, or Adomian polynomial construction, instead converting differential equations into algebraic recurrence relations that are easier to solve. The method has also shown effectiveness in modeling nonlinear heat transfer phenomena in porous fins subjected to magnetic fields [36]. In this study, we harness the robustness and computational efficiency of DTM to construct a convergent series solution for the considered class of highly nonlinear singular boundary value problems, aiming to address the limitations of existing approaches.

Following this introduction, the paper is divided into six sections. Section 2 demonstrates the problem statement of mathematical model of unsteady state oxygen diffusion in a spherical cell. Section 3 deals with the definition and fundamental concepts of semi-analytical approach. The implementation of semi-analytical approach is discussed in Section 4 followed by result and discussion in Section 5. Our findings are summarized in the concluding Section 6.

2. Problem Statement

The following equation can be used to model the unsteady state oxygen diffusion in a spherical cell, studied by Lin [3]:

$$\frac{\partial Q_T}{\partial t} = D\left(\frac{\partial^2 Q_T}{\partial r^2} + \frac{2}{r}\frac{\partial Q_T}{\partial r}\right) - \frac{R_r\, Q_T}{K_m + Q_T}, \tag{2.1}$$

with initial condition

$$Q_T(r,0) = 0, \quad 0 \le r \le R_0, \tag{2.2}$$

and the following prescribed boundary conditions

$$\frac{\partial Q_T}{\partial r}(0,t) = 0, \quad t > 0, \tag{2.3}$$

$$D\frac{\partial Q_T}{\partial r}(R_0,t) = h[Q_T^0 - Q_T(R_0,t)], \quad t > 0, \tag{2.4}$$

where D is the diffusion coefficient of the oxygen in the protoplasm, R_0 the radius of cell, h denotes the permeability of membrane, r the radial coordinate and t is the time, and the oxygen tension just outside the cell membrane is denoted by Q_T^0.

To facilitate computation, Eq. (2.1) and the corresponding initial and boundary conditions (Eqs. (2.1)-(2.3)) are cast in dimensionless form by employing the following variables and parameters:

$$c = \frac{Q_T}{Q_T^0}, \quad R = \frac{r}{R_0}, \quad \phi = \frac{t\,D}{R_0^2}, \quad \alpha = \frac{R_r R_0^2}{D\,Q_T^0}, \quad \mathbb{K} = \frac{K_m}{Q_t^0}, \quad H = \frac{hR_0}{D}.$$

Using the dimensional variables, Eqs. (2.1)-(2.3) are transformed to the following

$$\begin{cases} \dfrac{\partial c}{\partial \phi} = \dfrac{\partial^2 c}{\partial R^2} + \dfrac{2}{R}\dfrac{\partial c}{\partial R} - \dfrac{\alpha c}{\mathbb{K} + c}, \\ c(R,0) = 0, \quad 0 \le R \le 1, \\ \frac{\partial c}{\partial R}(0,\phi) = 0, \quad \phi > 0, \\ -\frac{\partial c}{\partial R}(1,\phi) = H[c(1,\tau) - 1], \quad \phi > 0. \end{cases} \tag{2.5}$$

McElwain considered the steady state and reduced the partial differential equations (2.5) into nonlinear SBVP in the Lane-Emden form [4] as

$$\frac{d^2 c}{dR^2} + \frac{2}{R}\frac{dc}{dR} = \frac{\alpha c}{\mathbb{K} + c}, \quad 0 \le R \le 1, \tag{2.6}$$

subject to the boundary conditions

$$\frac{dc}{dR}(0) = 0, \tag{2.7}$$

$$\frac{dc}{dR}(1) + H\,c(1) = H. \tag{2.8}$$

Equation (2.6) serves as the governing equation for a steady-state reaction-diffusion equation that represents oxygen transport via linear diffusion in a sphere with spherical symmetry with Michaelis-Menten constant ($\mathbb{K}$), maximum response rate and the half-saturation concentration $\mathbb{K}$ describes the absorption of oxygen. At the centre of the sphere ($R = 0$), the boundary condition ensures a symmetric oxygen distribution, while at $R = 1$, it specifies the oxygen flux across the cell membrane. The change in oxygen content across the cell membrane directly relates to this flow. The membrane permeability is represented by the proportionality coefficient, H [3,4]. Due to the oxygen uptake term's non-linearity, Eq. (2.6) supplemented with both initial and boundary conditions (2.7) face difficulty in find the analytical solution with less computational and time.

3. A Semi-Analytic Approach Based on DTM

The fundamental principle of the one-dimensional differential transform method (DTM) is introduced below.

Definition 3.1 *If a function $u(t)$ is analytic in the time domain D, then its differential transform at a point t_0 is defined as:*

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}, \tag{3.1}$$

evaluated at $t = t_0$, where $k \in \mathbb{Z}_{\geq 0}$ (i.e., k is a non-negative integer). This set of k values is referred to as the K-domain. Consequently, Eq. (3.1) becomes:

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_0)^k. \tag{3.2}$$

Here, $U(k)$ is referred to as the spectrum of the function $u(t)$ at $t = t_0$ in the K-domain.

Definition 3.2 *If $u(t)$ is analytic, then*

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_0)^k. \tag{3.3}$$

Equation (3.3) is called the inverse transformation of $U(k)$.

If $U(k)$ is defined as

$$U(k) = M(k)\left[\frac{d^k\{q(t)u(t)\}}{dt^k}\right]_{t=t_0}, \tag{3.4}$$

then the original function $u(t)$ can be described by the infinite series:

$$u(t) = \frac{1}{q(t)}\sum_{k=0}^{\infty}\frac{U(k)}{M(k)}(t-t_0)^k, \tag{3.5}$$

where, $M(k) \neq 0$ and $q(t)$ serves as the kernel for the transformation of $u(t)$. If $M(k) = H_k/k!$ and $q(t) = 1$ then Eqs. (3.2) and (3.4) are equivalent. Applying the differential transform maps the differential equation onto the K-domain, reducing it to an algebraic equation. The solution in the original domain, $u(t)$, is subsequently reconstructed from the transformed solution via a finite-term Taylor series expansion, accounting for a remainder term:

$$u(t) = \sum_{k=0}^{n}\frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}(t-t_0)^k + R_{n+1}(t),$$

where

$$R_{n+1}(t) = \frac{1}{(n+1)!}\left[\frac{d^{n+1}u(t)}{dt^{n+1}}\right]_{t=\xi}(t-t_0)^{n+1}, \quad t_0 < \xi < t. \tag{3.6}$$

Using the definition of DTM, fundamental operation are derived and mention in Table 1.

4. Implementation of DTM

The general solution of Eq. (2.6), satisfying the prescribed initial and boundary conditions, is derived in this section. Let $C(k)$ shows the differential transform of the solution $c(R)$ of the model (2.6). Rearranging

Table 1: Fundamental Properties of DTM

Original Function	**Transformed Function**
$v(t) = c_1\, u_1(t) \pm c_2\, u_2(t)$	$V(k) = c_1\, U_1(k) \pm c_2\, U_2(k)$, $(c_1, c_2$ are constants$)$
$v(t) = \dfrac{d^m u(t)}{dt^m}$	$V(k) = (n+1)(n+2)\ldots(n+m)U(n+m)$
$v(t) = u_1(t)u_2(t)$	$V(k) = \sum_{n=0}^{k} U_1(n)U_2(k-n)$
$v(t) = e^{mt}$	$V(k) = \dfrac{m^k}{k!}$
$v(t) = t^m$	$V(k) = \dfrac{m!}{\kappa!}\delta(k-m)$
$v(t) = t^m u(t)$	$V(k) = \dfrac{\kappa!}{(\kappa-m)!}U(k-m)$
$v(t) = u(t-a)$	$V(k) = e^{-a\kappa}U(k)$

the terms and applying fundamental properties DTM to Eq. (2.6), it reduces to

$$\begin{aligned}
&\sum_{\iota=0}^{k}\sum_{\ell=0}^{\iota}\delta(\ell-1)c(\ell-\iota)(k-\iota+1)(k-\iota+2)C(k-\iota+2)\\
&+\mathbb{K}\sum_{\iota=0}^{k}\delta(\iota-1)(k-\iota+1)(k-\iota+2)C(k-\iota+2)\\
&+2\sum_{\iota=0}^{k}C(\iota)(k-\iota+1)C(k-\iota+1)+2\mathbb{K}(k+1)C(k+1)-\alpha\sum_{\iota=0}^{k}\delta(\iota-1)C(k-\iota)=0,
\end{aligned}\tag{4.1}$$

with transformed initial condition

$$C(0) = a, \quad C(1) = 0, \tag{4.2}$$

where a represent the initial concentration. The coefficients $C(k)$ of the series solution are computed by successively evaluating the recurrence relation for $k = 0, 1, 2, \ldots$. The initial coefficient, corresponding to $k = 0$, is obtained from Eq. (4.1):

$$\begin{aligned}
&\sum_{\iota=0}^{1}\sum_{\ell=0}^{\iota}\delta(\ell-1)C(\ell-\iota)(2-\iota)(3-\iota)C(3-\iota)+\mathbb{K}\sum_{\iota=0}^{1}\delta(\iota-1)(2-\iota)(3-\iota)C(3-\iota)\\
&+2\sum_{\iota=0}^{1}C(\iota)(2-\iota)C(2-\iota)+4\mathbb{K}C(2)-\alpha\sum_{\iota=0}^{1}\delta(\iota-1)C(1-\iota)=0.
\end{aligned}\tag{4.3}$$

Substituting the transformed initial condition Eq. (4.2) into former Eq. (4.3), one has

$$C(2) = \frac{\alpha\, a}{6\mathbb{K}+6a}. \tag{4.4}$$

For $k = 2$, Eq. (4.1) simplifies to

$$\begin{aligned}
&\sum_{\iota=0}^{2}\sum_{\ell=0}^{\iota}\delta(\ell-1)C(\ell-\iota)(3-\iota)(4-\iota)C(4-\iota)+\mathbb{K}\sum_{\iota=0}^{k=2}\delta(\iota-1)(3-\iota)(4-\iota)C(4-\iota)\\
&+2\sum_{\iota=0}^{2}C(\iota)(3-\iota)C(3-\iota)+6\mathbb{K}C(3)-\alpha\sum_{\iota=0}^{2}\delta(\iota-1)C(2-\iota)=0.
\end{aligned}\tag{4.5}$$

Inserting the expression from Eqs. (4.2) and (4.4) into Eq. (4.5), we get

$$C(3) = 0. \tag{4.6}$$

For $k = 3$, Eq. (4.1) reduces to

$$\sum_{\iota=0}^{3}\sum_{\ell=0}^{\iota}\delta(\ell-1)C(\ell-\iota)(4-\iota)(5-\iota)C(5-\iota)+\mathbb{K}\sum_{\iota=0}^{3}\delta(\iota-1)(4-\iota)(5-\iota)C(5-\iota)$$
$$+2\sum_{\iota=0}^{3}C(\iota)(4-\iota)C(4-\iota)+8\mathbb{K}C(4)-\alpha\sum_{\iota=0}^{3}\delta(\iota-1)C(3-\iota)=0, \tag{4.7}$$

Substituting Eq. (4.2), (4.4), and (4.6) into Eq. (4.7), we get

$$C(4)=\frac{\alpha^2\mathbb{K}a}{120\mathbb{K}^3+360\mathbb{K}^2a+360\mathbb{K}a^2+120a^3}. \tag{4.8}$$

For $k = 4$, Eq. (4.1) reduces to

$$\sum_{\iota=0}^{4}\sum_{\ell=0}^{\iota}\delta(\ell-1)C(\ell-\iota)(5-\iota)(6-\iota)C(6-\iota)+\mathbb{K}\sum_{\iota=0}^{4}\delta(\iota-1)(5-\iota)(6-\iota)C(6-\iota)$$
$$+2\sum_{\iota=0}^{4}C(\iota)(5-\iota)C(5-\iota)+10\mathbb{K}\,C(5)-\alpha\sum_{\iota=0}^{4}\delta(\iota-1)C(4-\iota)=0. \tag{4.9}$$

Substituting Eqs. (4.2), (4.4), (4.6), and (4.8) in Eq. (4.9), we get

$$C(5)=0. \tag{4.10}$$

For $k = 4$, Eq. (4.1) reduces to

$$\sum_{\iota=0}^{5}\sum_{\ell=0}^{\iota}\delta(\ell-1)C(\ell-\iota)(6-\iota)(7-\iota)C(7-\iota)+\mathbb{K}\sum_{\iota=0}^{5}\delta(\iota-1)(6-\iota)(7-\iota)C(7-\iota)$$
$$+2\sum_{\iota=0}^{5}C(\iota)(6-\iota)C(6-\iota)+12\mathbb{K}\,C(6)-\alpha\sum_{\iota=0}^{5}\delta(\iota-1)C(5-\iota)=0. \tag{4.11}$$

Substituting Eqs. (4.2), (4.4), (4.6), (4.8), and (4.10) in Eq. (4.11), we get

$$C(6)=\frac{10\mathbb{K}\alpha^3a-3\alpha^2\mathbb{K}^2a}{15120\mathbb{K}^5+75600\mathbb{K}^4a+151200\mathbb{K}^3a^2+75600\mathbb{K}^2a^4+15120\mathbb{K}a^5}. \tag{4.12}$$

We computed the remaining coefficients of the series solution using the MATLAB software package. The approximation series solution of oxygen diffusion for nonlinear kinetics is given by

$$c(R)=\sum_{k=0}^{\infty}C(k)R^k.$$

By using the obtained coefficients $C(k)$ we have:

$$c(R)=a+\left(\frac{\alpha a}{6(\mathbb{K}+a)}\right)R^2+\left(\frac{\mathbb{K}\alpha^2a}{120(\mathbb{K}+a)^3}\right)R^4-\left(\frac{10\mathbb{K}\alpha^3a^2-3\mathbb{K}^2\alpha^2a}{15120\mathbb{K}(\mathbb{K}+a)^4}\right)R^6+\dots. \tag{4.13}$$

5. Result and Discussion

This study employs the Differential Transform Method (DTM) to conduct parametric simulations, investigating the influence of key biological parameters on the dimensionless oxygen concentration. Table 2 shows the comparison of DTM with Adomian decomposition method with Green's function (ADMGF) and optimal homotopy analysis method (OHAM) both reported in [11] for $c_{10}(R)$ parameter value $\alpha = 0.76129, \mathbb{K} = 0.03119$, and $H = 4$.

Table 2: Comparison of solution obtained using DTM with ADMGF and OHAM.

R	ADMGF	OHAM	DTM
0.0	0.828483290	0.828483290	0.828483105
0.1	0.829706092	0.829706092	0.829705907
0.2	0.833374734	0.833374734	0.833374548
0.3	0.839489914	0.839489914	0.839489728
0.4	0.848052785	0.848052785	0.848052599
0.5	0.859064927	0.859064927	0.859064741
0.6	0.872528320	0.872528320	0.872528134
0.7	0.888445306	0.888445306	0.888445119
0.8	0.906818548	0.906818548	0.906818361
0.9	0.927650988	0.927650988	0.927650799
1.0	0.950945798	0.950945798	0.950945603

Table 3: Comparison of residual error of ADMGF, OHAM and DTM for parameter value $\alpha = 0.76129, \mathbb{K} = 0.03119$, and $H = 4$.

R	ADMGF (5 terms)	OHAM	DTM	ADMGF (5 terms)	OHAM	DTM
0.0	2.91×10^{-6}	8.32×10^{-7}	0	2.13×10^{-10}	1.14×10^{-10}	0
0.1	2.80×10^{-6}	7.95×10^{-7}	0	1.95×10^{-10}	1.04×10^{-10}	0
0.2	2.49×10^{-6}	6.94×10^{-7}	0	1.50×10^{-10}	7.76×10^{-11}	0
0.3	2.03×10^{-6}	5.54×10^{-7}	0	9.42×10^{-11}	4.61×10^{-11}	0
0.4	1.50×10^{-6}	4.07×10^{-7}	0	4.56×10^{-11}	2.00×10^{-11}	0
0.5	9.93×10^{-7}	2.83×10^{-7}	0	1.45×10^{-11}	4.59×10^{-12}	0
0.6	5.66×10^{-7}	2.01×10^{-7}	0	6.60×10^{-13}	1.23×10^{-12}	0
0.7	2.63×10^{-7}	1.59×10^{-7}	0	2.52×10^{-12}	1.82×10^{-12}	0
0.8	8.70×10^{-8}	1.49×10^{-7}	0	1.77×10^{-12}	9.98×10^{-13}	0
0.9	1.13×10^{-8}	1.52×10^{-7}	0	7.45×10^{-13}	4.01×10^{-13}	0
1.0	5.97×10^{-9}	1.54×10^{-7}	3.25×10^{-4}	2.54×10^{-13}	1.69×10^{-13}	8.76×10^{-7}

Table 4: Variation of the dimensionless oxygen concentration with the maximal reaction rate parameter α. The Michaelis constant and membrane permeability are fixed at $\mathbb{K} = 0.03119$ and $H = 4$, respectively.

R	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
0.0	0.991701	0.959175	0.882433	0.811927	0.747278
0.1	0.991756	0.959445	0.883199	0.813131	0.748865
0.2	0.991922	0.960255	0.885497	0.816746	0.753636
0.3	0.992198	0.961607	0.889333	0.822786	0.761618
0.4	0.992585	0.963500	0.894715	0.831274	0.772853
0.5	0.993082	0.965937	0.901654	0.842243	0.787405
0.6	0.993690	0.968919	0.910165	0.855732	0.805353
0.7	0.994409	0.972447	0.920268	0.871793	0.826793
0.8	0.995239	0.976525	0.931982	0.890484	0.851841
0.9	0.996180	0.981155	0.945334	0.911874	0.880628
1.0	0.997232	0.986340	0.960351	0.936041	0.913306

Using the DTM approach from Section 2, Table 4 presents an analysis of the maximal reaction

Table 5: Variation of the dimensionless oxygen concentration with the Michaelis constant $\mathbb{K}$. The maximum reaction rate and membrane permeability are fixed at $\alpha = 0.76129$ and $H = 4$, respectively.

R	$\mathbb{K} = 0.1$	$\mathbb{K} = 0.5$	$\mathbb{K} = 1.5$	$\mathbb{K} = 2.5$	$\mathbb{K} = 3.5$
0.0	0.562570	0.695543	0.817111	0.867418	0.895659
0.1	0.565401	0.697483	0.818286	0.868277	0.896338
0.2	0.573901	0.703311	0.821818	0.870855	0.898378
0.3	0.588095	0.713051	0.827716	0.875161	0.901782
0.4	0.608019	0.726745	0.836002	0.881205	0.906558
0.5	0.633729	0.744449	0.846703	0.889002	0.912716
0.6	0.665294	0.766236	0.859853	0.898574	0.920270
0.7	0.702800	0.792196	0.875497	0.909946	0.929235
0.8	0.746346	0.822435	0.893686	0.923146	0.939632
0.9	0.796050	0.857074	0.914479	0.938210	0.951481
1.0	0.852041	0.896253	0.937942	0.955175	0.964809

rate parameter α's influence on the dimensionless oxygen concentration, with fixed parameters $\mathbb{K} = 0.03119$ and $H = 4$. It is clear from Table 4, as the reaction rate increases the dimensionless oxygen concentration decreases. Using the DTM method, Table 5 analyzes the influence of the Michaelis constant $\mathbb{K}$ on the dimensionless oxygen concentration. The maximum reaction rate ($\alpha = Y$) and cell membrane permeability ($H = 4$) are held constant.

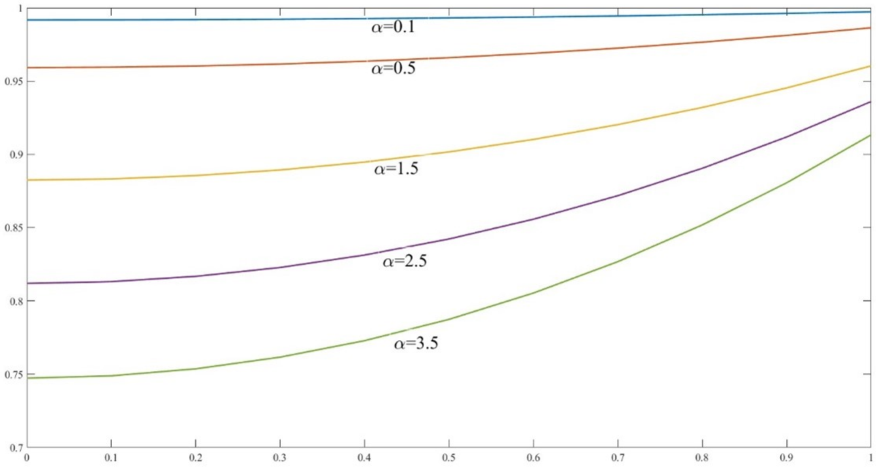

Figure 1: Variation of the dimensionless oxygen concentration with the maximal reaction rate parameter α. The Michaelis constant and membrane permeability are fixed at $\mathbb{K} = 0.03119$ and $H = 4$, respectively.

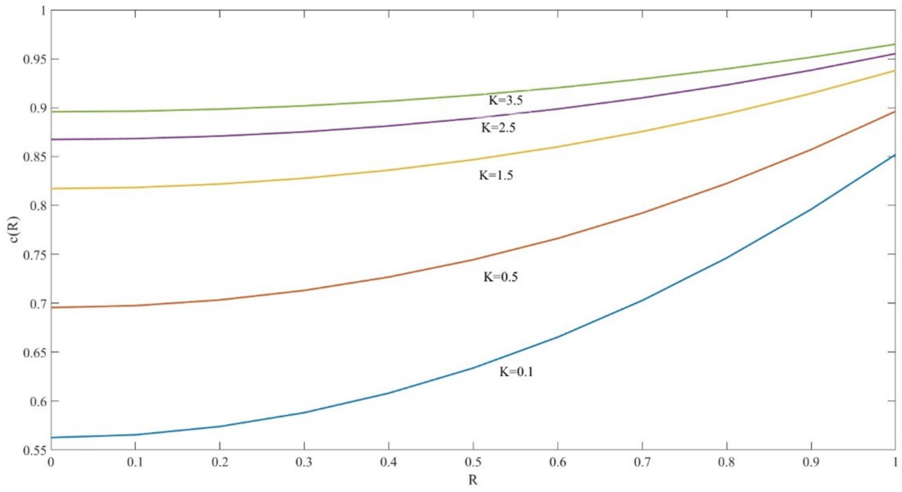

Figure 2: Variation of the dimensionless oxygen concentration with the Michaelis constant $\mathbb{K}$. The maximum reaction rate and membrane permeability are fixed at $\alpha = 0.76129$ and $H = 4$, respectively.

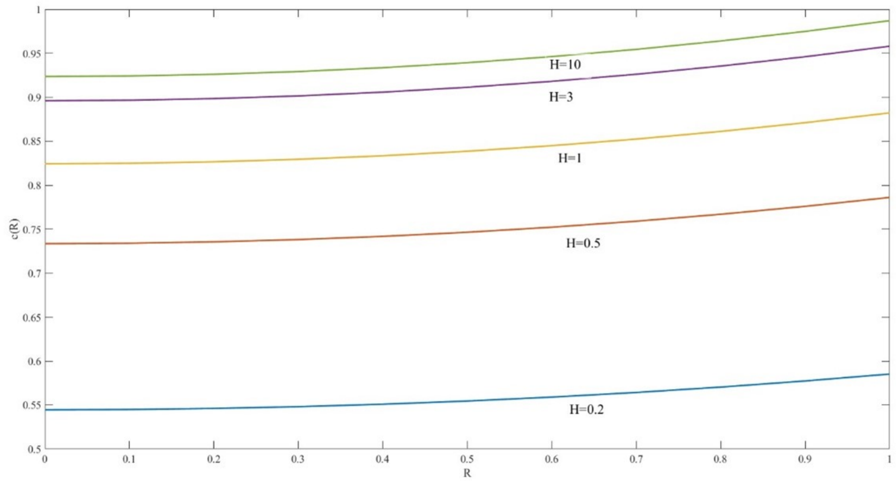

Figure 3: Influence of the cell membrane permeability parameter H on the dimensionless oxygen concentration. The maximum rate and Michaelis constant are fixed at $\alpha = 0.76129$ and $\mathbb{K} = 0.03119$, respectively.

6. Conclusion

This work examines the Michaelis-Menten kinetics-based nonlinear oxygen uptake effect on the Lane-Emden equation, which models oxygen diffusion in an idealised spherical cell. By transforming the highly nonlinear equation into algebraic equations, the proposed method simultaneously addresses the singularity at $x = 0$ and significantly enhances computational efficiency. By varying reaction parameter values, comparisons between the DTM, OHAM and ADMGF have been made. Also, based on the results, the proposed method displays high reliability and accuracy in solving and analysis the how the system behaves. The DTM's correctness, efficiency, and convergence make it clear that it is a practical analytical technique. This method is well-suited for solving differential equations with strongly nonlinear dynamics.

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