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Inversion of Double Fourier Integral of Non-Lebesgue Integrable Bounded Variation Functions*

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ABSTRACT: This work proves pointwise convergence of the truncated Fourier double integral of non-Lebesgue integrable bounded variation functions. This leads to the Dirichlet-Jordan theorem proof for non-Lebesgue integrable functions, which has not been sufficiently studied. Note that recent contributions regarding this subject consider Lebesgue integrable functions, [F. Móricz, 2015], [B. Ghodadra-V. Fülop, 2016].

Key Words: Dirichlet-Jordan theorem, KP-Fourier transform, Kurzweil-Henstock integral, double Fourier integral, bounded variation over $\mathbb{R}^2$, improper Riemann-Stieltjes integral, point-wise convergence.

Contents

1. Introduction

One of the most relevant and significant subjects in the Fourier analysis theory is the inversion problem. This means, given the Fourier transform $\hat{f}$ of a function f on $\mathbb{R}^n$, provides conditions such that the function

$$C\int_{\mathbb{R}^n}\hat{f}(\omega)e^{i<\omega,\overline{x}>}d\omega,\ \ \overline{x}\in\mathbb{R}^n,$$

approximates to $f(\overline{x})$, where C is a normalization constant and $<,>$ is the Euclidean inner product.

The Dirichlet-Jordan Theorem solves the pointwise inversion problem. For $n=1$, this states that if $f\in L^1(\mathbb{R})\cap BV(\mathbb{R})$ then, for each $x\in\mathbb{R}$,

$$\lim_{M\to\infty}\frac{1}{2\pi}\int_{-M}^{M}\hat{f}(\omega)e^{ix\omega}d\omega=\frac{f(x+)+f(x-)}{2}. \tag{1.1}$$

The integral at left side of (1.1) is called the truncated Fourier integral, also known as the Dirichlet integral of f. In [4, Corollary 3], F. Móricz proved that if $f\in L^1(\mathbb{R})\cap BV(\mathbb{R})$, then the convergence (1.1) is locally uniform at every point of continuity of f.

For the case $n=2$, considering the classical Lebesgue integral theory, in [5], F. Móricz proved locally uniform convergence of the truncated double Fourier integral

$$\frac{1}{4\pi^2}\int_{|\xi|\le u}\int_{|\eta|\le v}\hat{f}(\xi,\eta)e^{i(\xi x+\eta y)}d(\xi,\eta),$$

to $f(x,y)$, as $u,v\to\infty$, under the conditions: $f\in L^1(\mathbb{R}^2)\cap BV_H(\mathbb{R}^2)$,

$$\hat{f}\in L^1((\mathbb{R})\times[-\delta,\delta])\cup([-\delta,\delta]\times\mathbb{R}),\ \ \delta>0, \tag{C1}$$

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and $(x, y) \in \mathbb{R}^2$ a point of continuity of f. The set of bounded variation functions in the sense of Hardy over $\mathbb{R}^2$ is denoted as $BV_H(\mathbb{R}^2)$. In [2], it is proved that the integrability condition (C1) about $\hat{f}$ can be omitted to get locally uniform convergence of the truncated double Fourier integral.

We have that

$$L^1(\mathbb{R}^2) \cap BV_H(\mathbb{R}^2) \subsetneq BV_{||0||}(\mathbb{R}^2) \not\subseteq L^1(\mathbb{R}^2), \tag{1.2}$$

where $BV_{||0||}(\mathbb{R}^2)$ denotes the set of bounded variation functions in the sense of Vitali that vanish when the norm of their arguments tends to infinity. Thus, previous results and expression (1.2) motivate us to consider the set $BV_{||0||}(\mathbb{R}^2)$ to study the inversion problem.

The relations in (1.2) presupposes the use of integrals other than the Lebesgue one. Here, we consider locally Kurzweil-Henstock integrable functions over $\mathbb{R}^2$. Thus, we will show that if $f \in BV_{||0||}(\mathbb{R}^2)$ and $(\xi, \eta) \in \mathbb{R}^2$, where $\xi \neq 0$ and $\eta \neq 0$, then the map

$$(\xi, \eta) \longrightarrow \lim_{\substack{a,c \to -\infty \\ b,d \to \infty}} \iint_{[a,b]\times[c,d]} f(t_1, t_2) e^{-i(\xi t_1 + \eta t_2)} d(t_1, t_2)$$

is well defined. We call this limit the KP-Fourier transform of f at (ξ, η) and denote by $\mathcal{F}(f)(\xi, \eta)$. Of course, the KP-Fourier transform is defined in a more general sense than the classical Fourier transform, see Definition 3.2.

One important result in this article is the Dirichlet-Jordan theorem for the KP-Fourier transform. That is, if $f \in BV_{||0||}(\mathbb{R}^2)$, then, for $x \neq 0$ and $y \neq 0$,

$$\frac{1}{4\pi^2} \int_{\alpha_1 \leq |\xi| \leq \beta_1} \int_{\alpha_2 \leq |\eta| \leq \beta_2} \mathcal{F}(f)(\xi, \eta) e^{i(\xi x + \eta y)} d(\xi, \eta) \tag{1.3}$$

converges pointwise to

$$\frac{f(x+, y+) + f(x+, y-) + f(x-, y+) + f(x-, y-)}{4},$$

as $\alpha_1, \alpha_2 \to 0$ and $\beta_1, \beta_2 \to \infty$. Apparently, in mathematical literature there is no a similar space on which the proof of this theorem has been analyzed.

This article is organized as follows. In Section 2, we present the improper Riemann-Stieltjes integral definition over $\mathbb{R}^2$ and some of its properties, the concepts of bounded variation in the sense of Vitali and Hardy. In Section 3, we recall the Kurzweil-Henstock integral over rectangles. Also, we introduce the definition of the KP-Fourier transform which was defined in [3], and we provide an alternative proof of its continuity property which was demonstrated in [3] and some auxiliary results. In section 4, we present our main contributions; we prove a version of the Dirichlet-Jordan Theorem of non-Lebesgue integrable bounded variation functions, see Theorem 4.1. Moreover, we extend Theorem 1 from [4] and Theorem 2.1 in [2]. This leads us to consider the validity of the locally uniform convergence for the truncated double Fourier integral of functions in $BV_{||0||}(\mathbb{R}^2)$.

2. Preliminary Topics

Let us recall that a partition of the bounded interval $[a, b]$ is a finite collection of subintervals $\{[x_{i-1}, x_i] : i = 1, ..., n\}$, where $a = x_0 < ... < x_n = b$. Now, we consider $R = [a, b] \times [c, d]$ a bounded rectangle of $\mathbb{R}^2$. A partition of R is a finite collection of the form

$$\{R_{i,j}\} = \{[x_{i-1}, x_i] \times [y_{j-1}, y_j] \mid i = 1, .., n \text{ and } j = 1, ..., m\},$$

where $a = x_0 < ... < x_n = b$ and $c = y_0 < ... < y_m = d$, and the set of such partitions is defined by $\mathcal{P}(R)$. Furthermore, we define the norm of $P = \{R_{i,j}\} \in \mathcal{P}(R)$ as

$$||P|| = \max\{D_{i,j}\}, \tag{2.1}$$

where $D_{i,j}$ is the diagonal length of $R_{i,j}$.

Definition 2.1 *Let $f, g : R \to \mathbb{R}$ be functions. It is said that g is* ***Riemann-Stieltjes integrable*** *with respect to f over R, if there exists $A \in \mathbb{R}$ such that for $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $P = \{R_{i,j}\}$ with $||P|| < \delta_\varepsilon$ and $(\eta_{i,j}, \tau_{i,j}) \in R_{i,j}$, then*

$$\left| \sum_{i,j} g(\eta_{i,j}, \tau_{i,j}) \Delta f(x_i, y_j) - A \right| < \varepsilon,$$

where

$$\Delta f(x_i, y_j) := f(x_i, y_j) - f(x_{i-1}, y_j) - f(x_i, y_{j-1}) + f(x_{i-1}, y_{j-1}).$$

Further, A is the value of the Riemann-Stieltjes integral and is denoted as

$$A = \iint_R g df = \iint_R g(t_1, t_1)\, df(t_1, t_2).$$

Remark 2.1

i) The norm in (2.1) is equivalent to the norm used in [7] and [2], hence the integrals defined by each norm are equal.

ii) The Riemann-Stieltjes integral of a complex function g with respect to a real valued function f is the sum of the integrals of the real and imaginary part of g with respect to f.

From [6, Theorem 5.4.3] for functions of one variable, we obtain the following result for functions of two variables.

Lemma 2.1 *Let f be a Riemann integrable function over R and suppose that g is bounded and Riemann-Stieltjes integrable with respect to h over R, where*

$$h(t_1, t_2) = \iint_{[a,t_1]\times[c,t_2]} f(s_1, s_2) d(s_1, s_2),$$

for each $(t_1, t_2) \in R$. Then

$$\iint_R g(t_1, t_2) dh(t_1, t_2) = \iint_R g(t_1, t_2) f(t_1, t_2) d(t_1, t_2).$$

In the following definition, we will consider rectangles Q where their sides I_1 and I_2 can be of the form $(-\infty, \infty)$, $[a, \infty)$, $(-\infty, a]$ or bounded intervals. In addition, we will use the concept of bounded variation on intervals, [4].

Definition 2.2

i) A function $f : Q \to \mathbb{R}$ is said to be of ***bounded variation in the Vitali sense*** *over Q and is denoted as $f \in BV_V(Q)$, if*

$$Var(f, Q) := \sup_{R \subset Q} \sup_{\{R_{i,j}\} \in \mathcal{P}(R)} \left\{ \sum_{i,j} |\Delta f(x_i, y_j)| \right\} < \infty,$$

where the rectangles R are compacts contained in Q.

ii) A function $f : Q \to \mathbb{R}$ is said to be of ***bounded variation in the Hardy sense*** *over Q and is denoted as $f \in BV_H(Q)$, if $f \in BV_V(Q)$ and, for each x, y, $f(\cdot, y)$ y $f(x, \cdot)$ are of bounded variation over I_1 and I_2, respectively.*

It is evident that $BV_H(\mathbb{R}^2)$ is properly contained in $BV_V(\mathbb{R}^2)$ since $f(x, y) = x + y \in BV_V(\mathbb{R}^2) \setminus BV_H(\mathbb{R}^2)$, [5].

Remark 2.2 *We can observe that if $f \in BV_V(\mathbb{R}^2)$, then*

$$\lim_{m\to\infty}(Var(f,[m,\infty)\times\mathbb{R}) + Var(f,(-\infty,-m]\times\mathbb{R}) + Var(f,\mathbb{R}\times[m,\infty)) + Var(f,\mathbb{R}\times(-\infty,-m])) = 0.$$

We will show some properties that relate the previous concepts, but first we recall the following spaces of continuous functions defined on $\mathbb{R}^2$.

i) $C_b(\mathbb{R}^2)$ is the space of real valued functions which are bounded.

ii) $C_0(\mathbb{R}^2)$ is the space of functions which vanish at infinity.

iii) $C_c(\mathbb{R}^2)$ is the space of functions whose support is compact.

It is well known that $C_c(\mathbb{R}^2) \subsetneq C_0(\mathbb{R}^2) \subsetneq C_b(\mathbb{R}^2)$ and $C_c(\mathbb{R}^2)$ is dense in $C_0(\mathbb{R}^2)$ with respect to the supremum norm, [11, Theorem 3.17].

Lemma 2.2 ([2, Lema 3.9]) *Let $f \in BV_V(\mathbb{R}^2)$ and $(x,y) \in \mathbb{R}^2$ be a continuity point of f. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$Var(f,[x-\delta,x+\delta]\times[y-\delta,y+\delta]) < \varepsilon.$$

Now we define the improper Riemann-Stieltjes integral.

Definition 2.3 *Let $f,g:\mathbb{R}^2\to\mathbb{R}$ be functions. The multiple limit*

$$\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \iint_{[a,b]\times[c,d]} g(t_1,t_2)df(t_1,t_2)$$

is called the ***improper Riemann-Stieltjes integral of** g **with respect to** f, *when it exists in the Pringsheim sense [1], and is denoted by*

$$\iint_{\mathbb{R}^2} g(t_1,t_2)df(t_1,t_2).$$

From [5], if $g \in C_b(\mathbb{R}^2)$ and $f \in BV_V(\mathbb{R}^2)$, then the improper Riemann-Stieltjes integral of g with respect to f exists.

Lemma 2.3 ([2, Lema 3.6]) *If $g \in C_b(\mathbb{R}^2)$ and $f \in BV_V(\mathbb{R}^2)$, then*

i) The function $V(f;t_1,t_2) = Var(f,(-\infty,t_1]\times(-\infty,t_2))$ defined for each $(t_1,t_2)\in\mathbb{R}^2$, belongs to $BV_V(\mathbb{R}^2)$

ii) $\left|\iint_{\mathbb{R}^2} g(t_1,t_2)df(t_1,t_2)\right| \le \iint_{\mathbb{R}^2} |g(t_1,t_2)|dV(f;t_1,t_2).$

Theorem 2.1 *Suppose that $g \in C_0(\mathbb{R}^2)$ and $f \in BV_V(\mathbb{R}^2)$. Then, there exists a σ-algebra $\mathbb{M}$ of $\mathbb{R}^2$ containing the Borelians and there exists a unique positive finite measure μ on $\mathbb{M}$ such that*

i) $\mu((a,b)\times(c,d)) \le Var(f,[a,b]\times[c,d])$, *for* $(a,b)\times(c,d)\subset\mathbb{R}^2$.

ii) $\iint_{\mathbb{R}^2} g(t_1,t_2)dV(f;t_1,t_2) = \iint_{\mathbb{R}^2} g(t_1,t_2)d\mu(t_1,t_2).$

Proof: From Lemma 2.3, over $C_b(\mathbb{R}^2)$ we define the bounded positive linear functional

$$\Lambda(g) = \iint_{\mathbb{R}^2} g(t_1,t_2)dV(f;t_1,t_2).$$

By [11, Theorem 2.14] (Riesz Representation Theorem), there exists a σ-algebra $\mathbb{M}$ of $\mathbb{R}^2$ containing the Borelians and there exists a unique positive measure μ on $\mathbb{M}$ such that

$$\Lambda(g) = \iint_{\mathbb{R}^2} g(t_1, t_2) d\mu(t_1, t_2), \text{ for all } g \in C_c(\mathbb{R}^2). \tag{2.2}$$

Let $R = [a, b] \times [c, d] \subset \mathbb{R}^2$. From [11, Theorem 2.14], it follows that $\mu(int(R)) \leq Var(f, R)$. Thus, μ is a finite measure. This proves i).

Now, let $g \in C_0(\mathbb{R}^2)$ be. Then, there exists a sequence of functions (g_n) that belongs to $C_c(\mathbb{R}^2)$ which converges uniformly to g. Moreover,

$$\lim_{n\to\infty} \iint_{\mathbb{R}^2} g_n(t_1, t_2) dV(f; t_1, t_2) = \iint_{\mathbb{R}^2} g(t_1, t_2) dV(f; t_1, t_2). \tag{2.3}$$

and

$$\lim_{n\to\infty} \iint_{\mathbb{R}^2} g_n(t_1, t_2) d\mu(t_1, t_2) = \iint_{\mathbb{R}^2} g(t_1, t_2) d\mu(t_1, t_2). \tag{2.4}$$

By (2.2), (2.3) and (2.4) we conclude that, for $g \in C_0(\mathbb{R}^2)$,

$$\iint_{\mathbb{R}^2} g(t_1, t_2) dV(f; t_1, t_2) = \iint_{\mathbb{R}^2} g(t_1, t_2) d\mu(t_1, t_2).$$

□

2.1. The space $BV_{||0||}(\mathbb{R}^2)$

The space of functions f in $BV_V(\mathbb{R}^2)$ which satisfy that $\lim_{||(x,y)||\to\infty} f(x, y) = 0$ is denoted by $BV_{||0||}(\mathbb{R}^2)$.

The following lemma can be proved from the definition of bounded variation, see Definition 2.2 and [4].

Lemma 2.4 *Let $f \in BV_{||0||}(\mathbb{R}^2)$ be. Then, for each $(x, y) \in \mathbb{R}^2$,*

i) $|f(x, y)| \leq Var(f(x, \cdot), \mathbb{R}) \leq Var(f, [x, \infty) \times \mathbb{R})$,
ii) $|f(x, y)| \leq Var(f(x, \cdot), \mathbb{R}) \leq Var(f, (-\infty, x] \times \mathbb{R})$,
iii) $|f(x, y)| \leq Var(f(\cdot, y), \mathbb{R}) \leq Var(f, \mathbb{R} \times [y, \infty))$,
iv) $|f(x, y)| \leq Var(f(\cdot, y), \mathbb{R}) \leq Var(f, \mathbb{R} \times (-\infty, y])$.

The space $BV_{H_0}(\mathbb{R}^2)$ is defined in [3] as

$$BV_{H_0}(\mathbb{R}^2) = \{f \in BV_H(\mathbb{R}^2) : \lim_{a\to\pm\infty} f(a, y) = 0 = \lim_{b\to\pm\infty} f(x, b), \ \forall x, y \in \mathbb{R}\}.$$

From Lemma 2.4 and Remark 2.2, we obtain the following characterization theorem that confirms the equality (2.3) in [3].

Theorem 2.2 *The function $f \in BV_{||0||}(\mathbb{R}^2)$ if and only if $f \in BV_{H_0}(\mathbb{R}^2)$.*

It is important to note that

$$L^1(\mathbb{R}^2) \cap BV_H(\mathbb{R}^2) \subsetneq BV_{||0||}(\mathbb{R}^2) \not\subset L^1(\mathbb{R}^2). \tag{2.5}$$

Example 2.1 *The function*

$$f(x, y) = \begin{cases} (1/x)(1/y) & si \ x, y \geq 1 \\ 0 & si \ x < 1 \ o \ y < 1, \end{cases}$$

belongs to $BV_{||0||}(\mathbb{R}^2) \setminus L^1(\mathbb{R}^2)$. This function illustrates the contention relationships in (2.5).

Proposition 2.1 *If $g \in C_b(\mathbb{R}^2)$ and $f \in BV_{||0||}(\mathbb{R}^2)$, then*

$$\iint_{\mathbb{R}^2} g(t_1,t_2)df(t_1,t_2) = \iint_{\mathbb{R}^2} f(t_1,t_2)dg(t_1,t_2).$$

Proof: Let $R = [a,b] \times [c,d] \subset \mathbb{R}^2$ be. By Theorem 2.2 and the Integration by Parts Theorem in [5], we have that

$$\begin{aligned}\iint_R g(t_1,t_2)df(t_1,t_2) &= f(b,d)g(b,d) - f(b,c)g(b,c) - f(a,d)g(a,d) + f(a,c)g(a,c)\\ &- \int_a^b g(t_1,d)df(t_1,d) + \int_a^b g(t_1,c)df(t_1,c) - \int_c^d g(b,t_2)df(b,t_2)\\ &+ \int_c^d g(a,t_2)df(a,t_2) + \iint_R f(t_1,t_2)dg(t_1,t_2).\end{aligned} \tag{2.6}$$

It is immediate

$$\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} f(b,d)g(b,d) = \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} f(b,c)g(b,c) = \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} f(a,d)g(a,d) = \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} f(a,c)g(a,c) = 0.$$

In addition, according to Lemma 2.4, it is satisfied that

$$\left|\int_a^b g(t_1,d)df(t_1,d)\right| \leq Var(f, \mathbb{R}\times[d,\infty))||g||_\infty.$$

By Remark 2.2, it follows that

$$\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \int_a^b g(t_1,d)df(t_1,d) = 0.$$

Similarly, it shows that

$$\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \int_a^b g(t_1,c)df(t_1,c) = \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \int_c^d g(b,t_2)df(b,t_2) = \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \int_c^d g(a,t_2)df(a,t_2) = 0.$$

By [2, Lemma 3.5], the improper Riemann-Stieltjes integral of g with respect to f exists. Therefore, applying the limit in (2.6) as $a,c \to -\infty$ and $b,d \to -\infty$ we conclude that

$$\iint_{\mathbb{R}^2} g(t_1,t_2)df(t_1,t_2) = \iint_{\mathbb{R}^2} f(t_1,t_2)dg(t_1,t_2).$$

□

Proposition 2.2 *Let $f \in BV_{||0||}(\mathbb{R}^2)$ be. Then, for $u_1 < u_2$ and $v_1 < v_2$,*

$$\begin{aligned}&\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \iint_{[a,b]\times[c,d]} f(t_1,t_2)\left(\int_{u_1}^{u_2} cos(t_1\tau)d\tau\right)\left(\int_{v_1}^{v_2} cos(t_2\tau)d\tau\right)d(t_1,t_2)\\ &\qquad = \iint_{\mathbb{R}^2} f(t_1,t_2)d\left(\int_{u_1}^{u_2}\frac{sin(t_1\tau)}{\tau}d\tau\right)\left(\int_{v_1}^{v_2}\frac{sin(t_2\tau)}{\tau}d\tau\right).\end{aligned}$$

Proof: Let $R = [a,b]\times[c,d] \subset \mathbb{R}^2$ be. By [4, Lemma 4] we define, for each $(t_1,t_2) \in \mathbb{R}^2$, the functions

$$g(t_1,t_2) := \left(\int_{u_1}^{u_2} cos(t_1\tau)d\tau\right)\left(\int_{v_1}^{v_2} cos(t_2\tau)d\tau\right) \in C(\mathbb{R}^2)$$

and

$$h(t_1,t_2) := \left(\int_{u_1}^{u_2} \frac{sin(t_1\tau)}{\tau} d\tau\right)\left(\int_{v_1}^{v_2} \frac{sin(t_2\tau)}{\tau} d\tau\right) \in C_0(\mathbb{R}^2),$$

which satisfy the relation

$$\int_a^{t_1}\int_c^{t_2} g(x,y)dxdy = h(t_1,t_2) - h(t_1,c) - h(a,t_2) + h(a,c). \tag{2.7}$$

Since $h \in C(R)$ and $f \in BV_H(R)$, applying the Integration by Parts Theorem [5], we have that $\iint_R f dh$ exists. Similary, we prove that the following integrals

$$\iint_R f(t_1,t_2)dh(t_1,c), \quad \iint_R f(t_1,t_2)dh(a,t_2) \text{ and } \iint_R f(t_1,t_2)dh(a,c) \tag{2.8}$$

exist and are equal to zero.

From (2.7) and Lemma 2.1, we have that

$$\iint_R f(t_1,t_2)dh(t_1,t_2) - \iint_R f(t_1,t_2)dh(t_1,c) - \iint_R f(t_1,t_2)dh(a,t_2) + \iint_R f(t_1,t_2)dh(a,c) = \iint_R f(t_1,t_2)g(t_1,t_2)d(t_1,t_2). \tag{2.9}$$

According to (2.8) and (2.9),

$$\iint_R f(t_1,t_2)g(t_1,t_2)d(t_1,t_2) = \iint_R f(t_1,t_2)dh(t_1,t_2).$$

By Proposition 2.1, $\iint_{\mathbb{R}^2} f dh$ exists, then applying the limit to the above equality as $a,c \to -\infty$ and $b,d \to \infty$ we conclude that

$$\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \iint_{[a,b]\times[c,d]} f(t_1,t_2)g(t_1,t_2)d(t_1,t_2) = \iint_{\mathbb{R}^2} f(t_1,t_2)dh(t_1,t_2).$$

□

2.2. Sequences in (L)

The following definition can be found in [8] and [4].

Definition 2.4 *An increasing sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ is said to satisfy* ***Lacunary's condition (L)*** *and is denoted as $\{u_j\}_{j\in\mathbb{N}} \in (L)$, if there exists $A > 1$ such that*

$$u_m \sum_{j=m}^{\infty} \frac{1}{u_j} \le A, \quad m = 1,2,3,....$$

Example 2.2 *The sequence $\{2^j\}_{j\in\mathbb{N}}$ satisfies condition (L) with $A = 2$.*

Some results of our interest associated with sequences in (L) are the following.

Lemma 2.5 ([4, Lemma 2]) *If $\{u_j\}_{j\in\mathbb{N}} \in (L)$, then*

$$\sum_{j=1}^{\infty} \max_{u_{j-1}\le v\le u_j} \left| \int_{u_{j-1}}^{v} \frac{sin(tu)}{u} du \right| \le 3A + 4, \quad t \ne 0,$$

where $u_0 := 0$ and A is derived from Definition 2.4.

Lemma 2.6 ([4, Lema 3]) *If $\{u_j\}_{j\in\mathbb{N}} \in (L)$, then*

$$\sum_{j=m+1}^{\infty} \max_{u_{j-1}\le v\le u_j} \left| \int_{u_{j-1}}^{v} \frac{sin(tu)}{u} du \right| \le \frac{3A}{|t|\, u_m}, \quad m \in \mathbb{N};\ t \ne 0.$$

3. KP-Fourier Transform

In this section we define a transform that generalizes the Fourier transform operator since it can be applied to non-absolutely integrable functions. For this, we begin defining the Kurzweil-Henstock integral over a bounded rectangle $R = [a, b] \times [c, d]$, [10], [9].

i) Let $\delta : R \longrightarrow \mathbb{R}$ be a function. It is said that δ is a **gauge** on R, if $\delta(t_1, t_2) \geq 0$ for all $(t_1, t_2) \in R$.

ii) Let $P = \{R_{i,j}\} \in \mathcal{P}(R)$ (defined in Section 2) and $(\xi_{i,j}, \eta_{i,j}) \in R_{i,j}$. It is said that P is $\delta-$**fine**, if $[x_{i-1}, x_i] \subset (\xi_{i,j} - \delta(\xi_{i,j}, \eta_{i,j}), \xi_{i,j} + \delta(\xi_{i,j}, \eta_{i,j}))$ and $[y_{j-1}, y_j] \subset (\eta_{i,j} - \delta(\xi_{i,j}, \eta_{i,j}), \eta_{i,j} + \delta(\xi_{i,j}, \eta_{i,j}))$.

iii) Let $P = \{R_{i,j}\} \in \mathcal{P}(R)$ and $(\xi_{i,j}, \eta_{i,j}) \in R_{i,j}$. Given a function $f : R \longrightarrow \mathbb{R}$, the **Riemann sum of** f **over** P is defined as

$$S(f; P) = \sum_{i,j} f(\xi_{i,j}, \eta_{i,j})(x_i - x_{i-1})(y_j - y_{j-1}).$$

Definition 3.1 *A function $f : R \to \mathbb{R}$ is said to be **Kurzweil-Henstock (KH) integrable over** R and we denote it by $f \in KH(R)$, if there exists $A \in \mathbb{R}$ such that for $\varepsilon > 0$, there exists a gauge δ_ε on R such that for $P = \{R_{i,j}\} \in \mathcal{P}(R)$ which is $\delta_\varepsilon-$fine, then*

$$|S(f; P) - A| < \varepsilon.$$

Moreover, A is the Kurzweil-Henstock integral of f and is denoted by

$$A = \iint_R f(t_1, t_2) d(t_1, t_2).$$

The space of KH integrable functions over compact rectangles is denoted by $KH_{loc}(\mathbb{R}^2)$. In [10], it is proved that the space of locally Lebesgue integrable functions $L^1_{loc}(\mathbb{R}^2)$ is properly contained in $KH_{loc}(\mathbb{R}^2)$.

Definition 3.2 *[3, Definition 4.1] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(\cdot)e^{-i<\cdot,v>} \in KH_{loc}(\mathbb{R}^2)$. The **KP-Fourier Transform** of f at $(\xi, \eta) \in \mathbb{R}^2$ is defined as*

$$\mathcal{F}(f)(\xi, \eta) := \lim_{\substack{a,c \to -\infty \\ b,d \to \infty}} \iint_{[a,b]\times[c,d]} f(t_1, t_2) e^{-i(\xi t_1 + \eta t_2)} d(t_1, t_2), \tag{3.1}$$

when the limit in (3.1) exists in the Pringsheim sense.

Remark 3.1 *Let us observe that if $f \in L^1(\mathbb{R}^2)$, then the KP-Fourier transform $\mathcal{F}(f)$ is well-defined for each $(\xi, \eta) \in \mathbb{R}^2$ and is equal to the Fourier transform $\hat{f}$.*

The following result shows that the KP-Fourier transform operator is well defined on $BV_{||0||}(\mathbb{R}^2)$ which is not contained in $L^1(\mathbb{R}^2)$ according to (2.5).

Theorem 3.1 *Suppose that $f \in BV_{||0||}(\mathbb{R}^2)$. Then, for $\xi \neq 0$ and $\eta \neq 0$,*

$$\mathcal{F}(f)(\xi, \eta) = -\frac{1}{\xi\eta} \iint_{\mathbb{R}^2} e^{-i(\xi t_1 + \eta t_2)} df(t_1, t_2).$$

Proof:

Let $R = [a, b] \times [c, d] \subset \mathbb{R}^2$ and $\xi \neq 0$, $\eta \neq 0$. Let $g(t_1, t_2) = e^{-i(\xi t_1 + \eta t_2)}$ and $G(t_1, t_2) = -e^{-i(\xi t_1 + \eta t_2)}/\xi\eta$, for each $(t_1, t_2) \in \mathbb{R}^2$. By [10, Theorem 6.5.9], we have that $fg \in KH(R)$ and

$$\begin{aligned} \iint_R f(t_1, t_2) g(t_1, t_2) d(t_1, t_2) =& f(b, d)G(b, d) - f(b, c)G(b, c) - f(a, d)G(a, d) + f(a, c)G(a, c) \\ & - \int_a^b G(t_1, d) df(t_1, d) + \int_a^b G(t_1, c) df(t_1, c) - \int_c^d G(b, t_2) df(b, t_2) \\ & + \int_c^d G(a, t_2) df(a, t_2) + \iint_R G(t_1, t_2) df(t_1, t_2). \end{aligned} \tag{3.2}$$

It is easy to prove that

$$\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} f(b,d)G(b,d) = \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} f(b,c)G(b,c) = \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} f(a,d)G(a,d) = \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} f(a,c)G(a,c) = 0.$$

From Lemma 2.4,

$$\left|\int_a^b G(t_1,d)df(t_1,d)\right| \leq \frac{2}{|\xi\eta|} Var(f,\mathbb{R}\times[d,\infty)).$$

By Remark 2.2, it follows that

$$\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \int_a^b G(t_1,d)df(t_1,d) = 0.$$

Similarly, we have that

$$\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \int_a^b G(t_1,c)df(t_1,c) = \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \int_c^d G(b,t_2)df(b,t_2) = \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \int_c^d G(a,t_2)df(a,t_2) = 0.$$

Since the real and imaginary part of G belong to $C_b(\mathbb{R}^2)$, according to Proposition 2.1, we can claim that $\iint_{\mathbb{R}^2} Gdf$ exists. Therefore, applying the limit to (3.2) as $a,c\to$ and $b,d\to$ we concluded that

$$\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}} \iint_R f(t_1,t_2)e^{-i(\xi t_1+\eta t_2)}d(t_1,t_2) = -\frac{1}{\xi\eta}\iint_{\mathbb{R}^2} e^{-i(\xi t_1+\eta t_2)}df(t_1,t_2).$$

□

Example 3.1 *Considering the function defined in the Example 2.1 and by Theorem 3.1 we prove that, for* $(\xi,\eta)\in\mathbb{R}^2$ *with* $\xi\neq 0$ *and* $\eta\neq 0$*, its KP-Fourier transform is*

$$\mathcal{F}(f)(\xi,\eta) = \Gamma(0,i\xi)\Gamma(0,i\eta),$$

where $\Gamma(\cdot,\cdot)$ *is the incomplete Gamma function. Let us note that when* $\xi=0$ *or* $\eta=0$, $\mathcal{F}(f)(\xi,\eta)$ *does not exist.*

In [3, Corollary 4.1] the following result is proved, however we provide an alternative proof.

Proposition 3.1 *Let* $f\in BV_{||0||}(\mathbb{R}^2)$ *be. Then* $\mathcal{F}(f)$ *is continuous at* (ξ,η) *with* $\xi\neq 0$ *and* $\eta\neq 0$*.*

Proof: By Theorem 3.1, for $f\in BV_{||0||}(\mathbb{R}^2)$ and $(\xi_0,\eta_0)\in\mathbb{R}^2$ with $\xi_0\neq 0$ and $\eta_0\neq 0$, it is satisfied that

$$\mathcal{F}(f)(\xi_0,\eta_0) = -\frac{1}{\xi_0\eta_0}\iint_{\mathbb{R}^2} cos(\xi_0 t_1+\eta_0 t_2)df(t_1,t_2) + \frac{i}{\xi_0\eta_0}\iint_{\mathbb{R}^2} sin(\xi_0 t_1+\eta_0 t_2)df(t_1,t_2). \tag{3.3}$$

Let $(\xi,\eta)\in\{(x,y)\in\mathbb{R}^2:\ x\neq 0 \text{ and } y\neq 0\}$. By Lemma 2.3 and the Mean Value Theorem, we obtain the following inequality

$$\left|\iint_{[a,b]\times[c,d]} cos(\xi t_1+\eta t_2)df(t_1,t_2) - \iint_{[a,b]\times[c,d]} cos(\xi_0 t_1+\eta_0 t_2)df(t_1,t_2)\right|$$
$$\leq |\xi-\xi_0|\iint_{[a,b]\times[c,d]} |t_1|dV(f;t_1,t_2) + |\eta-\eta_0|\iint_{[a,b]\times[c,d]} |t_2|dV(f;t_1,t_2).$$

Then, for each $[a,b]\times[c,d]\subset\mathbb{R}^2$,

$$\lim_{(\xi,\eta)\to(\xi_0,\eta_0)} \iint_{[a,b]\times[c,d]} cos(\xi t_1+\eta t_2)df(t_1,t_2) = \iint_{[a,b]\times[c,d]} cos(\xi_0 t_1+\eta_0 t_2)df(t_1,t_2). \tag{3.4}$$

By Remark 2.2, given $\varepsilon > 0$ there exists $M > 0$ such that

$$Var(f, [M, \infty) \times \mathbb{R}) < \varepsilon/4,$$

$$Var(f, \mathbb{R} \times [M, \infty)) < \varepsilon/4,$$

$$Var(f, (-\infty, -M] \times \mathbb{R}) < \varepsilon/4$$

$$\text{and } Var(f, \mathbb{R} \times (-\infty, -M]) < \varepsilon/4.$$

Suppose that $[a, b] \times [c, d], [a_1, b_1] \times [c_1, d_1] \supset [-M, M]^2$. Applying $ii)$ of Theorem 2.3 on compact rectangles and the previous inequalities we have that, for each $(\xi, \eta) \in \mathbb{R}^2$,

$$\left| \iint_{[a_1,b_1]\times[c_1,d_1]} cos(\xi t_1 + \eta t_2) df(t_1, t_2) - \iint_{[a,b]\times[c,d]} cos(\xi t_1 + \eta t_2) df(t_1, t_2) \right| < \varepsilon. \tag{3.5}$$

Thus, the limit

$$\lim_{\substack{a,c\to-\infty \\ b,d\to\infty}} \iint_{[a,b]\times[c,d]} cos(\xi t_1 + \eta t_2) df(t_1, t_2) \tag{3.6}$$

is uniform with respect to $(\xi, \eta) \in \mathbb{R}^2$.

Applying [12, Theorem 1], (3.4) and (3.6),

$$\lim_{(\xi,\eta)\to(\xi_0,\eta_0)} -\frac{1}{\xi\eta} \iint_{\mathbb{R}^2} cos(\xi t_1 + \eta t_2) df(t_1, t_2) = -\frac{1}{\xi_0\eta_0} \iint_{\mathbb{R}^2} cos(\xi_0 t_1 + \eta_0 t_2) df(t_1, t_2). \tag{3.7}$$

Similarly, it is proved that

$$\lim_{(\xi,\eta)\to(\xi_0,\eta_0)} \frac{i}{\xi\eta} \iint_{\mathbb{R}^2} sin(\xi t_1 + \eta t_2) df(t_1, t_2) = \frac{i}{\xi_0\eta_0} \iint_{\mathbb{R}^2} sin(\xi_0 t_1 + \eta_0 t_2) df(t_1, t_2). \tag{3.8}$$

According to (3.3), (3.7) and (3.8), we conclude that $\mathcal{F}(f)$ is continuous at (ξ_0, η_0). □

For $0 < \alpha_i < \beta_i < \infty$ with $i = 1, 2$, we denote

$$R^{\beta_1,\beta_2}_{\alpha_1,\alpha_2} = \{(x, y) \in \mathbb{R}^2 : \alpha_1 \le |x| \le \beta_1,\ \alpha_2 \le |y| \le \beta_2\}$$

and

$$h_{\alpha_i,\beta_i}(t) = (sin(\beta_i t) - sin(\alpha_i t))/\pi t.$$

Proposition 3.2 *Suppose that $f \in BV_{||0||}(\mathbb{R}^2)$ and $(x, y) \in \mathbb{R}^2$. Then*

$$\frac{1}{4\pi^2} \iint_{R^{\beta_1,\beta_2}_{\alpha_1,\alpha_2}} \mathcal{F}(f)(\varepsilon, \eta) e^{i(x\varepsilon+y\eta)} d(\varepsilon, \eta) = \frac{1}{\pi^2} \lim_{\substack{a,c\to-\infty \\ b,d\to\infty}} \iint_{[a,b]\times[c,d]} f(x + t_1, y + t_2)$$
$$\times \left(\frac{sin(\beta_1 t_1) - sin(\alpha_1 t_1)}{t_1} \right) \left(\frac{sin(\beta_2 t_2) - sin(\alpha_2 t_2)}{t_2} \right) d(t_1, t_2).$$

Proof: Theorem 3.1 and Proposition 3.1 give us conditions to apply Lebesgue's Dominated Convergence Theorem and Fubini's Theorem in the following equalities

$$\frac{1}{4\pi^2}\iint\limits_{R^{\beta_1,\beta_2}_{\alpha_1,\alpha_2}} \mathcal{F}(f)(\varepsilon,\eta)e^{i(x\varepsilon+y\eta)}d(\varepsilon,\eta)$$

$$
\begin{aligned}
&= \frac{1}{4\pi^2}\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}}\iint\limits_{R^{\beta_1,\beta_2}_{\alpha_1,\alpha_2}}\iint_{[a,b]\times[c,d]} f(t_1,t_2)e^{-i(t_1\varepsilon+t_2\eta)}d(t_1,t_2)e^{i(x\varepsilon+y\eta)}d(\varepsilon,\eta)\\
&= \frac{1}{4\pi^2}\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}}\iint_{[a,b]\times[c,d]}\iint\limits_{R^{\beta_1,\beta_2}_{\alpha_1,\alpha_2}} f(t_1,t_2)e^{-i(t_1-x)\varepsilon}e^{-i(t_2-y)\eta}d(\varepsilon,\eta)d(t_1,t_2) \quad (3.9)\\
&= \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}}\iint_{[a,b]\times[c,d]} f(t_1,t_2)h_{\alpha_1,\beta_1}(x-t_1)h_{\alpha_2,\beta_2}(y-t_2)d(t_1,t_2)\\
&= \lim_{\substack{a,c\to-\infty\\ b,d\to\infty}}\iint_{[a,b]\times[c,d]} f(x+\mu,y+\tau)h_{\alpha_1,\beta_1}(\mu)h_{\alpha_2,\beta_2}(\tau)d(\mu,\tau).
\end{aligned}
$$

The last equality is obtained by making the change of variable $\mu = t_1 - x$ and $\tau = t_2 - y$. □

Proposition 3.3 *Let* $f \in BV_{||0||}(\mathbb{R}^2)$ *and* $(x,y)\in\mathbb{R}^2$. *Then the function*

$$g_{(x,y)}(t_1,t_2) = f(x-t_1,y-t_2)+f(x-t_1,y+t_2)+f(x+t_1,y-t_2)+f(x+t_1,y+t_2)$$

belongs to $BV_{||0||}(\mathbb{R}^2)$ *and the limit*

$$\lim_{a\to\infty}\iint\limits_{[0,a]\times[0,a]} g_{(x,y)}(t_1,t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2) \tag{3.10}$$

exists and is uniform with respect to $0<\alpha_i<\beta_i<\infty$ *for* $i=1,2$.

Proof: Making the change of variable $\mu = x - t_1$ and $\tau = y - t_2$ in (3.9) from Proposition 3.2 we have that, for $0<\alpha_i<\beta_i<\infty$, $i=1,2$, the limit

$$\lim_{a\to\infty}\iint\limits_{[-a,a]\times[-a,a]} f(x-t_1,y-t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2)$$

exists.

Considering the appropriate change of variables and for $0<\alpha_i<\beta_i<\infty$, $i=1,2$,

$$\lim_{a\to\infty}\iint\limits_{[-a,a]\times[-a,a]} f(x-t_1,y-t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2)$$
$$= \lim_{a\to\infty}\iint\limits_{[0,a]\times[0,a]} g_{(x,y)}(t_1,t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2). \tag{3.11}$$

Now, let us prove that the limit in (3.10) is uniform. Since $g_{(x,y)}$ belongs to $BV_{||0||}(\mathbb{R}^2)$, given $\varepsilon>0$ there exists $\delta_0>0$ such that

i) If $(t_1,t_2)\in(0,\infty)^2$ with $||(t_1,t_2)||>\delta_0$, then $|g_{(x,y)}(t_1,t_2)|<\varepsilon$,

ii) $Var(g_{(x,y)},[\delta_0,\infty)\times[0,\infty))<\varepsilon$,

iii) $Var(g_{(x,y)},[0,\infty)\times[\delta_0,\infty))<\varepsilon$.

For each $(z_1, z_2) \in \mathbb{R}^2$ and $0 < \alpha_i < \beta_i < \infty$, $i = 1, 2$, we define the function

$$H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(z_1, z_2) = \int_0^{z_1}\int_0^{z_2} h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)dt_1dt_2.$$

Let $a_2 > a_1 \geq \delta_0$. Applying the Integration by Parts Theorem [10, Theorem 6.5.9] and the above statements, we obtain that

$$\begin{aligned}
&\Bigg| \iint_{[0,a_1]\times[0,a_1]} g_{(x,y)}(t_1, t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1, t_2) \\
&\qquad\qquad - \iint_{[0,a_2]\times[0,a_2]} g_{(x,y)}(t_1, t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1, t_2)\Bigg| \\
&\leq \Bigg|\iint_{[0,a_2]\times[a_1,a_2]} g_{(x,y)}(t_1, t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1, t_2)\Bigg| \\
&\quad + \Bigg|\iint_{[a_1,a_2]\times[0,a_1]} g_{(x,y)}(t_1, t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1, t_2)\Bigg| \\
&= \Big|H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(a_2, a_2)g_{(x,y)}(a_2, a_2) - H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(0, a_2)g_{(x,y)}(0, a_2) \\
&\quad + H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(0, a_1)g_{(x,y)}(0, a_1) - H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(a_2, a_1)g_{(x,y)}(a_2, a_1) \\
&\quad - \int_0^{a_2} H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(\cdot, a_2)dg_{(x,y)}(\cdot, a_2) + \int_0^{a_2} H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(\cdot, a_1)dg_{(x,y)}(\cdot, a_1) \\
&\quad - \int_{a_1}^{a_2} H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(a_2, \cdot)dg_{(x,y)}(a_2, \cdot) + \int_{a_1}^{a_2} H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(0, \cdot)dg_{(x,y)}(0, \cdot) \\
&\quad + \iint_{[0,a_2]\times[a_1,a_2]} H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2} dg_{(x,y)}\Big| + \Big|H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(a_2, a_1)g_{(x,y)}(a_2, a_1) \\
&\quad - H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(a_1, a_1)g_{(x,y)}(a_1, a_1) + H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(a_2, 0)g_{(x,y)}(a_2, 0) \\
&\quad - H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(a_1, 0)g_{(x,y)}(a_1, 0) - \int_{a_1}^{a_2} H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(\cdot, a_1)dg_{(x,y)}(\cdot, a_1) \\
&\quad + \int_{a_1}^{a_2} H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(\cdot, 0)dg_{(x,y)}(\cdot, 0) - \int_0^{a_1} H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(a_2, \cdot)dg_{(x,y)}(a_2, \cdot) \\
&\quad + \int_0^{a_1} H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2}(a_1, \cdot)dg_{(x,y)}(a_1, \cdot) + \iint_{[a_1,a_2]\times[0,a_1]} H^{\beta 1,\beta_2}_{\alpha_1,\alpha_2} dg_{(x,y)}\Big| \\
&\leq Si(\pi)^2 \big(Var(g_{(x,y)}(\cdot, a_2), [0, a_2]) + Var(g_{(x,y)}(\cdot, a_1), [0, a_2]) \\
&\quad + Var(g_{(x,y)}(a_2, \cdot), [a_1, a_2]) + Var(g_{(x,y)}, [0, a_2] \times [a_1, a_2]) \\
&\quad + Var(g_{(x,y)}(\cdot, a_1), [a_1, a_2]) + Var(g_{(x,y)}(a_2, \cdot), [0, a_1]) \\
&\quad + Var(g_{(x,y)}(a_1, \cdot), [0, a_1]) + Var(g_{(x,y)}, [a_1, a_2] \times [0, a_1]) + 4\varepsilon\big) \\
&< 12Si(\pi)^2\varepsilon,
\end{aligned}$$

where Si is the Sine Integral function.

Therefore, given $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$\Bigg| \iint\limits_{[0,a_1]\times[0,a_1]} g_{(x,y)}(t_1,t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2) - \iint\limits_{[0,a_2]\times[0,a_2]} g_{(x,y)}(t_1,t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2)\Bigg| < 12Si(\pi)^2\varepsilon,$$

for $a_2, a_1 \geq \delta_0$ and $0 < \alpha_i < \beta_i < \infty$ with $i = 1, 2$. □

4. An Extension of the Dirichlet-Jordan Theorem on $BV_{||0||}(\mathbb{R}^2)$.

The first of our mains results is a two-dimensional extension of the Dirichlet-Jordan Theorem for functions in $BV_0(\mathbb{R})$.

Theorem 4.1 *Suppose that $f \in BV_{||0||}(\mathbb{R}^2)$ and $(x, y) \in \mathbb{R}^2$. Then,*

$$\frac{1}{4\pi^2} \lim_{\substack{\alpha_1,\alpha_2\to 0\\ \beta_1,\beta_2\to\infty}} \iint_{R^{\beta_1,\beta_2}_{\alpha_1,\alpha_2}} \mathcal{F}(f)(\varepsilon,\eta)e^{i(x\varepsilon+y\eta)}d(\varepsilon,\eta) = \frac{f(x+,y+)+f(x+,y-)+f(x-,y+)+f(x-,y-)}{4}. \tag{4.1}$$

Proof: By Proposition 3.2 and (3.11), for $0 < \alpha_i < \beta_i < \infty$ with $i = 1, 2$,

$$\frac{1}{4\pi^2} \iint_{R^{\beta_1,\beta_2}_{\alpha_1,\alpha_2}} \mathcal{F}(f)(\varepsilon,\eta)e^{i(x\varepsilon+y\eta)}d(\varepsilon,\eta) = \lim_{a\to\infty} \iint\limits_{[0,a]\times[0,a]} g_{(x,y)}(t_1,t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2). \tag{4.2}$$

We know that for $a > 0$, the following statements are satisfied.

i) $\lim\limits_{\substack{\alpha_2\to 0\\ \beta_2\to\infty}} \int_0^a g_{(x,y)}(t_1,t_2)h_{\alpha_2,\beta_2}(t_2)dt_2 = g_{(x,y)}(t_1,0+)/2$ for $t_1 \in [0,a]$,

ii) $\left|\int_0^a g_{(x,y)}(t_1,t_2)h_{\alpha_2,\beta_2}(t_2)dt_2\right| \leq 2Si(\pi)\left[||g_{(x,y)}||_\infty + Var(g_{(x,y)},\mathbb{R}^2)\right]$ for $t_1 \in [0,a]$ and $0 < \alpha_2 < \beta_2 < \infty$,

iii) $\left\{\int_0^a g_{(x,y)}(\cdot,t_2)h_{\alpha_2,\beta_2}(t_2)dt_2 : 0 < \alpha_2 < \beta_2 < \infty\right\} \subset L^1([0,a])$.

From Fubini's Theorem and Dominated Convergence Theorem we have, for $a > 0$, that

$$\lim_{\substack{\alpha_2\to 0\\ \beta_2\to\infty}} \iint\limits_{[0,a]\times[0,a]} g_{(x,y)}(t_1,t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2) = \int_0^a \frac{g_{(x,y)}(t_1,0+)}{2}h_{\alpha_1,\beta_1}(t_1)dt_1. \tag{4.3}$$

From Proposition 3.3, [12, Theorem 1] and (4.3), it follows that

$$\lim_{\substack{\alpha_2\to 0\\ \beta_2\to\infty}} \lim_{a\to\infty} \iint\limits_{[0,a]\times[0,a]} g_{(x,y)}(t_1,t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2) = \lim_{a\to\infty} \lim_{\substack{\alpha_2\to 0\\ \beta_2\to\infty}} \iint\limits_{[0,a]\times[0,a]} g_{(x,y)}(t_1,t_2)h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2). \tag{4.4}$$

By (4.2) and (4.4), we obtain that

$$\frac{1}{4\pi^2}\lim_{\substack{\alpha_1,\alpha_2\to 0\\ \beta_1,\beta_2\to\infty}}\iint_{R^{\beta_1,\beta_2}_{\alpha_1,\alpha_2}}\mathcal{F}(f)(\varepsilon,\eta)e^{i(x\varepsilon+y\eta)}d(\varepsilon,\eta)=\lim_{\substack{\alpha_1\to 0\\ \beta_1\to\infty}}\lim_{\substack{\alpha_2\to 0\\ \beta_2\to\infty}}\lim_{a\to\infty}\iint_{[0,a]\times[0,a]}g_{(x,y)}(t_1,t_2)$$
$$\times h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2)$$
$$=\lim_{\substack{\alpha_1\to 0\\ \beta_1\to\infty}}\lim_{a\to\infty}\lim_{\substack{\alpha_2\to 0\\ \beta_2\to\infty}}\iint_{[0,a]\times[0,a]}g_{(x,y)}(t_1,t_2)$$
$$\times h_{\alpha_1,\beta_1}(t_1)h_{\alpha_2,\beta_2}(t_2)d(t_1,t_2)$$
$$=\lim_{\substack{\alpha_1\to 0\\ \beta_1\to\infty}}\lim_{a\to\infty}\int_0^a\frac{g_{(x,y)}(t_1,0+)}{2}h_{\alpha_1,\beta_1}(t_1)dt_1$$
$$=\lim_{\substack{\alpha_1\to 0\\ \beta_1\to\infty}}\int_0^\infty\frac{g_{(x,y)}(t_1,0+)}{2}h_{\alpha_1,\beta_1}(t_1)dt_1$$
$$=\frac{g_{(x,y)}(0+,0+)}{4},$$

where $g_{(x,y)}(0+,0+)=f(x+,y+)+f(x+,y-)+f(x-,y+)+f(x-,y-)$. □

Theorems 1 and 3 in [5] and Theorem 2.1 in [2] claim that Fourier series of functions f in $L^1(\mathbb{R}^2)\cap BV_H(\mathbb{R}^2)$ are bounded and converge locally uniform at points of continuity of f. Thus, the extensions of such results for functions in $BV_{||0||}(\mathbb{R}^2)$ are stated as follows.

Theorem 4.2 *Let $f\in BV_{||0||}(\mathbb{R}^2)$ and $(u_n),(v_m)\in(L)$ with constants A_1 and A_2, respectively. Then, for each $(x,y)\in\mathbb{R}^2$,*

$$\sum_{i,j=2}^{\infty}\sup_{u\in[u_{i-1},u_i]}\sup_{v\in[v_{j-1}v_j]}\left|\frac{1}{4\pi^2}\iint_{R^{u,v}_{u_{i-1},v_{j-1}}}\mathcal{F}(f)(\xi,\eta)e^{i(x\xi+y\eta)}d(\xi,\eta)\right|\le k,$$

where $k=(3A_1+4)(3A_2+4)Var(f,\mathbb{R}^2)/\pi^2$.

Proof: Let $(x,y)\in\mathbb{R}^2$ be. For $u\in[u_{i-1},u_i]$, $v\in[v_{j-1},v_j]$ with $i,j\ge 2$, let

$$A_{i,j}(u,v):=\frac{1}{4\pi^2}\iint_{R^{u,v}_{u_{i-1},v_{j-1}}}\mathcal{F}(f)(\xi,\eta)e^{i(x\xi+y\eta)}d(\xi,\eta).$$

By Proposition 3.2, we have that

$$A_{i,j}(u,v)=\frac{1}{\pi^2}\lim_{\substack{a,c\to-\infty\\ b,d\to\infty}}\iint_{[a,b]\times[c,d]}f(x+t_1,y+t_2)\left(\int_{u_{i-1}}^{u}cos(t_1\tau)d\tau\right)\left(\int_{v_{j-1}}^{v}cos(t_2\tau)d\tau\right)d(t_1,t_2).$$

Since $f(x+\cdot,y+\cdot)\in BV_{||0||}(\mathbb{R}^2)$ and from Propositions 2.1 and 2.2, we obtain that

$$A_{i,j}(u,v)=\frac{1}{\pi^2}\iint_{\mathbb{R}^2}f(x+t_1,y+t_2)d\left(\int_{u_{i-1}}^{u}\frac{sin(t_1\tau)}{\tau}d\tau\right)\left(\int_{v_{j-1}}^{v}\frac{sin(t_2\tau)}{\tau}d\tau\right)$$
$$=\frac{1}{\pi^2}\iint_{\mathbb{R}^2}\left(\int_{u_{i-1}}^{u}\frac{sin(t_1\tau)}{\tau}d\tau\right)\left(\int_{v_{j-1}}^{v}\frac{sin(t_2\tau)}{\tau}d\tau\right)df(x+t_1,y+t_2).$$

Now, $ii)$ of Lemma 2.3 assures that

$$|A_{i,j}(u,v)|\le\frac{1}{\pi^2}\iint_{\mathbb{R}^2}\left|\int_{u_{i-1}}^{u}\frac{sin(t_1\tau)}{\tau}d\tau\right|\left|\int_{v_{j-1}}^{v}\frac{sin(t_2\tau)}{\tau}d\tau\right|dV_{x,y}(f;t_1,t_2)$$
$$=:B(u,v), \tag{4.5}$$

where $V_{x,y}(f;t_1,t_2) = Var(f,(-\infty,x+t_1]\times(-\infty,x+t_2])$.

Applying Theorem 2.1, there exists a $\sigma-$algebra $\mathbb{M}$ of $\mathbb{R}^2$ and a measure $\mu_{x,y}$ on $\mathbb{M}$ such that

$$\begin{aligned} B(u,v) &= \frac{1}{\pi^2}\iint_{\mathbb{R}^2}\left|\int_{u_{i-1}}^{u}\frac{sin(t_1\tau)}{\tau}d\tau\right|\left|\int_{v_{j-1}}^{v}\frac{sin(t_2\tau)}{\tau}d\tau\right|d\mu_{x,y}(t_1,t_2) \\ &\leq \frac{1}{\pi^2}\iint_{\mathbb{R}^2}\left(\max_{u\in[u_{i-1},u_i]}\left|\int_{u_{i-1}}^{u}\frac{sin(t_1\tau)}{\tau}d\tau\right|\right)\left(\max_{v\in[v_{j-1},v_j]}\left|\int_{v_{j-1}}^{v}\frac{sin(t_2\tau)}{\tau}d\tau\right|\right)d\mu_{x,y}(t_1,t_2). \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), for each $u\in[u_{i-1},u_i]$ and $v\in[v_{j-1},v_j]$, the following is satisfied

$$|A_{i,j}(u,v)| \leq \frac{1}{\pi^2}\iint_{\mathbb{R}^2}\left(\max_{u\in[u_{i-1},u_i]}\left|\int_{u_{i-1}}^{u}\frac{sin(t_1\tau)}{\tau}d\tau\right|\right)\left(\max_{v\in[v_{j-1},v_j]}\left|\int_{v_{j-1}}^{v}\frac{sin(t_2\tau)}{\tau}d\tau\right|\right)d\mu_{x,y}(t_1,t_2).$$

We define, for each $i,j\geq 2$,

$$M_{i,j}(f;x,y) := \sup_{u\in[u_{i-1},u_i]}\sup_{v\in[v_{j-1}v_j]}|A_{i,j}(u,v)|.$$

One can prove, for each $i,j\geq 2$,

$$M_{i,j}(f;x,y) \leq \frac{1}{\pi^2}\iint_{\mathbb{R}^2}\left(\max_{u\in[u_{i-1},u_i]}\left|\int_{u_{i-1}}^{u}\frac{sin(t_1\tau)}{\tau}d\tau\right|\right)\left(\max_{v\in[v_{j-1},v_j]}\left|\int_{v_{j-1}}^{v}\frac{sin(t_2\tau)}{\tau}d\tau\right|\right)d\mu_{x,y}(t_1,t_2).$$

Considering the above inequalities and by Lemma 2.5, for $i,j\geq 2$, we obtain that

$$\begin{aligned} \sum_{i=2}^{\infty}\sum_{j=2}^{\infty}M_{i,j}(f;x,y) &\leq \frac{1}{\pi^2}\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\iint_{\mathbb{R}^2}\left(\max_{u\in[u_{i-1},u_i]}\left|\int_{u_{i-1}}^{u}\frac{sin(t_1\tau)}{\tau}d\tau\right|\right) \\ &\times\left(\max_{v\in[v_{j-1},v_j]}\left|\int_{v_{j-1}}^{v}\frac{sin(t_2\tau)}{\tau}d\tau\right|\right)d\mu_{x,y}(t_1,t_2) \\ &= \frac{1}{\pi^2}\iint_{\mathbb{R}^2}\left(\sum_{i=2}^{\infty}\max_{u\in[u_{i-1},u_i]}\left|\int_{u_{i-1}}^{u}\frac{sin(t_1\tau)}{\tau}d\tau\right|\right) \\ &\times\left(\sum_{j=2}^{\infty}\max_{v\in[v_{j-1},v_j]}\left|\int_{v_{j-1}}^{v}\frac{sin(t_2\tau)}{\tau}d\tau\right|\right)d\mu_{x,y}(t_1,t_2) \\ &\leq \frac{1}{\pi^2}\iint_{\mathbb{R}^2}(3A_1+4)(3A_2+4)d\mu_{x,y}(t_1,t_2) \\ &= \frac{1}{\pi^2}(3A_1+4)(3A_2+4)Var(f,\mathbb{R}^2). \end{aligned} \tag{4.7}$$

□

Theorem 4.3 *Let $f\in BV_{||0||}(\mathbb{R}^2)$ and $(x,y)\in\mathbb{R}^2$ a point of continuity of f. If $(u_n),(v_m)\in(L)$ with constants A_1 y A_2, respectably, then the series*

$$\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\sup_{u\in[u_{i-1},u_i]}\sup_{v\in[v_{j-1}v_j]}\left|\frac{1}{4\pi^2}\iint_{R^{u,v}_{u_{i-1},v_{j-1}}}\mathcal{F}(f)(\xi,\eta)e^{i(\cdot\xi+\cdot\eta)}d(\xi,\eta)\right|$$

converges locally uniform at (x,y).

Proof: Let $n, m \in \mathbb{N}$ be. From (4.7), for $(x,y) \in \mathbb{R}^2$, the following holds

$$\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} M_{i,j}(f;x,y) \leq \frac{1}{\pi^2}\iint_{\mathbb{R}^2}\left(\sum_{i=n+1}^{\infty}\max_{u\in[u_{i-1},u_i]}\left|\int_{u_{i-1}}^{u}\frac{sin(t_1\tau)}{\tau}d\tau\right|\right) \times \left(\sum_{j=m+1}^{\infty}\max_{v\in[v_{j-1},v_j]}\left|\int_{v_{j-1}}^{v}\frac{sin(t_2\tau)}{\tau}d\tau\right|\right) d\mu_{x,y}(t_1,t_2), \quad (4.8)$$

where $\mu_{x,y}$ is a measure satisfying

$$\mu_{x,y}((a,b)\times(c,d)) \leq Var(f,[a,b]\times[c,d]+(x,y)) \quad (4.9)$$

$$\text{and } \mu_{x,y}(\mathbb{R}^2) \leq Var(f,\mathbb{R}^2). \quad (4.10)$$

Suppose that $(x_0,y_0)\in\mathbb{R}^2$ is a point of continuity of f and $\varepsilon > 0$. By Lemma 2.2, there exits $\delta > \delta_0 = \delta/2 > 0$, such that

$$Var(f,[x_0-2\delta_0, x+2\delta_0]\times[y_0-2\delta_0,y_0+2\delta_0]) < \varepsilon.$$

Now, if $(x',y')\in(x_0-\delta_0,x_0+\delta_0)\times(y_0-\delta_0,y_0+\delta_0)$, we can claim that

$$[x'-\delta_0,x'+\delta_0]\times[y'-\delta_0,y'+\delta_0]\subset[x_0-2\delta_0,x_0+2\delta_0]\times[y_0-2\delta_0,y_0+2\delta_0].$$

Then,

$$Var(f,[x'-\delta_0,x'+\delta_0]\times[y'-\delta_0,y'+\delta_0]) < \varepsilon. \quad (4.11)$$

Let $0<\delta_1<\delta_0$ and $(x',y')\in(x_0-\delta_0,x_0+\delta_0)\times(y_0-\delta_0,y_0+\delta_0)$. From (4.8), it follows that

$$\begin{aligned}\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} M_{i,j}(f;x',y') \leq \frac{1}{\pi^2}&\left(\iint\limits_{\substack{|t_1|\leq\delta_1\\|t_2|\leq\delta_1}}+\iint\limits_{\substack{|t_1|\geq\delta_1\\|t_2|\leq\delta_1}}+\iint\limits_{\substack{|t_1|\leq\delta_1\\|t_2|\geq\delta_1}}+\iint\limits_{\substack{|t_1|\geq\delta_1\\|t_2|\geq\delta_1}}\right)\\ &\left(\sum_{i=n+1}^{\infty}\max_{u\in[u_{i-1},u_i]}\left|\int_{u_{i-1}}^{u}\frac{sin(t_1\tau)}{\tau}d\tau\right|\right)\\ &\times\left(\sum_{j=m+1}^{\infty}\max_{v\in[v_{j-1},v_j]}\left|\int_{v_{j-1}}^{v}\frac{sin(t_2\tau)}{\tau}d\tau\right|\right)d\mu_{x',y'}(t_1,t_2)\\ =:&I_1+I_2+I_3+I_4.\end{aligned} \quad (4.12)$$

By (4.9), (4.11) and Lemma 2.5,

$$\begin{aligned} I_1 \leq& \frac{(3A_1+4)(3A_2+4)}{\pi^2}\iint\limits_{\substack{|t_1|\leq\delta_1\\|t_2|\leq\delta_1}} d\mu_{x',y'}(t_1,t_2)\\ \leq&\frac{(3A_1+4)(3A_2+4)}{\pi^2}\mu_{x',y'}([-\delta_1,\delta_1]\times[-\delta_1,\delta_1])\\ \leq&\frac{(3A_1+4)(3A_2+4)}{\pi^2}\mu_{x',y'}((-\delta_0,\delta_0)\times(-\delta_0,\delta_0))\\ \leq&\frac{(3A_1+4)(3A_2+4)}{\pi^2}Var(f,[x'-\delta_0,x'+\delta_0]\times[y'-\delta_0,y'+\delta_0])\\ \leq&\frac{(3A_1+4)(3A_2+4)}{\pi^2}\varepsilon.\end{aligned} \quad (4.13)$$

From Lemmas 2.5, 2.6 and inequality in (4.10),

$$\begin{aligned} I_2 \leq & \frac{(3A_2+4)}{\pi^2} \iint\limits_{\substack{|t_1|\geq\delta_1 \\ |t_2|\leq\delta_1}} \frac{3A_1}{|t_1|u_n} d\mu_{x',y'}(t_1,t_2) \\ \leq & \frac{3A_1(3A_2+4)}{\delta_1 u_n \pi^2} Var(f,\mathbb{R}^2). \end{aligned} \tag{4.14}$$

Using similar arguments, we conclude that

$$\begin{aligned} I_3 \leq & \frac{(3A_1+4)}{\pi^2} \iint\limits_{\substack{|t_1|\leq\delta_1 \\ |t_2|\geq\delta_1}} \frac{3A_2}{|t_2|v_m} d\mu_{x',y'}(t_1,t_2) \\ \leq & \frac{3A_2(3A_1+4)}{\delta_1 v_m \pi^2} Var(f,\mathbb{R}^2). \end{aligned} \tag{4.15}$$

Applying Lemma 2.6 it follows that

$$\begin{aligned} I_4 \leq & \frac{1}{\pi^2} \iint\limits_{\substack{|t_1|\geq\delta_1 \\ |t_2|\geq\delta_1}} \left(\frac{3A_1}{|t_1|u_n}\right)\left(\frac{3A_2}{|t_2|v_m}\right) d\mu_{x',y'}(t_1,t_2) \\ \leq & \frac{9A_1A_2}{\delta_1^2 u_n v_m \pi^2} Var(f,\mathbb{R}^2). \end{aligned} \tag{4.16}$$

Adding the expressions in (4.13), (4.14), (4.15) and (4.16), we have that

$$\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} M_{i,j}(f;x',y') \leq \frac{(3A_1+4)(3A_2+4)}{\pi^2}\left(\varepsilon + \left(\frac{1}{\delta_1 u_n} + \frac{1}{\delta_1 u_m} + \frac{1}{\delta_1^2 u_n v_m}\right) Var(f,\mathbb{R}^2)\right),$$

for each $(x',y') \in (x_0-\delta_0, x_0+\delta_0) \times (y_0-\delta_0, y_0+\delta_0)$ and $n,m \in \mathbb{N}$.

Considering (4.12) and $u_n, v_m \to \infty$, as $n,m \to \infty$, respectively, and given $\varepsilon > 0$, there exist $N, M \in \mathbb{N}$ such that $n \geq N$ and $m \geq M$, then

$$\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} M_{i,j}(f;x',y') \leq \frac{(3A_1+4)(3A_2+4)}{\pi^2} 4\varepsilon,$$

for each $(x',y') \in (x_0-\delta_0, x_0+\delta_0) \times (y_0-\delta_0, y_0+\delta_0)$. This prove the theorem. □

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