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Advances in Trigonometric and Hyperbolic Inequalities Using Mittag-Leffler Function Approaches

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ABSTRACT: In this research, we investigate significant inequalities related to trigonometric and hyperbolic functions. We introduce new definitions of trigonometric and hyperbolic sinc and tanc functions grounded in the Mittag-Leffler function framework. Building on these, we establish several important inequalities involving the nested function H_{pj}. Additionally, we provide a solution to a previously open problem in this area, contributing novel insights to the theory of special functions and inequalities.

Keywords: Mittage-Leffler Functions, Nested functions, Wilker and Huygen's inequalities.

Contents

1. Introduction

In recent years, inequalities involving trigonometric and hyperbolic functions have garnered considerable attention due to their wide-ranging applications in mathematical analysis, differential equations, and approximation theory. Among these, inequalities related to the *sinc* function (also known as the sine cardinal or sampling function) and the *tanc* function have emerged as significant topics of study. These functions have been the subject of numerous investigations, leading to various refinements and generalizations of classical results such as Wilker's and Huygens-type inequalities [14,18,22] and [25,26,27,28,29].

This study is structured into two main sections. In the first section, we revisit the classical Wilker and Huygens inequalities, along with several of their generalizations [6,8,17,16]. We then introduce a novel class of trigonometric and hyperbolic functions defined via *Mittag-Leffler-type kernels*, leading to the so-called *nested functions* $T_{pj}(x)$ and $H_{pj}(x)$. These generalizations are inspired by fractional calculus and provide a powerful framework for constructing new analytic inequalities [1,2,20,24].

In this work, we define new versions of the *sinc* and *tanc* functions based on the nested structures of T_{pj} and H_{pj}. Then, we establish several inequalities involving these generalized forms. Section 2 is devoted to further analysis of these inequalities, focusing on nested function H_{pj} and prove various results. We also address an open problem highlighted in [4], contributing a new result to this growing body of literature.

For additional background on Mittag-Leffler functions and related developments in the theory of trigonometric and hyperbolic inequalities (including refinements of the Wilker and Huygens-type inequalities) readers may refer to [2,3,5,9,12,19] and [23].

We begin this section by stating the following inequalities, which lay the foundation for the forthcoming analysis:

Lemma 1.1 *[20] For each $x \neq 0$ the following inequalities hold:*

$$\frac{\tanh(x)}{x} < 1$$

$$\frac{T_{p1}(x)}{x} > 1.$$

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Theorem 1.1 *[20] For each $x \neq 0$ and $n \geq 1$ the following inequality holds:*

$$\left(\frac{\sinh(x)}{x}\right)^n + \frac{n}{2}\frac{\tanh(x)}{x} > \frac{n+2}{2} .$$

Lemma 1.2 *[28] Let $x > 0$, then*

$$\left(\frac{\sinh x}{x}\right)^n > \cosh x.$$

holds if and only if $n \geq 3$.

The Wilker's-Anglesio inequality is given as (see [22], [25]).

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 + \frac{8}{45}x^3 \tanh x.$$

Theorem 1.2 *[22] For$x > 0$, we have*

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 3 + \frac{3}{20}x^3 \tanh x.$$

Lemma 1.3 *Let x and y be positive real numbers. Then,*

i. (Mitrinovic et al. [14]) For $\mu \in [0,1]$,

$$\mu x + (1-\mu) \geqslant x^{\mu} y^{1-\mu}.$$

ii. (Issa and Ibrahimov [9]) For $x \geq y$ and $\mu \in \left[\frac{1}{2}, 1\right]$,

$$\mu x + (1-\mu)\, y \geqslant x^{1-\mu} y^{\mu} + (2\mu - 1)\,(x-y) \geqslant x^{\mu} y^{1-\mu}.$$

iii. (Issa and Ibrahimov [9]) For $x \geq y$ and $\mu \in \left[\frac{1}{2}, \frac{3}{4}\right]$,

$$\mu x + (1-\mu)\, y \geqslant x^{\mu-\frac{1}{2}} y^{\frac{3}{2}-\mu} + \frac{(x-y)}{2} \geqslant x^{\mu} y^{1-\mu}.$$

Definition 1.1 *[11], [12] The sinc and tanc functions are defined as follows:*

$$\begin{aligned} \mathit{sinc}\ z &= \{ \begin{array}{c} \frac{\sin z}{z}, z \neq 0 \\ 1, z = 0 \end{array} , \ \mathit{tanc}\ z = \{ \begin{array}{c} \frac{\tan z}{z}, z \neq 0 \\ 1, z = 0 \end{array} \\ \mathit{sinhc}\ z &= \{ \begin{array}{c} \frac{\sinh z}{z}, z \neq 0 \\ 1, z = 0 \end{array} , \ \mathit{tanhc}\ z = \{ \begin{array}{c} \frac{\tanh z}{z}, z \neq 0 \\ 1, z = 0. \end{array} \end{aligned}$$

Definition 1.2 *[1], [19] H and T nested functions $T_{p\ j}, H_{p\ j} : \mathbb{R} \to \mathbb{R}$, $j = 0, 1, 2, \cdots, p-1$, $p \in \mathbb{N}$, are defined as follows:*

$$T_{pj}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{pn+j}}{(pn+j)!}, \quad H_{pj}(t) = \sum_{n=0}^{\infty} \frac{t^{pn+j}}{(pn+j)!}.$$

Definition 1.3 *[20] The functions ${}_p\tan_{i\ j}$, ${}_p\tanh_{i\ j} : \mathbb{R} \to \mathbb{R}$, $i, j = 0, 1, 2, \cdots, p-1$, $p \in \mathbb{N}, i \neq j$ are defined as follows:*

$${}_p\tan_{i\ j}(t) = \frac{T_{pi}(t)}{T_{pj}(t)}, \quad {}_p\tanh_{i\ j}(t) = \frac{H_{pi}(t)}{H_{pj}(t)}.$$

For example, for $i = 1$ and $j = 0$, we get

$${}_p\tan_{1\ 0}(t) = \frac{T_{p1}(t)}{T_{p0}(t)}, \quad {}_p\tanh_{1\ 0}(t) = \frac{H_{p1}(t)}{H_{p0}(t)}.$$

Definition 1.4 *We define ${}_p tanhc_{i\ 0}(x)$ as follows:*

$$ {}_p tanhc_{i\ 0}(x) = \{ \begin{matrix} \frac{{}_p \tan_{i\ 0}(x)}{x^i}, x \neq 0. \\ 1, \quad x = 0 \end{matrix} $$

For example, for $i = 1$, we have

$$ {}_p tanhc_{1\ 0}(x) = \{ \begin{matrix} \frac{{}_p tanhc_{i\ 0}(x)}{x}, x \neq 0 \\ 1, \quad x = 0. \end{matrix} $$

Lemma 1.4 *[20] For each $x \neq 0$, the following inequalities hold:*

$$ \begin{aligned} \frac{H_{pi}(x)}{x^i} &> 1, \ i = 0, 1, 2, \cdots, p-1, \\ H_{p0}(x) &> 1. \end{aligned} $$

And

$$ \begin{aligned} \frac{{}_p tanhc_{i\ 0}(x)}{x^i} &< 1 \quad , \ i = 0, 1, 2, \cdots, p-1, \\ \frac{{}_p \tanh_{i\ 0}(x)}{x^i} &< 1 \ , \ i = 0, 1, 2, \cdots, p-1. \end{aligned} $$

Lemma 1.5 *[20] For each $x \neq 0$, the following inequalities are valid:*

$$ \frac{{}_p \tanh_{1\ 0}(x)}{x} < 1 \quad \text{and} \quad \frac{H_{p1}(x)}{x} > 1. $$

Lemma 1.6 *[20] For each $x \neq 0$, the following inequalities hold:*

$$ (\frac{H_{p1}(x)}{x})^{p+1} > H_{p0}(x), \ \text{for} \ \ p \geq 2, \ , \ i = 0, 1, 2, \cdots, p-1. $$

If we put $p = 2$, then, we obtain the following corollary.

Corollary 1.1 *For each $x \neq 0$, the following inequality is valid.*

$$ (\frac{\sinh x}{x})^3 > \cosh(x). $$

Lemma 1.7 *[20] For each $x \neq 0$, the following inequality holds.*

$$ (\frac{H_{pi}(x)}{x})^{p+i} > H_{p0}(x), \ \text{for} \ \ p \geq 2, \ , \ i = 0, 1, 2, \cdots, p-1. $$

Lemma 1.8 *[20] For each $x \neq 0$, the following inequality holds.*

$$ p\left(\frac{H_{pi}(x)}{x}\right) + \frac{{}_p \tanh_{i\ 0}(x)}{x^i} > p + 1. \qquad (x \neq 0) $$

Lemma 1.9 *[28] Let $x > 0$, then*

$$ \left(\frac{\sinh x}{x}\right)^m > \cosh^n x, $$

holds if and only if $m \geq 3n$.

Proof. By Corollary 1.1, we have

$$ \left(\frac{\sinh x}{x}\right)^3 > \cosh x. $$

So,

$$ \left(\frac{\sinh x}{x}\right)^m > \left(\frac{\sinh x}{x}\right)^{3n} > \cosh^n x. $$

Lemma 1.10 *Let $x > 0$, then the following inequalities hold.*

$$\begin{aligned} 2xH_{p\ 0}(x)H_{p\ p-1}(x) &> p^2(H_{p\ 0}(x)-1), \\ 2H_{p\ 0}(x)H_{p\ p-1}(x) &> p\sum_{n=1}^{\infty}\frac{x^{pn-1}}{(pn)!}. \end{aligned}$$

Proof. We prove the second inequality. By expanding the left hand side product, we have

$$H_{p\ 0}(x)H_{p\ p-1}(x) = \sum_{n=1}^{\infty} x^{pn-1}\sum_{m=0}^{n-1}\frac{1}{(pm)!(p(n-m)-1)!}.$$

Hence, to prove the inequality

$$2H_{p\ 0}(x)H_{p\ p-1}(x) > p\sum_{n=1}^{\infty}\frac{x^{pn-1}}{(pn)!},$$

it suffices to prove that for each $n \geq 1$, the following inequaliy holds:

$$2\sum_{m=0}^{n-1}\frac{1}{(pm)!(p(n-m)-1)!} > \frac{p}{(pn)!}.$$

Since $(pm)!(p(n-m)-1)! < (pn)!$, each term of this series exceeds $\frac{1}{(pn)!}$. As, there are n such terms, so we get that

$$\sum_{m=0}^{n-1}\frac{1}{(pm)!(p(n-m)-1)!} > \frac{n}{(pn)!}.$$

Hence, we obtain

$$2\sum_{m=0}^{n-1}\frac{1}{(pm)!(p(n-m)-1)!} > \frac{2n}{(pn)!} \geq \frac{p}{(pn)!},$$

for all n satisfying $2n \geq p$. The cases where n is small can be proved directly.

Cosequently, summing over n shows that the inequality holds for all $x > 0$.

Similarly, we can prove the first inequality.

2. More inequalities and answer to an open problem

In this section, we first consider several important algebraic inequalities, which we then use to prove new inequalities involving trigonometric and hyperbolic functions. Subsequently, we study further inequalities related to the new sinc and tanc functions defined through the nested functions T_{pj} and H_{pj}. Additionally, the authors present various inequalities that provide bounds for sums involving these two new hyperbolic sinc and tanc functions. Finally, an open problem posed in [11,12] is addressed and resolved.

Lemma 2.1 *For real variables a and b with $b \neq 0$, $b \neq \pm\frac{1}{2}$, and such that*

$$\frac{4b^2-2b-1}{4b^2-1} \geq 0, \quad \textit{that is,} \quad b \in (-\infty, -\tfrac{1}{2}) \cup \left(-\tfrac{\sqrt{5}-1}{4}, \tfrac{1}{2}\right) \cup \left(\tfrac{\sqrt{5}+1}{4}, \infty\right).$$

Then, the following inequality holds:

$$a^2 + \frac{4b^2-2b-1}{2b} \quad \geq \quad \frac{2b}{4b^2-1}\left(a + \frac{4b^2-2b-1}{2b}\right)^2.$$

Proof. Let

$$Y = \frac{4b^2 - 2b - 1}{2b},$$

where $b \neq 0, \pm\frac{1}{2}$.

We aim to prove the following inequality:

$$a^2 + Y \geq \frac{2b}{4b^2 - 1}(a + Y)^2.$$

After rearranging, we consider the following difference:

$$a^2 - \frac{2b}{4b^2 - 1}(a + Y)^2 + Y.$$

By expanding and simplifying, this expression equals to the following expression:

$$\frac{(4b^2 - 2b - 1)(1 - a)^2}{4b^2 - 1}.$$

Since $(1 - a)^2 \geq 0$, the right-hand side is nonnegative whenever we have

$$\frac{4b^2 - 2b - 1}{4b^2 - 1} \geq 0,$$

and the denominator $4b^2 - 1 \neq 0$, which holds according to the given restrictions on b.

Thus, we get

$$a^2 + \frac{4b^2 - 2b - 1}{2b} \geq \frac{2b}{4b^2 - 1}\left(a + \frac{4b^2 - 2b - 1}{2b}\right)^2.$$

Hence, the proof is completed.

By Lemma 2.1, we have the following corollaries.

Corollary 2.1 *For real variables a, $b \neq 0, \frac{1}{2}, \frac{-1}{2}$ and $x \neq 0$, following inequalities hold.*

$$\begin{aligned} \cosh^2 x + \frac{4b^2 - 2b - 1}{2b} &\geq \frac{2b}{4b^2 - 1}(\cosh x + \frac{4b^2 - 2b - 1}{2b})^2, \\ H_{pi}^2 x + \frac{4b^2 - 2b - 1}{2b} &\geq \frac{2b}{4b^2 - 1}(H_{pi} x + \frac{31}{6})^2. \end{aligned}$$

Corollary 2.2 *For real variable $x \neq 0$, following inequalities hold.*

$$\begin{aligned} \cosh^2 x + \frac{11}{4} &\geq \frac{4}{15}(\cosh x + \frac{11}{4})^2 \quad [11], \\ H_{p\ i}^2 x + \frac{11}{4} &\geq \frac{4}{15}(H_{p\ i} x + \frac{11}{4})^2. \end{aligned}$$

Corollary 2.3 *For real variable $x \neq 0$, following inequalities hold.*

$$\begin{aligned} \cosh^2 x + \frac{55}{8} &\geq \frac{8}{63}(\cosh x + \frac{47}{8})^2, \\ H_{p\ i}^2 x + \frac{55}{8} &\geq \frac{8}{63}(H_{p\ i} x + \frac{55}{16})^2. \end{aligned}$$

Corollary 2.4 *For real variable $x \neq 0$, following inequalities hold.*

$$\begin{aligned} \cosh^2 x + \frac{31}{6} &\geq \frac{6}{37}(\cosh x + \frac{31}{6})^2, \\ H_{p\ i}^2 x + \frac{31}{6} &\geq \frac{31}{37}(H_{p\ i} x + \frac{31}{6})^2. \end{aligned}$$

Lemma 2.2 *For $x \neq 0$, the following inequalities hold.*

$$
\begin{aligned}
2+\cosh^2(x) &> \frac{\sinh(2x)}{2x}+2\frac{\sinh(x)}{x} > 1+2\cosh(x)\\
2+\cosh^2(x) &> \frac{\sinh(2x)}{2x}+2\frac{\sinh(x)}{x}+\frac{11}{60}x^3\tanh x.
\end{aligned}
$$

Proof. To prove $\frac{\sinh(2x)}{2x}+2\frac{\sinh(x)}{x} > 1+2\cosh(x))$, we have

$$
\begin{aligned}
\frac{\sinh(2x)}{2x}+2\frac{\sinh(x)}{x} &= \\
\sum_{n=0}^{\infty}\frac{2^{2n}x^{2n}}{(2n+1)!}+2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n+1)!} &= 3+\sum_{n=1}^{\infty}\frac{2^{2n}x^{2n}}{(2n+1)!}+\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n+1)!}\\
&= 3+\sum_{n=1}^{\infty}\frac{(2^{2n}+2)x^{2n}}{(2n+1)!} > 3+2\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\\
&= 1+2\cosh(x).
\end{aligned}
$$

And for ($2+\cosh^2(x) > \frac{\sinh(2x)}{2x}+2\frac{\sinh(x)}{x}$)

$$
\sum_{n=0}^{\infty}\frac{2^{2n}x^{2n}}{(2n+1)!}+2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n+1)!}=3+\sum_{n=1}^{\infty}\frac{(2^{2n}+1)x^{2n}}{(2n+1)!}
$$

$$
\begin{aligned}
2+(1+\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!})^2 &= 3+2\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}+(\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!})^2\\
&\quad \&\\
(\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!})^2-\sum_{n=1}^{\infty}\frac{2^{2n}-4n-1}{(2n+1)!}x^{2n} &> 0\\
&\Longrightarrow\\
(\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!})^2+\sum_{n=1}^{\infty}[\frac{2}{(2n)!}-\frac{1}{(2n+1)!}]x^{2n}-\sum_{n=1}^{\infty}\frac{(2^{2n})x^{2n}}{(2n+1)!} &> 0
\end{aligned}
$$

Theorem 2.1 *(theorem 2, [11]). For $x \in R$ and $n \geq 2$, we have*

$$
\frac{4}{15}(\cosh t+\frac{11}{4})^2-\frac{3}{4} > \frac{\sinh(2x)}{2x}+2\frac{\sinh(x)}{x} > 1+2\cosh(x)+\sum_{k=2}^{m}b_kx^{2k},
$$

where $b_k = \frac{(2^{2n}-4k)}{(2k+1)!}$ for $k \geq 2$.

Lemma 2.3 *For $x \neq 0$, following inequality holds.*

$$
p+H_{p\ 0}^{p}(x) > \frac{H_{p\ 1}(px)}{px}+p\frac{H_{p\ 1}(x)}{x} > 1+pH_{p\ 0}(x).
$$

Proof. (Simple proof). Note that ($\frac{1}{(pn+1)!}-\frac{1}{(p+1)(pn)!}) < 0\, n \geq 2)$

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{p^{pn}x^{pn}}{(pn+1)!}+p\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn+1)!} &= 1+p+\sum_{n=1}^{\infty}\frac{p^{pn}x^{pn}}{(pn+1)!}+p\sum_{n=1}^{\infty}\frac{x^{pn}}{(pn+1)!}\\
&= p+1+\sum_{n=1}^{\infty}\frac{(p^{pn}+1)x^{pn}}{(pn+1)!} > p+1+p\sum_{n=1}^{\infty}\frac{x^{pn}}{(pn)!}\\
&= 1+p\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn)!} = 1+pH_{p0}(x).
\end{aligned}
$$

Theorem 2.2 *For $x \in R$ and $n \geq 2$, following inequality is valid.*

$$\frac{2p}{4p^2-1}(H_{p1}t+\frac{4p^2-2p-1}{2p})^2-\frac{2p^2-2p-1}{2p} > \frac{H_{p\ 1}(px)}{px}+p\frac{H_{p\ 1}(x)}{x}$$
$$> 1+pH_{p\ 0}(x)+\sum_{k=p}^{m} b_k x^{pk},$$

where, $b_k = \frac{(p^{pk}-p^2k)}{(pk+1)!}$ *for* $k \geq p$.

Proof.

$$\begin{aligned} & \frac{H_{p1}(px)}{px}+p\frac{H_{p1}(x)}{x}-1-pH_{p0}(x)-\sum_{k=2}^{m} b_k x^{pk} \\ = & \sum_{n=0}^{\infty}\frac{p^{pn}x^{pn}}{(pn+1)!}+p\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn+1)!}-1-p\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn)!}-\sum_{k=p}^{m}\frac{(p^{pk}-p^2k)}{(pk)!}x^{pk} \\ = & \sum_{n=0}^{\infty}\frac{(p^{pk}-p^2k)}{(pk+1)!}x^{pk}-\sum_{k=p}^{m}\frac{(p^{pk}-p^2k)}{(pk)!}x^{pk} \\ = & \sum_{n=m+1}^{\infty}\frac{(p^{pk}-p^2k)}{(pk+1)!}x^{pk} > 0. \end{aligned}$$

Hence, we get

$$\frac{H_{p1}(px)}{px}+p\frac{H_{p1}(x)}{x} > 1+pH_{p0}(x)+\sum_{k=2}^{m} b_k x^{pk}$$

Thus, the left-hand side inequality is proved.

For the other side, we have

$$\begin{aligned} & \frac{H_{p1}(px)}{px}+p\frac{H_{p1}(x)}{x}-\frac{2p}{4p^2-1}(H_{p1}t+\frac{4p^2-2p-1}{2p})^2+\frac{2p^2-2p-1}{2p} \\ = & \sum_{n=0}^{\infty}\frac{p^{pn}x^{pn}}{(pn+1)!}+p\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn+1)!}-\frac{2(4p^2-2p-1)}{4p^2-1}\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn)!}-\frac{p}{4p^2-1}\sum_{k=p}^{m}\frac{(p^{pk}}{(pk)!}x^{pk} \\ = & -\sum_{n=3}^{\infty} a_k x^{pk} < 0, \end{aligned}$$

$a_k = \frac{(2p^2-2p-1)k-p+p^{pk-p}(p^2k-(2p^2-3p-1))}{(pk+1)!.}$

Hence, the inequality of the right-hand side is proved.

Lemma 2.4 *For $x \neq 0$, following inequality holds.*

$$\frac{H_{p1}(px)}{px}+p\frac{H_{p1}(x)}{x} > 1+pH_{p0}(x).$$

Proof. We have

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{p^{pn}x^{pn}}{(pn+1)!}+p\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn+1)!} &= 1+p+\sum_{n=1}^{\infty}\frac{p^{pn}x^{pn}}{(pn+1)!}+p\sum_{n=1}^{\infty}\frac{x^{pn}}{(pn+1)!} \\ &= p+1+p\sum_{n=1}^{\infty}\frac{(p^{pn-1}+1)x^{pn}}{(pn+1)!} > p+1+p\sum_{n=1}^{\infty}\frac{x^{pn}}{(pn)!} \\ &= 1+p\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn)!} = 1+pH_{p0}(x). \end{aligned}$$

Lemma 2.5 *For $x \neq 0$, following inequalities are valid.*

$$\begin{aligned} H_{pi}(x) + p\frac{1}{H_{pi}(x)} &> 2 + (p-1)\frac{1}{H_{pi}(x)}, \\ pH_{pi}(x) + \frac{1}{H_{pi}(x)} &> p+1. \end{aligned}$$

Proof. It can be prove by direct calculations.

Lemma 2.6 *For $x \neq 0$, following inequality is valid.*

$$\frac{1}{4\cosh^2(x)} + \frac{3}{4}\frac{x\cosh(x)}{\sinh(x)} > \cosh^{\frac{1}{4}}(x)(\frac{x}{\sinh x})^{\frac{3}{4}}.$$

Proof. We set a, b as follows:

$$a = \frac{1}{\cosh^2(x)} > 0, \quad b = \frac{x\cosh(x)}{\sinh(x)} > 0, \quad \text{for } x \neq 0.$$

Note that both a and b are positive for all $x \neq 0$.

By the weighted arithmetic mean–geometric mean (AM–GM) inequality, for weights $\alpha = \frac{1}{4}$ and $\beta = \frac{3}{4}$ with $\alpha + \beta = 1$, we have

$$\alpha a + \beta b \geq a^{\alpha} b^{\beta},$$

and the equality is valid if and only if $a = b$.

By substituting a and b, we get

$$\frac{1}{4\cosh^2(x)} + \frac{3}{4}\frac{x\cosh(x)}{\sinh(x)} \geq \left(\frac{1}{\cosh^2(x)}\right)^{\frac{1}{4}} \left(\frac{x\cosh(x)}{\sinh(x)}\right)^{\frac{3}{4}}.$$

This inequality can be rewritten as follows:

$$\frac{1}{4\cosh^2(x)} + \frac{3}{4}\frac{x\cosh(x)}{\sinh(x)} \geq \cosh^{-\frac{1}{2}\cdot\frac{1}{2}}(x) \cdot \cosh^{\frac{3}{4}}(x) \left(\frac{x}{\sinh(x)}\right)^{\frac{3}{4}}.$$

Since $\left(\frac{1}{\cosh^2(x)}\right)^{\frac{1}{4}} = \cosh^{-\frac{1}{2}}(x)$, we can simplify the right-hand side as follows:

$$\cosh^{-\frac{1}{2}}(x) \cdot \left(\frac{x\cosh(x)}{\sinh(x)}\right)^{\frac{3}{4}} = \cosh^{-\frac{1}{2}}(x) \cdot \cosh^{\frac{3}{4}}(x) \cdot \left(\frac{x}{\sinh(x)}\right)^{\frac{3}{4}} = \cosh^{\frac{1}{4}}(x) \left(\frac{x}{\sinh(x)}\right)^{\frac{3}{4}}.$$

Thus, we obtain

$$\frac{1}{4\cosh^2(x)} + \frac{3}{4}\frac{x\cosh(x)}{\sinh(x)} \geq \cosh^{\frac{1}{4}}(x) \left(\frac{x}{\sinh(x)}\right)^{\frac{3}{4}}.$$

To prove the strict inequality for $x \neq 0$, we see that equality holds when $a = b$. This means the equality holds when we have

$$\frac{1}{\cosh^2(x)} = \frac{x\cosh(x)}{\sinh(x)}.$$

However, by considering the behavior near zero and for other values of x, it can be seen that the inequality is strict for all $x \neq 0$.

This completes the proof.

Lemma 2.7 *For $x \neq 0$, following inequality holds.*

$$\frac{1}{(p+2)H_{p0}^{p}(x)} + \frac{(p+1)}{p+2}\frac{xH_{p0}(x)}{H_{p1}(x)} > (H_{p0}(x))^{\frac{1}{p+2}}(\frac{x}{H_{p1}(x)})^{\frac{p+1}{p+2}}.$$

Proof. We have

$$\begin{aligned}\frac{1}{(p+2)H_{p0}^p(x)}+\frac{(p+1)}{p+2}\frac{xH_{p0}(x)}{H_{p1}(x)} &> (\frac{1}{H_{p0}^p(x)})^{\frac{1}{p+2}}(\frac{xH_{p0}(x)}{H_{p1}(x)})^{\frac{p+1}{p+2}}\\ &= (H_{p0}(x))^{\frac{-p}{p+2}}(H_{p0}(x))^{\frac{p+1}{p+2}}(\frac{x}{H_{p1}(x)})^{\frac{p+1}{p+2}}\\ &= (H_{p0}(x))^{\frac{1}{p+2}}(\frac{x}{H_{p1}(x)})^{\frac{p+1}{p+2}}.\end{aligned}$$

Theorem 2.3 *For $t \in (0,\infty)$, following inequality holds.*

$$\frac{1}{p+2}H_{p0}^p t+\frac{p+1}{p+2}\frac{H_{p1}t}{xH_{p0}t} > (\frac{1}{H_{p0}t})^{\frac{1}{p+2}}(\frac{H_{p1}t}{x})^{\frac{p+1}{p+2}}.$$

Proof. We have,

$$\begin{aligned}\frac{1}{p+2}H_{p0}^p t+\frac{p+1}{p+2}\frac{H_{p1}t}{xH_{p0}t} &> (H_{p0}^p t)^{\frac{1}{p+2}}(\frac{H_{p1}t}{xH_{p0}t})^{\frac{p+1}{p+2}} = (H_{p0}^p t)(\frac{H_{p1}t}{xH_{p0}t})^{p+1})^{\frac{1}{p+2}}\\ &= ((H_{p0}^p t)(\frac{1}{H_{p0}t})^{p+1})^{\frac{1}{p+2}}(\frac{H_{p1}t}{x})^{\frac{p+1}{p+2}}\\ &= (\frac{1}{H_{p0}t})^{\frac{1}{p+2}}(\frac{H_{p1}t}{x})^{\frac{p+1}{p+2}}.\end{aligned}$$

Theorem 2.4 *For $x \neq 0$ and $q \geq p \geq 2$, following inequality is valid.*

$$\frac{H_{p1}(qx)}{qx}+q\frac{H_{p1}(x)}{x} > 1+qH_{p0}(x).$$

Proof. By using $\frac{(q^{pn-2}+1)}{(pn+1)!} \geq \frac{1}{(pn)!}, n \geq 1$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{pn-1}x^{pn}}{(pn+1)!}+q\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn+1)!} &= 1+q+\sum_{n=1}^{\infty}\frac{q^{pn-1}x^{pn}}{(pn+1)!}+q\sum_{n=1}^{\infty}\frac{x^{pn}}{(pn+1)!}\\ &= 1+q+q\sum_{n=1}^{\infty}\frac{(q^{pn-2}+1)x^{pn}}{(pn+1)!} > \frac{1}{q}+q+q\sum_{n=1}^{\infty}\frac{x^{pn}}{(pn)!}\\ &= 1+q\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn)!} = 1+qH_{p0}(x).\end{aligned}$$

By Theorem 2.4 we have the following corollary.

Corollary 2.5 *For $x \neq 0$ and $q \geq 2$, following inequality is valid.*

$$\frac{\sinh(qx)}{qx}+q\frac{\sinh(x)}{x} > 1+q\cosh(x).$$

Remark 2.1 By Corollary 2.5, we provided an answer to the open problem (part (iv), inequality (1.6)) mentioned in the paper [11].

Theorem 2.5 *For $x \neq 0$ and $p \geq 3$, the following inequality holds.*

$$\frac{H_{p\ i}(px)}{(px)^i}+p^i\frac{H_{p\ i}(x)}{x^i} > 1+\frac{p^i}{x^{i-1}}H_{p\ i-1}(x), \quad i=1,...,p-2.$$

Proof. Note that $p^{pn-i}+1 \geq p+i, p \geq 3, , i=1,...,p-2$. Thus, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{p^{pn}x^{pn}}{(pn+i)!}+p^{i}\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn+i)!} &= 1+p^{i}+\sum_{n=1}^{\infty}\frac{p^{pn}x^{pn}}{(pn+i)!}+p^{i}\sum_{n=1}^{\infty}\frac{x^{pn}}{(pn+i)!}\\ &= p^{i}+1+\sum_{n=1}^{\infty}\frac{(p^{pn}+p^{i})x^{pn}}{(pn+i)!} > p^{i}+1+p^{i}\sum_{n=1}^{\infty}\frac{x^{pn}}{(pn+i-1)!}\\ &= 1+p^{i}\sum_{n=0}^{\infty}\frac{x^{pn}}{(pn+i-1)!}=1+\frac{p^{i}}{x^{i-1}}H_{p\ i-1}(x).\end{aligned}$$

By selecting $p=2$ and $i=1$, we have the following corollary that presents a Wilker-type inequality for the nested function $H_{p\,i}$.

Corollary 2.6 *For $x \neq 0$, the following inequality holds:*

$$\frac{H_{2\,1}(2x)}{2x}+2\frac{H_{2\,1}(x)}{x} > 1+\frac{2^{1}H_{2\,0}(x)}{x^{0}}.$$

As we know,

$$H_{2\,1}(t)=\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}=\sinh(t) \quad \textit{and} \quad H_{2\,0}(t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}=\cosh(t).$$

Thus, this inequality equivalently, can be written as the following inequality:

$$\frac{\sinh(2x)}{2x}+2\frac{\sinh(x)}{x} > 1+2\cosh(x).$$

Remark 2.2 For all $x>0$, the inequality of Corollary 2.6, equivalently can be written as the following inequality:

$$\frac{\sinh x}{x}(\cosh x+2) > 1+2\cosh x.$$

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