Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 2** : 1–15.
ISSN-0037-8712
doi:10.5269/bspm.78848

Matrix Pencil Method for Rectangular Polynomial Two-Parameter Eigenvalue Problem

Niranjan Bora* and Bharati Borgohain

ABSTRACT: Rectangular multiparameter eigenvalue problems (RMEP) consisting of a single multivariate polynomial have received interest among the researchers due to their applications in diverse scientific domain, particularly in optimal least square model problems. A common method for determining the optimal least squares of linear time-invariant dynamical systems (LTI) and autoregressive moving average (ARMA) models are obtained from the solution of the rectangular polynomial two-parameter eigenvalue problems (RPTEP). This makes it necessary to find effective solution methods for this particular kind of eigenvalue problem. Linearizing the matrix polynomial associated with RMEP followed by the conversion to a known form of linear two-parameter eigenvalue problem, and then using the Vandermonde compression is the currently available method in the literature. In this paper, we present a two-parameter matrix pencil method to obtain the solution of the RPTEP, that can be used as a ready reference to compute the solution of LTI and ARMA. Numerical works are performed to verify the computational efficiency of the method.

Key Words: Linearization, matrix polynomial, matrix pencil method, rectangular polynomial two-parameter eigenvalue problem.

Contents

1. Introduction

The eigenvalue problem associated with two-parameter matrix polynomials of grade k with square coefficient matrices has gained attention in numerous scientific domains, including the study of delay differential equations, see, [13,14,21]. To solve such type of eigenvalue problem, there are mainly two types of numerical approach. One that deals directly with the problem [5,25] and the other that computes the eigenvalues of the linearized form of the problem with high dimension [10,15,27]. Authors generally consider square matrices while dealing with eigenvalue problem consisting of matrix polynomials. However, study of eigenvalue problem with rectangular coefficient matrices also holds its significance. The works [4,6,7,20,29,30] contains theoretical and numerical insights of the eigenvalue problem that corresponds to rectangular one parameter matrix polynomials. The study on eigenvalue problem associated with rectangular multiparameter matrix polynomial with $m \times n$ coefficient matrices are found in the works of

* Corresponding author.
2010 *Mathematics Subject Classification:* 15A18, 15A03, 15A69, 47J10, 93B18.
Submitted September 08, 2025. Published September 30, 2025

[16]. The well-known survey paper [19] contains more details on RMEP. In [28], author examined the linear multiparameter eigenvalue problem of the form

$$\left(\sum_{i=1}^{k}(A_0 + \lambda_i A_i)\right) x = 0 \tag{1.1}$$

where $A_0, A_i \in \mathbb{C}^{(n+k-1)\times n}$. A general algorithm is available in [9] for the problem (1.1). A special category of eigenvalue problem associated with rectangular matrix polynomial is introduced in [12] for $k \geq 2$, and is given by

$$\left(\sum_{\tau} \lambda^{\tau} A_{\tau}\right) x = 0 \tag{1.2}$$

where, $\tau = (\tau_1, \ldots, \tau_k)$ is multi-index, $0 \neq x \in \mathbb{C}^n$ and $A_\tau = A_{\tau_1,\ldots,\tau_k} \in \mathbb{C}^{(n+k-1)\times n}$. In [2], author studied the problem (1.2) presenting its application in $\mathcal{H}_2$ Optimal Model Reduction and developed a new numerical method that relies on a reduction to compressed RMEP. Eigenvalue problems with suitably defined rectangular two-parameter matrix pencil are found in the works [17,18], and the refernces therein. Motivated by the rectangular eigenvalue problems defined in (1.1)-(1.2), we are interested to study the eigenvalue problem associated with two-parameter matrix polynomials with nonsquare coefficient matrices. The aim of the current paper, is to present a two-parameter based matrix pencil method to solve numerically the eigenvalue problem that corresponds to the two-parameter matrix polynomials with non-square coefficient matrices.

1.1. Problem formulation

We consider the rectangular two-parameter matrix polynomial (RTMP) of the form

$$R(\lambda, \mu) = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \lambda^i \mu^j M_{ij} \tag{1.3}$$

where, $M_{ij} \in \mathbb{C}^{m\times n}$, $m > n$. The associated eigenvalue problem is to find the parameters $\lambda, \mu \in \mathbb{C}$ such that

$$R(\lambda, \mu)x := \sum_{i=0}^{k} \sum_{j=0}^{k-i} \lambda^i \mu^j M_{ij} x = 0 \tag{1.4}$$

where, $0 \neq x \in \mathbb{C}^n$. The problem is termed as RPTEP, where we search the pair (λ, μ) and the corresponding nonzero vector x for which equation (1.4) is satisfied. The pair (λ, μ) is called an eigenvalue and the nonzero vector x is called the eigenvector of (1.4). Eigenvalue problem associated with RTMP are required to find the critical point of optimizing the error of the objective function of LTI [22,23] as well as ARMA [31] models. Numerical solution of RPTEP can be obtained by the algorithm proposed in [32], which uses certain Block Macaulay matrices. Here we investigate the problem (1.4) as a ready reckoner to get the critical points of the LTI and ARMA models.

Outlines of the article: In Section 2, we present some notation and preliminary definitions, which will help to obtain the main findings of the article. In Section 3, we discuss the linearization of RTMP. In Section 4, we discuss the deflation scheme used to solve the two-parameter matrix pencil problem (MPP), which in turn leads to the solution of our original problem (1.4). Section 5 contains two numerical examples and finally in section 6 a concluding remark of the work.

2. Preliminaries

Few notations and definitions will be used throughout the article. In the whole article, x^* denotes the conjugate transpose of a vector x. I_n and $O_{m\times n}$ denotes the identity matrix of order $n \times n$ and zero matrix of order $m \times n$, respectively. A^{-1} and A^T represent the inverse and transpose of any matrix A, respectively. Moreover, standard Kronecker product is denoted by the symbol $\otimes$.

Definition 2.1 *[4] A RTMP defined in (1.3) is said to be tall if $m > n$.*

Definition 2.2 *[4](Standard generalised linearization) A linear matrix polynomial of the form $L(\lambda, \mu) = N_0 + \lambda N_1 + \mu N_2$, of size $mk \times nk$ is said to be a standard generalized linearization of the RTMP specified in (1.3), if there exist two unimodular matrices (matrix with constant determinants) $U(\lambda, \mu) \in \mathbb{C}^{mk\times mk}$ and $V(\lambda, \mu) \in \mathbb{C}^{nk\times nk}$ such that,*

$$U(\lambda, \mu) L(\lambda, \mu) V(\lambda, \mu) = \begin{pmatrix} R(\lambda, \mu) & O \\ O & I_{(k-1)} \otimes I_{m,n} \end{pmatrix} \in \mathbb{C}^{mk\times nk} \tag{2.1}$$

where $I_{m,n} = \begin{pmatrix} I_n \\ O_{(m-n)\times n} \end{pmatrix}$ *if $m > n$.*

3. Linearizations of RTMP

Linearization [1,8,11,24] plays its role in solving such problem, where the problem is converted into two-parameter linear classes of high dimension. Consider a tall RTMP as defined in (1.3) of grade k. Also define the matrices as given below,

$$R = \begin{bmatrix} M_{k0} & M_{k-1,1} & \dots & M_{2,k-2} & M_{1,k-1} \end{bmatrix} \in \mathbb{C}^{m\times kn}, \tag{3.1}$$

$$S = \begin{bmatrix} O & \dots & O & M_{0k} \end{bmatrix} \in \mathbb{C}^{m\times kn}, \tag{3.2}$$

$$T_j = \begin{bmatrix} M_{j0} & M_{j-1,1} & \dots & M_{1,j-1} & M_{0j} \end{bmatrix} \in \mathbb{C}^{m\times (j+1)n}; \; j = 0, \dots, k, \tag{3.3}$$

$$\mathcal{I}_j := \begin{bmatrix} O_{n\times jn} \\ I_j \otimes I_n \end{bmatrix} \in \mathbb{C}^{(j+1)n\times jn}; j = 1, \dots, (k-1), \tag{3.4}$$

$$I_{m,n} = \begin{pmatrix} I_n \\ O_{(m-n)\times n} \end{pmatrix} \in \mathbb{C}^{m\times n} \tag{3.5}$$

$$\widehat{I}_j := \begin{bmatrix} O_{m\times jn} \\ I_j \otimes I_{m,n} \end{bmatrix} \in \mathbb{C}^{(j+1)m\times jn}; \; j = 1, \dots, (k-1), \tag{3.6}$$

$$\mathcal{I}_{js} := \begin{bmatrix} I_{m,n} & & \\ & \ddots & \\ & & I_{m,n} \end{bmatrix} \in \mathbb{C}^{jm\times jn}; \; j = 2, \dots, k, \tag{3.7}$$

$$\mathcal{Q}_j := \begin{bmatrix} 1 & O_{1\times (j-1)} \\ O_{j\times 1} & O_{j\times (j-1)} \end{bmatrix} \in \mathbb{C}^{(j+1)\times j}, \tag{3.8}$$

$$\widehat{J}_j := \mathcal{Q}_j \otimes I_{m,n} \in \mathbb{C}^{(j+1)m\times jn}; \; j = 1, \dots, (k-1), \tag{3.9}$$

$$\text{and} \quad \Im_j := \mathcal{Q}_j \otimes I_n \in \mathbb{C}^{(j+1)n\times jn}; \; j = 1, \dots, (k-1). \tag{3.10}$$

Define,

$$\Lambda_j := \begin{bmatrix} \lambda^j & \lambda^{j-1}\mu & \dots & \lambda\mu^{j-1} & \mu^j \end{bmatrix}^T \in \mathbb{C}^{j+1}; \; j = 0, \dots, (k-1) \tag{3.11}$$

and

$$\Lambda := \begin{bmatrix} \Lambda_0(\lambda, \mu) & \dots & \Lambda_{k-2}(\lambda, \mu) & \Lambda_{k-1}(\lambda, \mu) \end{bmatrix}^T \in \mathbb{C}^p. \tag{3.12}$$

We construct the matrices A, B and C with the help of (3.1), (3.2) and (3.3).

$$A = \left(\begin{array}{ccc|c} T_0 & T_1 & \dots & T_{k-1} \\ \hline -\mathcal{I}_{2s} & & & \\ & \ddots & & \\ & & & -\mathcal{I}_{ks} \end{array}\right), B = \left(\begin{array}{ccc|c} & & & R \\ \hline \widehat{J}_1 & & & \\ & \ddots & & \\ & & \widehat{J}_{k-1} & \end{array}\right) \text{ and } C = \left(\begin{array}{ccc|c} & & & S \\ \hline \widehat{I}_1 & & & \\ & \ddots & & \\ & & \widehat{I}_{k-1} & \end{array}\right) \tag{3.13}$$

Let $w = \Lambda \otimes x$. Then,

$$L(\lambda,\mu)\,(\Lambda \otimes x) = \begin{pmatrix} R(\lambda,\mu)x & 0 & \ldots 0 \end{pmatrix}^T \tag{3.14}$$

where $L(\lambda,\mu)$ denotes the linearization class

$$L(\lambda,\mu) := A + \lambda B + \mu C \tag{3.15}$$

The matrices A, B and C defined in (3.13) are of size $\frac{k(k+1)}{2}m \times \frac{k(k+1)}{2}n$. We consider $p = \frac{k(k+1)}{2}$. Then the matrices A, B and C are of size $p \times p$ with $m \times n$ matrix entries. Consider the unimodular matrix $V(\lambda,\mu)$ of order $pn \times pn$ given by

$$V(\lambda,\mu) = \begin{pmatrix} \mathcal{V}_{11} & O \\ \mathcal{V}_{21} & \mathcal{V}_{22} \end{pmatrix} \tag{3.16}$$

where, $\mathcal{V}_{11} = \Lambda_0 \otimes I_n; \mathcal{V}_{21} = \begin{pmatrix} \Lambda_1 \otimes I_n \\ \Lambda_2 \otimes I_n \\ \vdots \\ \Lambda_{k-2} \otimes I_n \\ \Lambda_{k-1} \otimes I_n \end{pmatrix}$ and

$$\mathcal{V}_{22} = \begin{pmatrix} -I_{2n} & & & & \\ -(\mu\mathcal{I}_2 + \lambda\Im_2) & -I_{3n} & & & \\ & \ddots & \ddots & & \\ & & -(\mu\mathcal{I}_{k-2} + \lambda\Im_{k-2}) & -I_{(k-1)n} & \\ & & & -(\mu\mathcal{I}_{k-1} + \lambda\Im_{k-1}) & -I_{kn} \end{pmatrix}.$$

Then it follows that,

$$L(\lambda,\mu)V(\lambda,\mu) = \left(\begin{array}{c|cccc} R(\lambda,\mu) & \mathcal{M}_2(\lambda,\mu) & \ldots & \mathcal{M}_k(\lambda,\mu) \\ \hline & \mathcal{I}_{2s} & & \\ & & \ddots & \\ & & & \mathcal{I}_{ks} \end{array}\right)_{pm \times pn}$$

where,

$$\begin{aligned} \mathcal{M}_k(\lambda,\mu) &= -(T_{k-1} + \lambda R + \mu S) \in \mathbb{C}^{m \times kn} \\ \mathcal{M}_2(\lambda,\mu) &= -T_1 - T_2.(\mu\mathcal{I}_2 + \lambda\Im_2) \in \mathbb{C}^{m \times 2n} \\ &\vdots \\ \mathcal{M}_{k-1}(\lambda,\mu) &= -T_{k-2} - T_{k-1}.(\mu\mathcal{I}_{k-1} + \lambda\Im_{k-1}) \in \mathbb{C}^{m \times (k-1)n} \end{aligned}$$

for $j = 2, \ldots, (k-2)$. Denote,

$$\mathcal{M}_d^{m,n} = \begin{pmatrix} \mathcal{M}_d & 0_{m \times d(m-n)} \end{pmatrix}_{m \times dm}; \text{ for } d = 2, \ldots, k \text{ and } m \geq n.$$

If we define the unimodular $U(\lambda,\mu)$ of size $pm \times pm$ such that,

$$U(\lambda,\mu) = \left(\begin{array}{c|cccc} I_m & -\mathcal{M}_2^{m,n}(\lambda,\mu) & \ldots & -\mathcal{M}_k^{m,n}(\lambda,\mu) \\ \hline & I_{2m} & & \\ & & \ddots & \\ & & & I_{km} \end{array}\right); \text{ for } m \geq n$$

the equation (2.1) holds for the matrices $U(\lambda,\mu)$, $L(\lambda,\mu)$ and $V(\lambda,\mu)$. Thus, by Definition 2.2, the linear matrix pencil defined in (3.15) is a linearization of RTMP defined in (1.3). The linear matrix polynomial specified in (3.15) is the standard generalized linearization of the RTMP defined in (1.3). For $m = n$, the linearization class derived in (3.15) reduces to the standard linearization presented in [3,8,24]. Equation (3.15) represents a rectangular linear two-parameter matrix pencil. Again, from (3.14) it follows that, if w is an eigenvector of the eigenvalue problem associated with $L(\lambda, \mu)$, i.e.

$$L(\lambda, \mu)w := (A + \lambda B + \mu C)(\Lambda \otimes x) = 0 \tag{3.17}$$

then, x is an eigenvector of (1.4). Now we are ready to define an equivalent problem of (1.4).

Problem 1 *Let A, B and C be defined in (3.13) and assume that*

$$Rank\left(\begin{bmatrix} A \\ B \\ C \end{bmatrix}\right) = pn \quad and \quad Rank\left(\begin{bmatrix} A & B & C \end{bmatrix}\right) = pm. \tag{3.18}$$

Then, find all the eigenvalues λ and its associated eigenvectors w such that

$$(\lambda_0 A + \lambda_1 B + \lambda_2 C)w = 0, 0 \neq w \in \mathbb{C}^{pn}, 0 \neq \lambda := \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \in \mathbb{C}^3, \tag{3.19}$$

where an eigenvalue λ is considered equivalent to $\alpha\lambda$ for any nonzero $\alpha \in \mathbb{C}$, and an eigenvector x is treated as equivalent to βx for any nonzero $\beta \in \mathbb{C}$.

This problem is related to well known class of linear two-parameter eigenvalue problem, except there is only one linear matrix pencil, and crucially the matrices are not necessarily square. There is no loss of generality in making assumptions (3.18). If these assumptions are not satisfied, then using unitary transformation, we can apply column compression on the first matrix in (3.18) and /or row compression on the second matrix of (3.18) so as to remove zero columns and/or rows. This induces an equivalent linear matrix pencil with smaller pn and/or pm.

4. Solution of RPTEP by Deflation Scheme

In this section, we define the Kronecker commutator operator and provide a deflation scheme to find solution of RTPEP.

Theorem 4.1 *[9] Let $P, Q \in \mathbb{C}^{pm\times pn}$ and define the Kronecker commutator operator $\Delta = P \otimes Q - Q \otimes P \in \mathbb{C}^{(pm)^2\times(pn)^2}$. Then, the one-parameter MPP*

$$(\gamma_1 P + \gamma_2 Q)x = 0, \quad 0 \neq x \in \mathbb{C}^{pn} \tag{4.1}$$

has a solution if and only if

$$\Delta z = 0 \tag{4.2}$$

has a solution, where $0 \neq z \in \mathbb{C}^{(pn)^2}$.

Hence, $(\gamma_1 P + \gamma_2 Q)x = 0$ has a solution when null space of Δ contains a strongly decomposable vector $z = x \otimes x$. To get the solution of the Problem 1, the following Theorem 4.2 will be useful.

Theorem 4.2 *[9] Let $A, B, C \in \mathbb{C}^{pm\times pn}$ and define the Kronecker commutator operators $\Delta_0 = B\otimes C - C \otimes B$, $\Delta_1 = C \otimes A - A \otimes C$, $\Delta_2 = A \otimes B - B \otimes A$, so that $\Delta_i \in \mathbb{C}^{(pm)^2\times(pn)^2}$, $i = 0, 1, 2$. Then the $pm \times pn$ matrix pencil*

$$(\lambda_0 A + \lambda_1 B + \lambda_2 C)w = 0, \quad 0 \neq w \in \mathbb{C}^{pn} \tag{4.3}$$

has a solution if and only if the following three $(pm)^2 \times (pn)^2$ matrix pencils,

$$(\lambda_i \Delta_j - \lambda_j \Delta_i)(w \otimes w) = 0 \tag{4.4}$$

for $(i, j) \in \{(0, 1), (0, 2), (1, 2)\}$ and $0 \neq z \in \mathbb{C}^{(pn)^2}$ have a simultaneous solution. So, if these three matrix pencils in (4.4) have a simultaneous solution, then there exists a strongly decomposable $0 \neq z = w \otimes w$ such that, $(\lambda_i \Delta_j - \lambda_j \Delta_i)(w \otimes w) = 0$ and $(\lambda_0, \lambda_1, \lambda_2) \neq 0$ such that $(\lambda_0 A + \lambda_1 B + \lambda_2 C)w = 0$.

The significance of theorem 4.2 is that it allows us to capture all the solutions of the two-parameter MPPs in (4.4). Thus, finding the solution of the two-parameter MPPs defined in (3.19) is equivalent to finding the solution of a set of three one-parameter matrix pencil problems

$$(\lambda_i \Delta_j - \lambda_j \Delta_i)z = 0 \quad \text{for} \quad (i,j) \in \{(0,1),(0,2),(1,2)\}$$

with

$$\Delta_0 = B \otimes C - C \otimes B \tag{4.5}$$

$$\Delta_1 = C \otimes A - A \otimes C \tag{4.6}$$

$$\Delta_2 = A \otimes B - B \otimes A \tag{4.7}$$

where $\Delta_0, \Delta_1, \Delta_2 \in \mathbb{C}^{(pm)^2 \times (pn)^2}$. Now, to solve these one-parameter MPPs we will use Theorem 4.1. The study of the properties of the matrix pencil problem containing the Kronecker commutator operator Δ was carried out in [9]. Based on this idea of [9], we Consider three selection matrices as M_d^p, M_l^p and M_u^p. The diagonal matrix $M_d^p \in \mathbb{R}^{p \times p^2}$, lower diagonal matrix $M_l^p \in \mathbb{R}^{\frac{p(p-1)}{2} \times p^2}$ and upper diagonal matrix $M_u^p \in \mathbb{R}^{\frac{p(p-1)}{2} \times p^2}$ are defined as,

$$M_d^p = \sum_{i=1}^{p} e_i^p \otimes e_i^{p\,T} \otimes e_i^{p\,T}$$

$$M_l^p = \sum_{j=1}^{p-1} \sum_{i=j+1}^{p} e_{k_{ij}}^{\frac{p(p-1)}{2}} \otimes e_j^{p\,T} \otimes e_i^{p\,T}$$

$$M_u^p = \sum_{j=1}^{p-1} \sum_{i=j+1}^{p} e_{k_{ij}}^{\frac{p(p-1)}{2}} \otimes e_i^{p\,T} \otimes e_j^{p\,T},$$

where e_i^p denotes the i^{th} column of I_p and let $k_{ij} = (j-1)p + i - \frac{j(j+1)}{2}$ for each $1 \leq j < i \leq p$. Using these three matrices M_d^p, M_l^p and M_u^p, we define two new associated matrices as,

$$U^p = (M_l^p - M_u^p)_{\frac{p(p-1)}{2} \times p^2} \text{ and } V^p = \begin{pmatrix} M_d^p \\ M_l^p + M_u^p \end{pmatrix}_{\frac{p(p+1)}{2} \times p^2}. \tag{4.8}$$

For $p = 3$, we will give the illustration for three selection matrices of order 3×9 and two associated matrices of orders 3×9 and 6×9 respectively.

$$M_d^3 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}, \quad M_l^3 = \begin{pmatrix} 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \end{pmatrix},$$

$$M_u^3 = \begin{pmatrix} 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \end{pmatrix};$$

$$U^3 = \begin{pmatrix} 0&1&0&-1&0&0&0&0&0 \\ 0&0&1&0&0&0&-1&0&0 \\ 0&0&0&0&0&1&0&-1&0 \end{pmatrix}, \quad V^3 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&1&0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0&1&0 \end{pmatrix}.$$

From Theorem 4.2, it is seen that, by solving the three set of the one-parameter MPPs as in (4.4) we can have the solution of (4.3). Hence, to solve our original problem (3.19), we will use the following Deflation scheme.

Theorem 4.3 *[9] Let us define the Kronecker determinants,*

$$\Gamma_0 = 2U^{pm}(B \otimes C)V^{(pn)^T} = U^{pm}\Delta_0 V^{(pn)^T} \tag{4.9}$$

$$\Gamma_1 = 2U^{pm}(C \otimes A)V^{(pn)^T} = U^{pm}\Delta_1 V^{(pn)^T} \tag{4.10}$$

$$\Gamma_2 = 2U^{pm}(A \otimes B)V^{(pn)^T} = U^{pm}\Delta_2 V^{(pn)^T} \tag{4.11}$$

with $\Gamma_i \in \mathbb{C}^{\tilde{m}\times\tilde{n}}$ for $i = 0, 1, 2$, where $\tilde{m} = \frac{m(m-1)}{2}$ and $\tilde{n} = \frac{n(n-1)}{2}$. Then equation (4.4) will have a solution (equivalently, the $pm \times pn$ rectangular two-parameter matrix pencil (4.3) will have a solution) if and only if the following three $\tilde{m} \times \tilde{n}$ one-parameter MPPs

$$(\lambda_i\Gamma_j - \lambda_j\Gamma_i)\tilde{z} = 0 \tag{4.12}$$

for $(i, j) \in (0, 1), (0, 2), (1, 2)$ and $0 \neq \tilde{z} \in \mathbb{C}^{\tilde{n}}$ have a simultaneous solution.

We will use the idea of Theorem 4.3 to deflate the one-parameter MPPs with $(pm)^2 \times (pn)^2$ ordered Δ_i matrices in (4.5)-(4.7). Comparing equation (3.17) with the general two-parameter MPPs in (4.3) leads to condition $\lambda_0 = 1$ as defined in (3.19) of Problem 1. Hence, in order to solve (1.4) we have to find λ and μ from the following pair of one-parameter MPPs,

$$(\Gamma_1 - \lambda\Gamma_0)\tilde{z} = 0, \tag{4.13}$$

$$(\Gamma_2 - \mu\Gamma_0)\tilde{z} = 0; \tag{4.14}$$

where $\Gamma_i \in \mathbb{C}^{\tilde{pm}\times\tilde{pn}}$, with $\tilde{pm} = \frac{pm(pm-1)}{2}$ and $\tilde{pn} = \frac{pn(pn-1)}{2}$.

We provide an Algorithm 1 to compute the solution of the RTPEP (1.4), based on Matrix Pencil Method as a summary of our findings.

Algorithm 1 Matrix Pencil Method for RPTEP

Input: The coefficient matrices $M_{ij} \in \mathbb{C}^{m\times n}$ where $0 \le i \le k$ and $0 \le j \le k - i$.
Output: The eigenvalue (λ, μ) satisfying the system (1.4).

1: Construct the coefficient matrices A, B and C as in (3.13).
2: Convert the given RTMP (1.3) into the two-parameter matrix pencil (3.15) of size $pm \times pn$
3: Define U^{pm} and V^{pn} as in equation (4.8).
4: Evaluate the matrices Γ_0, Γ_1 and Γ_2 using (4.9)-(4.11), where Δ_i's are as defined in (4.5)-(4.7) for $i = 0, 1, 2$.
5: Use `joint_delta_eig` from MultiParEig [26] to solve the one-parameter MPPs (4.13) and (4.14) of sizes $\frac{pm(pm-1)}{2} \times \frac{pn(pn-1)}{2}$.
6: Extract the simultaneous solution (λ, μ) to get the required solution of (1.4).

---

5. Numerical results and discussions

We present three numerical examples to check the efficiency of the technique discussed in the paper. All the computations are performed in the MATLAB $R2024a$ environment with Windows 11 operating system, AMD Ryzen 5 5500U 2.10 GHz processor.

5.1. Randomly generated matrices

Example 5.1 *Consider a quadratic rectangular two-parameter eigenvalue problem defined by*

$$(M_{00} + \lambda M_{10} + \mu M_{01} + \lambda^2 M_{20} + \lambda\mu M_{11} + \mu^2 M_{02})x = 0 \tag{5.1}$$

where the matrices $M_{ij} \in \mathbb{C}^{3\times 2}$ are randomly generated for $i, j = 0, 1, 2$ such that

$$M_{00} = \begin{pmatrix} 0.8147 & 0.9134 \\ 0.9058 & 0.6324 \\ 0.1270 & 0.0975 \end{pmatrix}; M_{10} = \begin{pmatrix} 0.2785 & 0.9649 \\ 0.5469 & 0.1576 \\ 0.9575 & 0.9706 \end{pmatrix}; M_{01} = \begin{pmatrix} 0.9572 & 0.1419 \\ 0.4854 & 0.4218 \\ 0.8003 & 0.9157 \end{pmatrix};$$

$$M_{20} = \begin{pmatrix} 0.7922 & 0.0357 \\ 0.9595 & 0.8491 \\ 0.6557 & 0.9340 \end{pmatrix}; M_{11} = \begin{pmatrix} 0.6787 & 0.3922 \\ 0.7577 & 0.6555 \\ 0.7431 & 0.1712 \end{pmatrix}; M_{02} = \begin{pmatrix} 0.7060 & 0.0462 \\ 0.0318 & 0.0971 \\ 0.2769 & 0.8235 \end{pmatrix}.$$

Here, $m = 3, n = 2, k = 2$ and $p = 3$. Since $m > n$, and therefore the problem is tall. For $x \in \mathbb{C}^2$ the matrices A, B and C in the linearization class $L(\lambda, \mu)$ defined in (3.15) are given by

$$A = \left(\begin{array}{cc|cccc} 0.8147 & 0.9134 & 0.2785 & 0.9649 & 0.9572 & 0.1419 \\ 0.9058 & 0.6324 & 0.5469 & 0.1576 & 0.4854 & 0.4218 \\ 0.1270 & 0.0975 & 0.9575 & 0.9706 & 0.8003 & 0.9157 \\ \hline & & 1 & 0.0000 & & \\ & & 0.0000 & 1 & & \\ & & 0.0000 & 0.0000 & & \\ & & & & -1 & 0.0000 \\ & & & & 0.0000 & -1 \\ & & & & 0.0000 & 0.0000 \end{array}\right)_{9\times 6},$$

$$B = \left(\begin{array}{cc|cccc} & & 0.7922 & 0.0357 & 0.6787 & 0.3922 \\ & & 0.9595 & 0.8491 & 0.7577 & 0.6555 \\ & & 0.6557 & 0.9340 & 0.7431 & 0.1712 \\ \hline 1 & 0.0000 & & & & \\ 0.0000 & 1 & & & & \\ 0.0000 & 0.0000 & & & & \\ 0 & 0.0000 & & & & \\ 0.0000 & 0 & & & & \\ 0.0000 & 0.0000 & & & & \end{array}\right)_{9\times 6}$$

$$C = \left(\begin{array}{cc|cccc} & & 0 & 0 & 0.7060 & 0.0462 \\ & & 0 & 0 & 0.0318 & 0.0971 \\ & & 0 & 0 & 0.2769 & 0.8235 \\ \hline 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 1 & 0 & & & & \\ 0 & 1 & & & & \\ 0 & 0 & & & & \end{array}\right)_{9\times 6}.$$

We use Algorithm 1 to find the eigenvalues. Then the Kronecker operators Δ_i's, $i = 0, 1, 2$ defined in Theorem 4.2 are found to be of order 81×36. The associated matrices are $U^9 \in \mathbb{C}^{36\times 81}$ and $V^6 \in \mathbb{C}^{21\times 36}$. Then the corresponding Kronecker determinants defined in (4.9), (4.10) and (4.11) are of order 36×21. The computed 12 eigenvalues are listed in Table 1, out of which two are real with $\lambda = -0.2575, \mu = 0.0705$ and $\lambda = -0.3701, \mu = -0.7140$. They are ploted in Figure 1.

Table 1: Eigenvalues of Example 5.1

Sl. No.	λ	μ
1	-0.2575 + 0.0000i	+0.0705 + 0.0000i
2	+0.0911 + 0.3560i	+0.8211 + 0.5512i
3	+0.0911 - 0.3560i	+0.8211 - 0.5512i
4	+0.3897 + 0.8631i	+1.5799 - 0.7295i
5	+0.3897 - 0.8631i	+1.5799 - 0.7295i
6	-0.3701 + 0.0000i	-0.7140 + 0.0000i
7	-0.6419 + 0.5624i	-1.4287 + 0.7097i
8	-0.6419 - 0.5624i	-1.4287 - 0.7097i
9	-0.4997 + 1.3236i	+0.2057 + 0.8928i
10	-0.4997 - 1.3236i	+0.2057 - 0.8928i
11	+1.2662 + 1.8337i	-1.2062 + 0.5893i
12	+1.2662 - 1.8337i	-1.2062 - 0.5893i

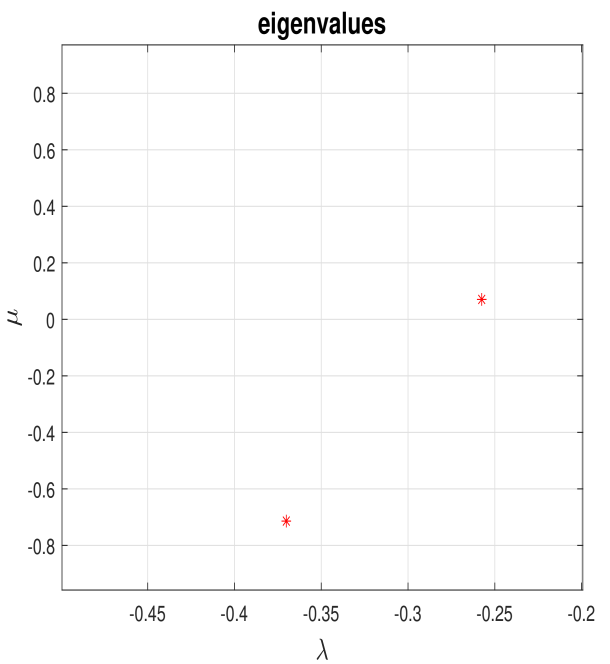

Figure 1: The red stars indicate the real eigenvalues of Example 5.1

5.2. ARMA and LTI model

Algorithm 1 allows us to calculate more efficiently the stationary points of LTI as well as ARMA models. Assume that $y_i \in \mathbb{R}, i := 1, \ldots, N$ represents a sequence of N values of a time series, that could be affected by noise. Various models are available in the existing literature for the statistical analysis of time series that uses either one or more than one parameters. Our aim is to compute the optimal values of these parameters in order to minimize the generated error. It is to be noted that critical points of ARMA as well as LTI models are the eigenvalues of RPTEP [22,23,31] and computational techniques for them are found in [32]. Although advanced numerical methods for identifying parameters in LTI and ARMA models, which rely on nonlinear optimization, achieve local convergence without a guarantee of yielding the optimal solution, all stationary points including the global minimizer, are provided by the solutions of the associated RPTEP. We consider the scalar ARMA(p,q) model studied in [12] and are given by

$$\sum_{i=0}^{p} \alpha_i y_{k-i} = \sum_{i=0}^{q} \gamma_i e_{k-j}, \text{with } k = p+1, \ldots, N. \tag{5.2}$$

Here p and q denotes the respective orders of the auto regressive(AR) and the moving-average (MA) part. We set $\alpha_0 = \gamma_0 = 1$. For a given $y \in \mathbb{R}^N$, the problem is to compute the values of the parameter (real) $\alpha_i, i = 1, \ldots, p$ and $\gamma_j, j = 1, \ldots, q$ which minimize the norm $\|e\|$ of the error $e \in \mathbb{R}^{N-p+q}$. The aim is to find solutions for which the roots of the characteristic polynomials of both the AR as well as MA components lie within the open unit disk. The conditions of optimality for minimizing $\|e\|$ are suggested in [31]. They result in a homogeneous system, where both the parameters α_i and γ_j appear as polynomial up to degree 2. These polynomial equations constitute a RPTEP of grade 2, with eigenvalues that represent critical values for the associated objective function. Here, we are searching only in real eigenvalues of the associated RPTEP due to the fact that the parameters of the ARMA model are real.

Example 5.2 *[12] Consider the ARMA(1,1) model with*

$$(M_{00} + \alpha M_{10} + \gamma M_{01} + \gamma^2 M_{02})x = 0 \tag{5.3}$$

where matrices M_{ij} *are of order* $(3N-1) \times (3N-2)$ *such that,*

$$M_{00} = \begin{pmatrix} \boldsymbol{y}_{(2)} & I & 0 & 0 \\ \boldsymbol{y}_{(1)} & 0 & I & 0 \\ 0 & R & 0 & I \\ 0 & \boldsymbol{y}_{(1)}^T & \boldsymbol{y}_{(2)}^T & 0 \\ 0 & 0 & 0 & \boldsymbol{y}_{(2)}^T \end{pmatrix}, M_{10} = \begin{pmatrix} \boldsymbol{y}_{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{y}_{(1)}^T & 0 \\ 0 & 0 & 0 & \boldsymbol{y}_{(1)}^T \end{pmatrix},$$

$$M_{01} = \begin{pmatrix} 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 2I & 0 & R \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_{02} = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\boldsymbol{y}_{(1)} = [y_1, \ldots, y_{N-1}]^T, \boldsymbol{y}_{(2)} = [y_2, \ldots, y_N]^T$, *and* R *represents a tridiagonal matrix along with stencil* $[1, 0, 1]$.

Consider $\mathbf{y} = [2.4130, 1.0033, 1.2378, -0.72191]$. Upon solving it with the help of Algorithm1, respective Kronecker determinants are found to be of order 528×465. The only real solution to this problem are $\alpha = -0.1159$ and $\gamma = 0.4708$. It lies within the domain of interest as shown in Figure 2. For $\mathbf{y} = [2.4130\ 1.0033\ 1.2378\ -0.72191\ -0.81745\ -2.2918]$, we get three real solutions. The contour plot for the eigenvalues $(-0.3410, -0.7537), (-0.5392, 0.0262)$and $(0.0304, 0.6761)$ is shown in Figure 3.

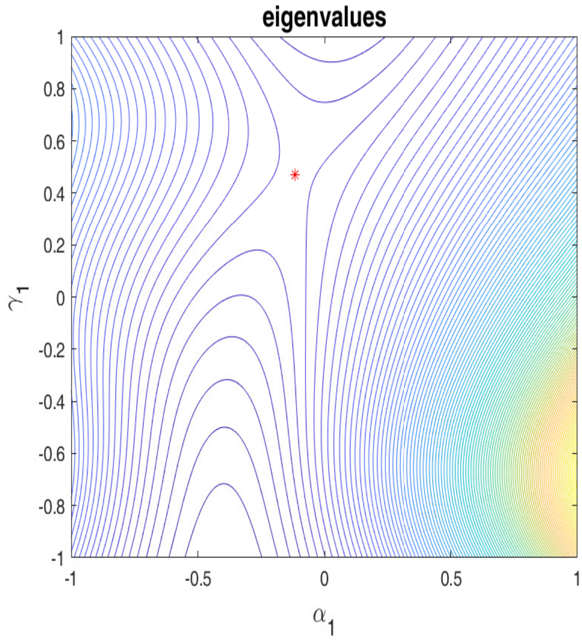

Figure 2: The critical points (red stars) for the eigenvalue problem with the contour plot of $\|e\|^2$ for $(\alpha, \gamma) \in [-1, 1]^2$.

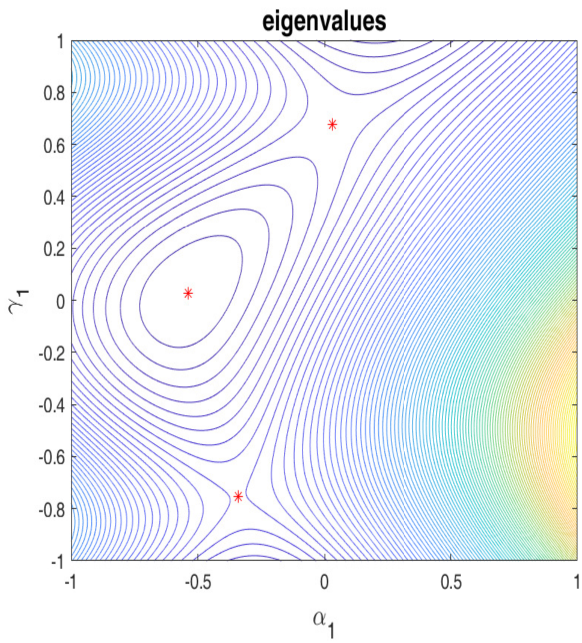

Figure 3: The critical points (red stars) for the eigenvalue problem with the contour plot of $\|e\|^2$ for $(\alpha, \gamma) \in [-1, 1]^2$

In the next example, we consider LTI (p) model, where p denotes the order of the LTI model. The problem is to find the parameters $\alpha_i, i = 1, \dots, p$ that provide the best 2-norm approximation of a specified $y \in \mathbb{R}^N$ by $\hat{y} \in \mathbb{R}^N$, with elements that satisfies the following difference equation [12],

$$\hat{y}_{k+p} + \sum_{i=1}^{k} \alpha_i \hat{y}_{k+p-j} = 0; j = 1, \dots, p. \tag{5.4}$$

where $k = 1, \dots, (N-p)$. Reference [23] provides the detailed outlines of the optimality conditions for minimizing the error norm $\|y - \hat{y}\|^2$. The critical values α_i of the objective function are eigenvalues of the associated RPTEP with grade 2. Here, we are interested in real eigenvalues only.

Example 5.3 *[12] The LTI(2) model can be considered as a RPTEP of grade 2 of the form,*

$$(M_{00} + \lambda M_{10} + \mu M_{01} + \lambda^2 M_{20} + \lambda\mu M_{11} + \mu^2 M_{02})x = 0 \tag{5.5}$$

where M_{ij} are $(3N-4) \times (3N-5)$ ordered matrices with,

$$M_{00} = \begin{pmatrix} \boldsymbol{y}_{(3)} & I & 0 & 0 \\ \boldsymbol{y}_{(2)} & R & I & 0 \\ \boldsymbol{y}_{(1)} & S & 0 & I \\ 0 & \boldsymbol{y}_{(2)}^T & \boldsymbol{y}_{(3)}^T & 0 \\ 0 & \boldsymbol{y}_{(1)}^T & 0 & \boldsymbol{y}_{(3)}^T \end{pmatrix}, M_{10} = \begin{pmatrix} \boldsymbol{y}_{(2)} & R & 0 & 0 \\ 0 & 2I & R & 0 \\ 0 & R & 0 & R \\ 0 & 0 & \boldsymbol{y}_{(2)}^T & 0 \\ 0 & 0 & 0 & \boldsymbol{y}_{(2)}^T \end{pmatrix}, M_{01} = \begin{pmatrix} \boldsymbol{y}_1 & S & 0 & 0 \\ 0 & R & S & 0 \\ 0 & 2I & 0 & S \\ 0 & 0 & \boldsymbol{y}_{(1)}^T & 0 \\ 0 & 0 & 0 & \boldsymbol{y}_{(1)}^T \end{pmatrix}.$$

The matrices $M_{20} = M_{02}$ are as defined in Example 5.2.

Here $M_{11} = M_{02}.\text{diag}(R, R, R, R)$, $\mathbf{y}_1 = [y_1, y_2, \dots, y_{N-2}]^T$, $\mathbf{y}_2 = [y_2, y_3, \dots, y_{N-1}]^T$, $\mathbf{y}_3 = [y_3, y_4, \dots, y_N]^T$. Consider R as tridiagonal with stencil $[1, 0, 1]$, and S as pentadiagonal with stencil $[1, 0, 0, 0, 1]$. For the same $\mathbf{y} = [2.4130\ 1.0033\ 1.2378\ -0.72191\ -0.81745\ -2.2918]$ as in Example 5.2, we get the eigenvalues listed in Table 2. The Figure 4 gives the plot of these eigenvalues.

Table 2: Eigenvalues of Example 5.3

Sl. No.	λ	μ
1	-0.5586	-10.8058
2	+1.5386	+1.0290
3	+0.8202	+0.9936
4	-0.1273	+0.9702
5	+0.4345	-1.8103
6	-0.0722	-0.1107
7	-0.1604	-0.9163
8	-2.3431	+1.1211
9	-0.2917	-0.7251

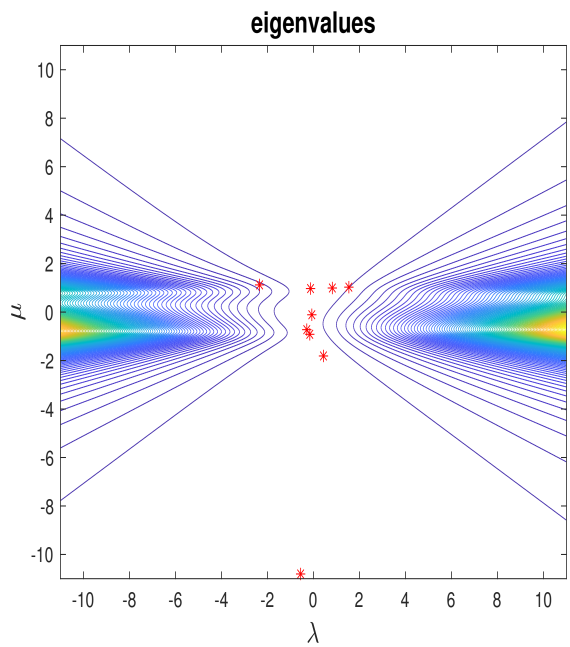

Figure 4: The critical points (red stars) for the eigenvalue problem with the contour plot for the objective function $\|y - \hat{y}\|^2$

6. Conclusion

We provided a general framework to solve the RPTEP by first adopting standard generalized linearizations on RTMP and then using two-parameter matrix pencil method. The use of Kronecker commutator operators to decompose the two-parameter MPPs into a pair of one-parameter MPP simplifies the computational works. The structured deflation help to reduce the size of the original problem with operators of size $(pm)^2 \times (pn)^2$ to $\frac{pm(pm-1)}{2} \times \frac{pn(pn-1)}{2}$. This method reduces computational effort and avoids redundant calculations. Numerical examples are provided to show the efficiency of the method.

Acknowledgments

All authors are thankful to Dibrugarh University, Assam, India for providing all facilities during the research works.

References

1. Adhikari, B., *Vector space of linearizations for the quadratic two-parameter matrix polynomial*, Linear Multilinear Algebra. 61(5), 603-616, (2013).

2. Alsubaie, F. F., *$\mathcal{H}_2$ Optimal Model Reduction for Linear Dynamic Systems and the Solution of Multiparameter Matrix Pencil Problems*, PhD thesis, Department of Electrical and Electronic Engineering, Imperical College, London, (2019).

3. Bora, N., Buragohain, B. and Sharma, B. Linearizations of cubic two-parameter eigenvalue problem and its vector space, Contemp. Math. 5(1), 930-948, (2024).

4. Das, B. and Bora, S., *Vector spaces of generalized linearizations for rectangular matrix polynomials*, Electron. J. Linear Algebra. 35, 116-155, (2019).

5. Dong, B., *The Homotopy Method for the Complete Solution of Quadratic Two-parameter Eigenvalue Problems*, J. Sci. Comput. 90, 18, (2022).

6. Dopico, F. M., Lawrence, P. W., Pérez, J. and Dooren, P. V., *Block Kronecker linearizations of matrix polynomials and their backward errors*, Numer. Math. 140, 373–426, (2018).

7. Faßbender, H., and Saltenberger, P., *Block Kronecker ansatz spaces for matrix polynomials*, Linear Algebra Appl. 542, 118-148, (2018).

8. Ghosh, A., *Linearization, sensitivity and backward perturbation analysis of multiparameter eigenvalue problems*, Ph.D. Thesis, Department of Mathematics, Indian Institute of Technology, India (2017).

9. Gungah, S. K., Alsubaie, F. F. and Jaimoukha, I. M., *On the Two-Parameter Matrix Pencil Problem*, SIAM J. Matrix Anal. Appl. 45(3), 1287-1309, (2024).

10. Hochstenbach, M. E., Muhič, A., and Plestenjak, B., *Jacobi-Davidson methods for polynomial two-parameter eigenvalue problems*, J. Comput. Applied Math. 288, 251-263, (2015).

11. Hochstenbach, M. E., Muhič, A., and Plestenjak, B., *On linearizations of the quadratic two-parameter eigenvalue problems*, Linear Algebra Appl. 436, 2725-2743, (2012).

12. Hochstenbach, M. E., Košir, T. and Plestenjak, B., *Numerical methods for rectangular multiparameter eigenvalue problems*, with applications to finding optimal ARMA and LTI models, Numer. Linear Algebra Appl. 31(2), Article No. e2540, (2024).

13. Jarlebring, E., *Critical delays and polynomial eigenvalue problems*, J. Comput. Applied Math. 224(1), 296-306, (2009).

14. Jarlebring, E., and Hochstenbach, M. E., *Polynomial two-parameter eigenvalue problems and matrix pencil methods for stability of delay-differential equations*, Linear Algebra Appl. 431(3-4), 369–380, (2009).

15. Kosir, T. and Plestenjak, B., *On the singular two-parameter eigenvalue problem II*, Linear Algebra Appl. 649, 433-451, (2022).

16. Khazanov, V. B., *On spectral properties of multiparameter polynomial matrices*, J. Math. Sci. 89, 1775–1800, (1998).

17. Kublanovskaya, V. N., *Solving eigenvalue problems for two-parameter polynomial matrices. Methods and algorithms. Part I.*, Russian J. Numer. Anal. Math. Model. 9(2), 111-120, (1994).

18. Kublanovskaya, V. N., On some factorizations of two-parameter polynomial matrices. J. Math. Sci. 86, 2866–2879, (1997).

19. Kublanovskaya, V. N., *Methods and algorithms of solving spectral problems for polynomial and rational matrices*, J. Math. Sci. 96, 3085-3287, (1999).

20. Kublanovskaya, V. N., *Solution of spectral problems for polynomial matrices*, J. Math. Sci. 127(3), 2024–2032, (2005).

21. Meerbergen, K., Schröder, C., and Voss, H., *A Jacobi–Davidson method for two-real-parameter nonlinear eigenvalue problems arising from delay-differential equations*, Linear Algebra Appl. 20(5), 852-868, (2013).

22. Moor, B. D., *Least squares realization of LTI models is an eigenvalue problem*, Paper presented at: 18th European control conference IEEE. 2270–2275, (2019).

23. Moor, B. D., *Least squares optimal realisation of autonomous LTI systems is an eigenvalue problem*, Communication Info. Syst. 20(2), 163-207, (2020).

24. Muhič, A. and Plestenjak, B., *On the quadratic two-parameter eigenvalue problem and its linearization*, Linear Algebra Appl. 432, 2529-2542, (2010).

25. Plestenjak, B., *Numerical methods for nonlinear two-parameter eigenvalue problems*, BIT Numer. Math. 56, 241-262, (2016).

26. Plestenjak, B., MultiParEig *https://www.mathworks.com/matlabcentral/fileexchange/47844-multipareig*, MATLAB Central File Exchange. Retrieved March 30, 2024.

27. Rodriguez, J. I., Du, J. H., You, Y. and Lim, L. H., *Fiber product homotopy method for multiparameter eigenvalue problems*, Numer. Math. 148, 853-888, (2021).

28. Shapiro, B. and Shapiro, M., *On eigenvalues of rectangular matrices*, Proc. Steklov Inst. Math. 267(1), 248-255, (2009).

29. Terán, F. D., Dopico, F. M. and Mackey, D. S., *Fiedler companion linearizations for rectangular matrix polynomials*, Linear Algebra Appl. 437(3), 957-991, (2012).

30. Terán, F. D., Dopico, F. M. and Mackey, D. S., *Spectral equivalence of matrix polynomials and the Index Sum Theorem*, Linear Algebra . Appl. 459, 264-333, (2014).

31. Vermeersch, C. and Moor, B. D., *Globally optimal least-squares ARMA model identification is an eigenvalue problem*, IEEE Control Syst. Lett. 3(4), 1062-1067, (2019).

32. Vermeersch, C. and Moor, B. D., *Two complementary block Macaulay matrix algorithms to solve multiparameter eigenvalue problems*, Linear Algebra Appl. 654(1), 177-209, (2022).

Niranjan Bora,
Department of Mathematics,
Dibrugarh University,
India.
E-mail address: niranjanbora@dibru.ac.in; niranjanbora11@gmail.com

and

Bharati Borgohain,
Department of Mathematics,
Dibrugarh University,
India.
E-mail address: bharatiborgohain3@gmail.com