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A Novel Method for Solving Fractional Order Delay Integro-Differential Equations via Optimal Auxiliary Function Method

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ABSTRACT: The method for determining the optimal auxiliary function is developed for non-linear delay integro differential equations. The proposed technique ensures high accuracy and allows for control over approximate solution convergence. OAFM ensures quick convergence with a single iteration, making it highly effective.The efficiency and reliability of the proposed approach are demonstrated by some numerical applications,which support the theoritical results of the implemented algorithm.

Key Words: Fractional integro-differential equations, optimal auxiliary function method, delay differential equations, least squares method.

Contents

1. Introduction

Fractional calculus is the process of integrating or dividing non-integer orders. Interestingly, this subject has a lengthy history in calculus. The first discussion on fractional calculus was between Leibniz and L'Hopital. He enquired about the differentiation of order half in certain functions. Mathematicians such as Riemann, Abel, Liouville, and Lacroix pioneered fractional calculus and dominated the field. Abel was the first to describe the integral system of order 1/2 in his renowned article on the even time issue see [1,2,3,4,5]. Recent research has focused on solving fractional differential and integro-differential problems. FIDEs are widely used in fields like as mechanical engineering, nuclear physics, chemistry, astronomy, biology, economics, potential theory, and electrostatics. In certain circumstances, the exact solution to FIDEs may only be found. Analytical solutions to integro-differential equations can be challenging, thus efficient approximation solutions are often necessary. Recently, many effective techniques have been presented to solve integro-differential equations having fractional-order see [6]. Several studies have examined the existence and uniqueness of fractional-order differential equation solutions [7,8,9,10,11, 13,12].In [14] , fractional-order integro-differential equations are used to simulate some fluid dynamics events. [15], proposes a numerical method based on basic functions.in [16,17,18], the Krasnoselskii fixed-point theorem is used to evaluate the solvability of fractional-order integral differential equations. In [19]. Riemann-Liouville integral and Caputo fractional derivative operators were used to solve nonlinear fractional-order differential equations. This paper investigates the asymptotic stability, boundedness of nonzero solutions, stability of Mittag-Le er zero solutions, and monotonic stability of fractional-order integral differential equations [20].This study examines the existence and uniqueness of fractional-order integro-differential equations in Banach spaces [21,22]. The authors of [23] reached identical findings with the Schauder fixed-point approach. Theorem and contraction map principle. [24], Legendre wavelet collation and discrete and stochastic operational matrices were used to assess uncertainty in solving

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fractional order integro-differential equations. In [25], the Schauder and Banach fixed-point theorem is used to build the conditions for existence and uniqueness using extended distances and piecewise constant functions. In [26], the Jacobi-Gauss collocation approach was used to solve fractional-order integro-differential equations .Similarly, we extended a new method, called as optimal auxuiliary .

function method (OAFM). Marince et al. proposed the approach to achieve an approximate analytical solution For the weak flow of fourth-grade urine At [27]. A Vertical cylinder.This approach adjusts the convergence of approximate solutions using convergence control parameters. Recently Researchers have focused on this process, highlighting its potential uses in natural sciences and engineering. As an example, at [28,29].

Delay differential and integro-differential equations are commonly employed to simulate issues in several fields of research, including ecology and epidemiology [30], immunology [31], physiology [32] and electrodynamics [33]. This research will employ the OAFM to find approximate solutions to fractional order delay integro-differential equations (FDIDE).

2. Preliminaries

This section defines and highlights significant properties of fractional integrals and derivatives [34,35] :

Definition 2.1 *Riemann-Liouville theory defines the fractional integral operator for a function u with order $\alpha > 0$ is:*

$$I_x^{\beta} u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} u(\tau) d\tau, \ x > 0, \ \alpha \in R^+$$

Definition 2.2 *The phrases listed below describe Caputo's fractional derivative with an order $\alpha > 0$*

$$^{C}D_x^{\alpha} u(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \frac{d^m}{dx^m} f(x), & \alpha = m \end{cases}$$

m is an integer.Integral operators by Caputo and Riemann-Liouville offer several advantages, including:
(1) $I_x^{\alpha} \ x^{\beta} = \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\right) x^{\alpha+\beta}, \beta > -1, \alpha > 0$
(2) $^{c}D_x^{\alpha} x^{\beta} = \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\right) x^{\beta-\alpha}, \ \beta > -1, \alpha > 0$
(3) $(I_x^{\alpha} \ (^{c}D_x^{\alpha} \ u(x)) = u(x) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} x^k, x \geq 0, n-1 < \alpha < n$

3. Implementation of the Method for Solving FDIDE

In this section an effective novel method which is so-called optimal auxiliary function method (OAFM) will be used for solving FDIDE as follows :

Consider the FDIDE :

$$^{c}D_x^{\alpha} z(x) = f(x) + H(z(x)) + z(x-\mu) + \int_a^x K(x,s).F(z(s)) \ ds \qquad x \geq \alpha \, , \ 0 < \alpha \leq 1 \tag{3.1}$$

s.t.

$$z(x) = \varphi(x) \quad , \quad a - \mu \ \leq \ x \ \leq a \tag{3.2}$$

$$z(a) = z_a \tag{3.3}$$

According to the (OAFM), the approximate solution of problem (3.1)-(3.3) ,can be written in the following formula :

$$\widetilde{z} = z_0(x) + z_1(x, c_i) \qquad , \ i = 1, 2, ..., p \tag{3.4}$$

To start the proposed approach and by using equation (3.2), equation (3.1) can be re-written in the following form :

$$L\ (z(x)) = \widetilde{f}\ (x) + N\ (z(x)) = 0 \tag{3.5}$$

where

$$L\ (z(x)) = {}^{c}D_x^{\alpha} z(x)\ ,\ \widetilde{f}\ (x)\ =\ f(x) + \varphi(x - \mu) \tag{3.6}$$

and

$$N\ (z(x)) = H\ (z(x)) + \int_a^x K(x, s)\ .\ F(z(s))\ ds \tag{3.7}$$

To obtain the initial and first approximation , substiuting equation (3.4) into equation (3.5) gives :

$$L\ (z_0(x)) + L\ (z_1(x, c_i))\ =\ \widetilde{f}\ (z) + N\ (z_0(x) + z_1(x, c_i)) \qquad i = 1, 2, ..., p \tag{3.8}$$

For finding the initial approximation $z_0(x)$, the following equation must be solved :

$$L\ (z_0(x))\ -\ \widetilde{f}\ (x)\ = 0\ ,\ z_0(a) = z_a \tag{3.9}$$

The function $z_1(x)$, will be found upon solving the below equation :

$$L\ (z_1(x\ , c_i) -\ N\ (z_0(x)) + z_1(x, c_i)) = 0 \quad , \ z_1\ (a) = z_1 \tag{3.10}$$

The nonlinear term in equation (3.10) is expanded in the form :

$$N\ (z_0(x) + z_1\ (x, c_i)) = N\ (z_0(x)) + \sum_{n=0}^{\infty} \frac{z_1^n\ {}_{(x,c_i)}}{n!}\ N^{(n)}\ (z_0(x)) \tag{3.11}$$

This building phase uses a sequence technique to produce a limit solution that is convergent .

To solve the nonlinear integro-differential equation (3.10) and accelerate the convergence of the first approximation $z_1(x, c_i)$ and the implicit solution $\widetilde{z}\ (x)$, an alternative expression is proposed. Equation (3.10) ,can now be expressed as :

$L\ (z_1(x, x_i)) - A_1(z_0(x), c_i)\ N(z_0(x)) - A_2(z_0(x), c_j) = 0$

A_1 and A_2 are arbitrary auxiliary functions based on the initial estimate $z_0(x)$ and unknown values c_i and c_j.Different numerical approaches, such as the Ritz, collocation, Galerkin's, or least squares methods, can be used to minimize the square residual error :

$z\ (c_i, c_j)\ =\ min \int_0^1 (Re(x, c_i, c_j))^2\ dx$

where

$Re\ (x, c_i, c_j) =\ L\ (z(x, c_i, c_j)) - \widetilde{f}\ (x)\ - N\ (z(x, c_i, c_j)) \qquad i = 1, 2, ..., s\ ,\ j = s + 1, s + 2, ..., p\ .$

Adding equation (3.9) and equation (3.10), We get the first approximate answer as :

$\widetilde{z}\ (x) = z_0\ (x) + z_1(x, c_i)\ \ .$

4. Illustrative Examples

In the present section, we investigate the approximate solution of several FDIDEs, by means of (OAFM), the suggested technique will be demonstrated through illustrative examples to ensure efficiency and correctness.

Example 4.1 *consider the following FDIDE:*

$${}^{c}D_x^{\alpha} z(x) = z\ (x) - z\ (x - \mu) - \int_{x-\mu}^{x} x.z\ (s) ds + \frac{\gamma\ (\alpha + 4)}{6} x^3 - x^{\alpha+3} + (x - \mu)^{\alpha+3} - \frac{x}{\alpha + 4} \left(x^{\alpha+4} - (x - \mu)^{\alpha+4}\right) \tag{4.1}$$

$$where\ z(x) = x^{\alpha+3} \qquad , \ -1 \leq x \leq 0,\ 0 < \alpha < 1 \tag{4.2}$$

The perfect answer to the problem (4.1)-4.2) is $z(x) = x^{\alpha+3}$. Following figure 1 represent the approximate solution of problem (4.1)-(4.2) For different alpha values, compared to the precise answer.

Example 4.2 *consider the following FDIDE:*

$$^{c}D_x^{\alpha} z(x) = z(x) - z(x-\mu) + \int_0^x (x+s).z(s)ds + \frac{\gamma(3)}{\gamma(3-\alpha)} x^{2-\alpha} - x^2 - (x-\mu)^2 - \frac{7}{6}x^4 \quad , 0 \le x \le 1 \quad (4.3)$$

$$where\ z(x) = x^2, \quad -\mu \le x \le 0, 0 < \alpha \le 1 \quad (4.4)$$

The perfect answer to the problem (4.3)-(4.4) is $z(x) = x^2$ see [36] . Figure 2 shows the approximate answer for issue (4.3)-(4.4) for different values of α compared with the exact solution when $\alpha = 1$

Example 4.3 *consider the following FDIDE:*

$$^{c}D_x^{\alpha} z(x) = z(x) + z(x-\mu) + \int_0^x x^2 z(s)ds \quad , 0 \le x \le 1 , 0 < \alpha \le 1 \quad (4.5)$$

$$where\ z(x) = x^2 \quad , -\mu \le x \le 0 \quad (4.6)$$

The exact solution of problem (4.5)-(4.6) is $z(x) = x^2$. Figure 3 shows an estimated solution for the problem (4.5)-(4.6) Different values of α are compared to the precise answer at $\alpha = 1$.

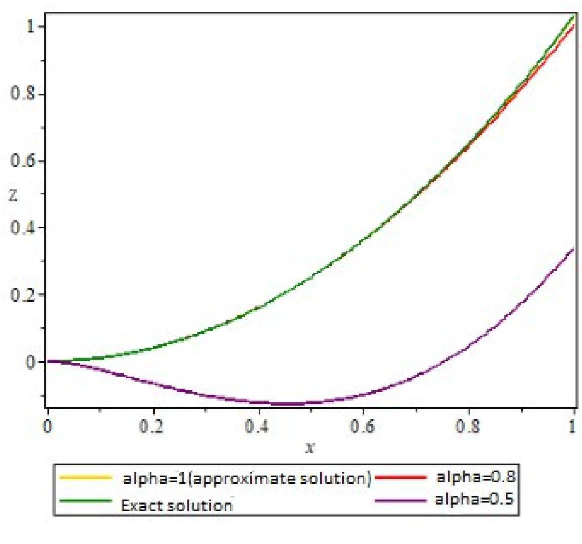

Figure 1: The approximate solution of Example 1 for various values of α

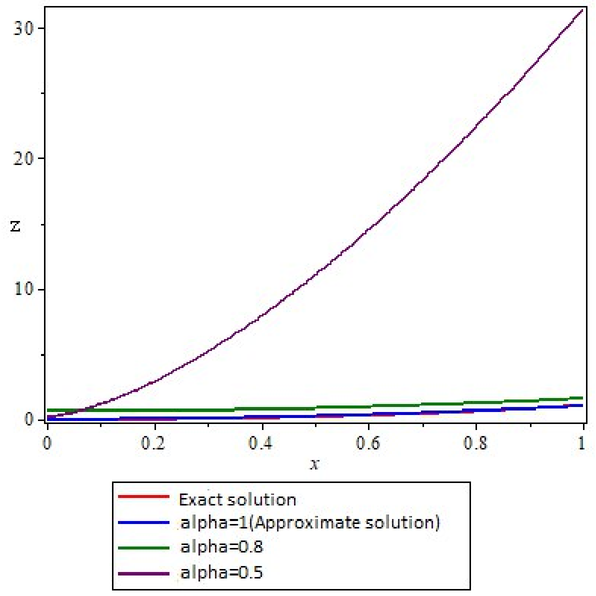

Figure 2: The approximate solution of Example 2 for various values of α

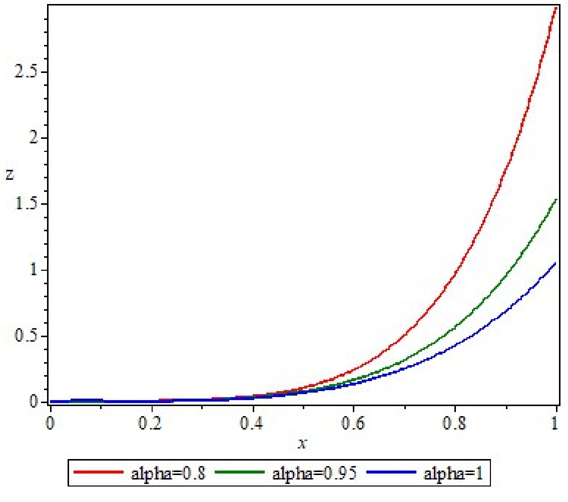

Figure 3: The approximate solution of Example 3 for various values of α

5. Conclusions

In this paper The OAFM has been effectively implemented in solve FDIDEs.

The numerical results from the planned procedure are compared to the precise answer. The suggested technique utilizes auxiliary functions A_1 and A_2, as well as parameters c_1 and c_2 ,... to achieve speedy solution convergence. Finally, to demonstrate the suggested method's accuracy and validity, we present instances with graphs.

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