Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 2** : 1–18.
ISSN-0037-8712
doi:10.5269/bspm.78869

On Common Fixed Point Theorems in S-multiplicative Metric Spaces

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ABSTRACT: In this paper, we study the existence and uniqueness of a common fixed point for two weakly compatible self-maps that satisfy different contractive conditions in S-multiplicative metric spaces. We also provide an example to support the results.

Key Words: S-metric space, multiplicative metric space, S-multiplicative metric space, fixed point.

Contents

1. Introduction

Fréchet [7] introduced the concept of a metric space. Banach [4] has shown the existence of a fixed point for contraction mapping in metric space. It has wide applications in the existence and uniqueness of solutions of differential equations under certain conditions. Since then, metric spaces have become central to functional analysis, topology, and nonlinear analysis, enabling significant contributions to both mathematical theory and its applications.

Many researchers generalized metric spaces in various angles namely G-metric spaces, b-metric spaces, p-metric or extended b metric spaces, quasi-metric spaces, S_p-metric spaces, parametric metric spaces(see [11], [6],[12], [14], [10], [8]). Sedghi et al. [13] introduced S-metric spaces by replacing the standard two-variable distance with a three-variable function $S(x, y, z)$. This generalization provides a richer framework for studying fixed points, convergence, and continuity.

Bashirov et al. [5] introduced the multiplicative metric space. Instead of using additive distances, it describes relationships with multiplicative inequalities, making it ideal for systems involving ratios, proportions, or exponential growth. Recently, Adewale et al. [3] introduced the concept of S-multiplicative metric space, which combines the multiplicative structure with the triple variable nature of S-metrics. This combined structure expands both frameworks and opens up new possibilities for research. In this paper, we will prove some common fixed point theorems in S-multiplicative metric spaces.

2. Preliminaries

Frechet [7] defined metric spaces as follows:

Definition 2.1 [7] Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is said to be metric on X, if it satisfies the following properties:

(i) $d(x, y) \geq 0$ for all $x, y \in X$;

(ii) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(iii) $d(x, y) = d(y, x),\ \forall x, y \in X$;

(iv) $d(x, y) \leq d(x, z) + d(z, y),\ \forall x, y, z \in X$.

2010 *Mathematics Subject Classification:* 54H25, 54E50, 47H10.
Submitted September 09, 2025. Published September 30, 2025

The pair (X, d) is called a metric space.

The concept of S-metric spaces was introduced by Sedghi et al. [13] as follows:

Definition 2.2 [13] Let X be a non empty set and $S : X \times X \times X \to [0, \infty)$ be a mapping satisfying following properties:

(i) $S(x, y, z) = 0$ if and only if $x = y = z$;

(ii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a), \ \ \forall\, a, x, y, z \in X$ (rectangle inequality).

Then (X, S) is called a S-metric space.

Bashirov et al. [5] introduced the concept of multiplicative metric spaces as follows:

Definition 2.3 [5] Let X be a nonempty set, and let $d : X \times X \to [0, \infty)$ be a function satisfying the following properties:

(i) $d(x, y) \geq 1$ for all $x, y \in X$;

(ii) $d(x, y) = 1$ if and only if $x = y$, for all $x, y \in X$;

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iv) $d(x, y) \leq d(x, z)\ d(z, y)$ for all $x, y, z \in X$.

Then d is called multiplicative metric on X and (X, d) is called a multiplicative metric space. By taking the logarithm of property (iv), the structure of a multiplicative metric space becomes equivalent to that of a standard metric space.

The notion of an S-multiplicative metric space was initially defined by Adewale et al. [3] in the following way:

Definition 2.4 [3] Let X be a nonempty set and $\overline{S} : X \times X \times X \to \mathbb{R}^+$, a function satisfying the following properties:

(i) $\overline{S}(x, y, z) \geq 1$;

(ii) $\overline{S}(x, y, z) = 1$ if and only if $x = y = z$;

(iii) $\overline{S}(x, y, z) \leq \overline{S}(x, x, a)\ \overline{S}(y, y, a)\ \overline{S}(z, z, a)$, for all $x, y, z, a \in X$.

Then, $(X, \overline{S})$ is called a S-multiplicative metric space.

Example 2.1 [3] Let $X = \mathbb{Z}$ and define $\overline{S} : X \times X \times X \to \mathbb{R}^+$ by:

$$\overline{S}(x, y, z) = \begin{cases} 1, & \text{if } \ x = y = z; \\ e^{x+y+z}, & \text{otherwise.} \end{cases}$$

Then $(X, \overline{S})$ is a S-multiplicative metric space.

Example 2.2 [3] Let $X = \mathbb{Z}$ and define $\overline{S} : X \times X \times X \to \mathbb{R}^+$ by:

$$\overline{S}(x, y, z) = \begin{cases} 1, & \text{if } \ x = y = z; \\ e^{x}, & \text{otherwise.} \end{cases}$$

Then $(X, \overline{S})$ is a S-multiplicative metric space.

Example 2.3 [3] Let $X = \mathbb{N} \cup \{0\}$ and define $\overline{S} : X \times X \times X \to \mathbb{R}^+$ by:

$$\overline{S}(x, y, z) = \begin{cases} 1, & \text{if } \ x = y = z; \\ xyz, & \text{otherwise.} \end{cases}$$

Then $(X, \overline{S})$ is a S-multiplicative metric space.

Example 2.4 Let $X=[1,\infty)$, and define the function $\overline{S}:X\times X\times X\to\mathbb{R}^+$ by

$$\overline{S}(x,y,z)=\max\left\{\frac{x}{y},\frac{y}{x},\frac{y}{z},\frac{z}{y},\frac{x}{z},\frac{z}{x}\right\}.$$

Then, $(X,\overline{S})$ is S-multiplicative metric space.

Proof: Let $X=[1,\infty)$ and $x,y,z\in X$.

(i) Then all fractions $\frac{x}{y},\frac{y}{x},\frac{y}{z},\frac{z}{x},\frac{x}{z},\frac{z}{x}$ are positive real numbers. Therefore, we have

$$\max\left\{\frac{x}{y},\frac{y}{x}\right\}\geq 1,\quad \max\left\{\frac{y}{z},\frac{z}{y}\right\}\geq 1,\quad \max\left\{\frac{x}{z},\frac{z}{x}\right\}\geq 1.$$

Therefore,

$$\overline{S}(x,y,z)=\max\left\{\frac{x}{y},\frac{y}{x},\frac{y}{z},\frac{z}{y},\frac{x}{z},\frac{z}{x}\right\}\geq 1.$$

(ii) If $x=y=z$, then $\overline{S}(x,y,z)=\max\{1,1,1,1,1,1\}=1$. Conversely, if $\overline{S}(x,y,z)=1$, implies that $\max\left\{\frac{x}{y},\frac{y}{x},\frac{y}{z},\frac{z}{y},\frac{x}{z},\frac{z}{x}\right\}=1$, which further implies that $\frac{x}{y}=\frac{y}{x}=\frac{y}{z}=\frac{z}{y}=\frac{x}{z}=\frac{z}{x}=1$. Therefore, we must have $x=y=z$. Hence, $\overline{S}(x,y,z)=1$ if and only if $x=y=z$.

(iii) Now to prove $\overline{S}(x,y,z)\leq\overline{S}(x,x,a)\ \overline{S}(y,y,a)\ \overline{S}(z,z,a)$, for $x,y,z\in X$, we have,

$$\overline{S}(x,x,a)=\max\left\{1,\frac{x}{a},\frac{a}{x}\right\}=\max\left\{\frac{x}{a},\frac{a}{x}\right\}.$$

Similarly,

$$\overline{S}(y,y,a)=\max\left\{\frac{y}{a},\frac{a}{y}\right\}\quad\text{and}\quad S(z,z,a)=\max\left\{\frac{z}{a},\frac{a}{z}\right\}.$$

Also, we have,

$$\frac{x}{y}=\frac{x}{a}\cdot\frac{a}{y}\leq\max\left\{\frac{x}{a},\frac{a}{x}\right\}\cdot\max\left\{\frac{a}{y},\frac{y}{a}\right\},$$

$$\frac{y}{x}=\frac{y}{a}\cdot\frac{a}{x}\leq\max\left\{\frac{y}{a},\frac{a}{y}\right\}\cdot\max\left\{\frac{a}{x},\frac{x}{a}\right\},$$

and

$$\frac{x}{z}=\frac{x}{a}\cdot\frac{a}{z}\leq\max\left\{\frac{x}{a},\frac{a}{x}\right\}\cdot\max\left\{\frac{a}{z},\frac{z}{a}\right\}.$$

Therefore,

$$\overline{S}(x,y,z)\leq\max\left\{\frac{x}{a},\frac{a}{x}\right\}\ \max\left\{\frac{y}{a},\frac{a}{y}\right\}\ \max\left\{\frac{z}{a},\frac{a}{z}\right\}=\overline{S}(x,x,a)\ \overline{S}(y,y,a)\ \overline{S}(z,z,a).$$

Hence, $(X,\overline{S})$ is S-multiplicative metric space. □

Definition 2.5 [3] Let $(X,\overline{S})$ be a S-multiplicative metric space. For $y\in X$, $r>0$, the S-sphere with center y and radius r is

$$S(y,r)=\{z\in X:\overline{S}(y,z,z)<r\}.$$

Definition 2.6 [3] Let $(X,\overline{S})$ be a S-multiplicative metric space. A sequence $\{x_n\}$ in X is S-convergent to z if it converges to z in the S-multiplicative metric topology.

Definition 2.7 [3] Let (X,S) and $(\overline{X},\overline{S})$ be two S-multiplicative metric spaces. A function $T : X \to \overline{X}$ is S-continuous at a point $x \in X$ if

$$T^{-1}\left(S_{\overline{S}}(T(x),r)\right) \in \tau(S) \quad \text{for all } r > 1.$$

T is S-continuous if it is S-continuous at all points of X.

Lemma 2.1 *[3] Let $(X,\overline{S})$ be a S-multiplicative metric space and $\{x_n\}$ a sequence in X. Then, $\{x_n\}$ converges to x if and only if*

$$\overline{S}(x_n,x,x) \to 1 \quad \text{as } n \to \infty.$$

Lemma 2.2 *[3] Let $(X,\overline{S})$ be a S-multiplicative metric space and $\{x_n\}$ a sequence in X. Then, $\{x_n\}$ is said to be a Cauchy sequence if and only if*

$$\overline{S}(x_n,x_m,x_l) \to 1 \quad \text{as } \; n,m,l \to \infty.$$

Definition 2.8 [1] Let f and g be self maps of a set X. If $w = fx = gx$, for some $x \in X$, then x is called coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 2.9 [9] Let f and g be self maps of a set X. Then f and g are said to be weakly compatible, if they commute at any coincidence point. That is $fgx = gfx$, for $x \in X$.

Proposition 2.1 *[2] Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is unique common fixed point of f and g.*

Proposition 2.2 *If $(X,\overline{S})$ is S-multiplicative metric space, then $\overline{S}(x,x,y) = \overline{S}(y,y,x)$, for all $x,y \in X$.*

Proof: Let $(X,\overline{S})$ is S-multiplicative metric space and $x,y \in X$. Then

$$\overline{S}(x,x,y) \le \left(\overline{S}(x,x,x)\right)^2 \; \overline{S}(y,y,x) = \overline{S}(y,y,x). \tag{2.1}$$

Also,

$$\overline{S}(y,y,x) \le \left(\overline{S}(y,y,y)\right)^2 \; \overline{S}(x,x,y) = \overline{S}(x,x,y). \tag{2.2}$$

From inequalities (2.1) and (2.2), we obtain, $\overline{S}(x,x,y) = \overline{S}(y,y,x)$. □

Proposition 2.3 *If $(X,\overline{S})$ is S-multiplicative metric space, then for $x,y \in X$, $\overline{S}(x,y,y) \le \overline{S}(x,x,y)$.*

Proof: Let $(X,\overline{S})$ is S-multiplicative metric space and $x,y \in X$. Then

$$\overline{S}(x,y,y) \le \overline{S}(x,x,y) \; \left(\overline{S}(y,y,y)\right)^2.$$

Therefore, $\overline{S}(x,y,y) \le \overline{S}(x,x,y)$. □

Proposition 2.4 *If $(X,\overline{S})$ is S-multiplicative metric space, then for $x,y \in X$, $\overline{S}(x,y,x) \le \overline{S}(y,y,x)$.*

Proof: Let $(X,\overline{S})$ is S-multiplicative metric space and $x,y \in X$. Then

$$\overline{S}(x,y,x) \le \overline{S}(x,x,x) \; \overline{S}(y,y,x) \; \overline{S}(x,x,x).$$

Therefore, $\overline{S}(x,y,x) \le \overline{S}(y,y,x)$. □

3. Main results

In this section, we present and prove some common fixed point theorems for mappings defined on S-multiplicative metric space.

Theorem 3.1 *Let $(X, \overline{S})$ be S-multiplicative metric space and $f, g : X \to X$ be the mappings for which there is a real number k satisfying $0 \le k < \frac{1}{2}$ such that*

$$\overline{S}(fx, fy, fz) \le \left(\overline{S}(gx, gy, gz)\right)^k . \tag{3.1}$$

If

(i) $f(X) \subseteq g(X)$;

(ii) $g(X)$ is complete,

then f and g have unique coincidence point in X. Moreover, if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$. This can be done because $f(X) \subseteq g(X)$. Continuing this process, for any $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. Then,

$$\begin{aligned} \overline{S}(gx_n, gx_n, gx_{n+1}) &= \overline{S}(fx_{n-1}, fx_{n-1}, fx_n) \\ &\le \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\right)^k \\ &\le \left(\overline{S}(gx_{n-2}, gx_{n-2}, gx_{n-1})\right)^{k^2} . \end{aligned}$$

Repeating this process, we obtain,

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \le \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n} . \tag{3.2}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have,

$$\begin{aligned} \overline{S}(gx_n, gx_m, gx_m) &\le \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_m, gx_m, gx_{n+1})\right)^2 \\ &\le \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^2 \ \overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \\ &= \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \ \left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^{2^2} . \end{aligned}$$

Continuing in the same way, we obtain,

$$\begin{aligned} \overline{S}(gx_n, gx_m, gx_m) &\le \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \cdots \left(\overline{S}(gx_{m-1}, gx_{m-1}, gx_m)\right)^{2^{n-m}} \\ &\le \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \ \left(\overline{S}(gx_{n+2}, gx_{n+2}, gx_{n+3})\right)^{2^2} \cdots . \end{aligned}$$

Using inequality (3.1), we obtain,

$$\begin{aligned} \overline{S}(gx_n, gx_m, gx_m) &\le \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n} \ \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^{n+1}}\right)^2 \ \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^{n+2}}\right)^{2^2} \cdots \\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n + 2k^{n+1} + 2^2 k^{n+2} + 2^3 k^{n+3} + \cdots} \\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n \left(1 + (2k) + (2k)^2 + (2k)^3 + \cdots\right)} \\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{k^n}{1-2k}} . \qquad \left(\because \ \ 0 \le k < \frac{1}{2}\right) \end{aligned}$$

Therefore,

$$\overline{S}(gx_n, gx_m, gx_m) \le \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{k^n}{1-2k}}.$$

Since, $0 \le k < \dfrac{1}{2} < 1$, letting $n, m \to \infty$, we have, $\dfrac{k^n}{1-2k} \to 0$ and hence,

$$\lim_{n,m\to\infty} \overline{S}(gx_n, gx_m, gx_m) = 1.$$

Now, for $n, m, l \in \mathbb{N}$ with $n > m > l$, we have,

$$\overline{S}(gx_n, gx_m, gx_l) \le \overline{S}(gx_n, gx_n, gx_{n-1})\ \overline{S}(gx_m, gx_m, gx_{n-1})\ \overline{S}(gx_l, gx_l, gx_{n-1}).$$

Letting $n, m, l \to \infty$, we obtain,

$$\lim_{n,m,l\to\infty} \overline{S}(gx_n, gx_m, gx_l) = 1.$$

This shows that the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists a point $q \in g(X)$ such that $gx_n \to q$. That is,

$$\lim_{n\to\infty} gx_n = q = \lim_{n\to\infty} fx_{n-1}.$$

Now, as $q \in g(X)$, there exists $p \in X$ such that $q = g(p)$. Then,

$$\overline{S}(gx_{n+1}, fp, fp) = \overline{S}(fx_n, fp, fp) \le \left(\overline{S}(gx_n, gp, gp)\right)^k = \left(\overline{S}(gx_n, q, q)\right)^k.$$

Letting $n \to \infty$, we obtain,

$$\overline{S}(q, fp, fp) \le \left(\overline{S}(q, q, q)\right)^k = 1.$$

But since $\overline{S}(q, fp, fp) \ge 1$, it must be that $\overline{S}(q, fp, fp) = 1$, which implies that $fp = q = gp$. Therefore, p is coincidence point of f and g. Now, we claim that f and g have unique coincidence point. Contrary, suppose that there exists another coincidence point say $r \ne p$ of f and g. Then,

$$\overline{S}(gr, gp, gp) = \overline{S}(fr, fp, fp) \le \left(\overline{S}(gr, gp, gp)\right)^k.$$

Therefore, $\left(\overline{S}(gr, gp, gp)\right)^{1-k} \le 1$, which implies that $\overline{S}(gr, gp, gp) \le 1$. But $\overline{S}(gr, gp, gp) \ge 1$. It implies that $\overline{S}(gr, gp, gp) = 1$. Hence, we conclude that $gr = gp$. This shows that f and g have unique coincidence point. If f and g are weakly compatible, then by Proposition 2.1, f and g have unique common fixed point in X.

□

Theorem 3.2 *Let $(X, \overline{S})$ be S-multiplicative metric space and $f, g : X \to X$ be the mappings for which there is a real number $0 \le b < \dfrac{1}{3}$ such that*

$$\overline{S}(fx, fy, fz) \le \left[\overline{S}(gx, fx, fx)\ \overline{S}(gy, fy, fy)\ \overline{S}(gz, fz, fz)\right]^b. \tag{3.3}$$

If

(i) $f(X) \subseteq g(X)$;

(ii) $g(X)$ is complete,

then f and g have unique coincidence point in X. Moreover, if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$ there exists $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing in this way, for any $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. Then, we have,

$$\begin{aligned}
\overline{S}(gx_n, gx_n, gx_{n+1}) &= \overline{S}(fx_{n-1}, fx_{n-1}, fx_n) \\
&\leq \left[\left(\overline{S}(gx_{n-1}, fx_{n-1}, fx_{n-1})\right)^2 \ \overline{S}(gx_n, fx_n, fx_n)\right]^b \\
&= \left(\overline{S}(gx_{n-1}, gx_n, gx_n)\right)^{2b} \ \left(\overline{S}(gx_n, gx_{n+1}, gx_{n+1})\right)^b \\
&\leq \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\right)^{2b} \ \left(\overline{S}(gx_n, gx_n, gx_{n+1})\right)^b \qquad \left(\because \overline{S}(x,y,y) \leq \overline{S}(x,x,y)\right).
\end{aligned}$$

It implies that

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \leq \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\right)^{\frac{2b}{1-b}}.$$

Assuming $k = \dfrac{2b}{1-b} < 1$, it follows that

$$\begin{aligned}
\overline{S}(gx_n, gx_n, gx_{n+1}) &\leq \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\right)^k \\
&\leq \left(\overline{S}(gx_{n-2}, gx_{n-2}, gx_{n-1})\right)^{k^2}
\end{aligned}$$

Continuing this process, we get,

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \leq \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n}. \tag{3.4}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have,

$$\begin{aligned}
\overline{S}(gx_n, gx_m, gx_m) &\leq \overline{S}(gx_n, gx_n, gx_{n+1}) \left(\overline{S}(gx_m, gx_m, gx_{n+1})\right)^2 \\
&\leq \overline{S}(gx_n, gx_n, gx_{n+1}) \left(\left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^2 \overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \\
&= \overline{S}(gx_n, gx_n, gx_{n+1}) \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^{2^2}.
\end{aligned}$$

Continuing in the same way, we obtain,

$$\begin{aligned}
\overline{S}(gx_n, gx_m, gx_m) &\leq \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \cdots \left(\overline{S}(gx_{m-1}, gx_{m-1}, gx_m)\right)^{2^{n-m}} \\
&\leq \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \ \left(\overline{S}(gx_{n+2}, gx_{n+2}, gx_{n+3})\right)^{2^2} \cdots.
\end{aligned}$$

Using inequality (3.4), we obtain,

$$\begin{aligned}
\overline{S}(gx_n, gx_m, gx_m) &\leq \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n} \ \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^{n+1}}\right)^2 \ \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^{n+2}}\right)^{2^2} \cdots \\
&= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n + 2k^{n+1} + 2^2k^{n+2} + 2^3k^{n+3} + \cdots} \\
&= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n\left(1 + (2k) + (2k)^2 + (2k)^3 + \cdots\right)} \\
&= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{k^n}{1-2k}}. \qquad (\because \ \ 0 \leq k < 1)
\end{aligned}$$

Therefore,

$$\overline{S}(gx_n, gx_m, gx_m) \leq \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{k^n}{1-2k}}.$$

Since, $0 \leq k < 1$, letting $n, m \to \infty$, we have, $\dfrac{k^n}{1-2k} \to 0$ and hence,

$$\lim_{n,m\to\infty} \overline{S}(gx_n, gx_m, gx_m) = 1.$$

Now, for $n, m, l \in \mathbb{N}$ with $n > m > l$, we have,

$$\overline{S}(gx_n, gx_m, gx_l) \leq \overline{S}(gx_n, gx_n, gx_{n-1})\ \overline{S}(gx_m, gx_m, gx_{n-1})\ \overline{S}(gx_l, gx_l, gx_{n-1}).$$

Letting $n, m, l \to \infty$, we obtain,

$$\lim_{n,m,l\to\infty} \overline{S}(gx_n, gx_m, gx_l) = 1.$$

This shows that the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists a point $q \in g(X)$ such that $gx_n \to q$. That is,

$$\lim_{n\to\infty} gx_n = q = \lim_{n\to\infty} fx_{n-1}.$$

Since, $q \in g(X)$, there exists $p \in X$ such that $q = g(p)$. Then,

$$\overline{S}(gx_{n+1}, fp, fp) = \overline{S}(fx_n, fp, fp) \leq \left[\overline{S}(gx_n, fx_n, fx_n)\ \left(\overline{S}(gp, fp, fp)\right)^2\right]^b$$

As $n \to \infty$, it follows that

$$\overline{S}(q, fp, fp) \leq \left[\overline{S}(q, q, q)\ \left(\overline{S}(q, fp, fp)\right)^2\right]^b.$$

This implies that $\left(\overline{S}(q, fp, fp)\right)^{1-2b} \leq 1$. Therefore, $\overline{S}(q, fp, fp) \leq 1$. But since $\overline{S}(q, fp, fp) \geq 1$, it follows that $S(q, fp, fp) = 1$. Hence, we conclude that $q = fp = gp$. Therefore, p is coincidence point of f and g. Now, it is claimed that f and g have unique coincidence point. Contrary, suppose that there exists another coincidence point say $r \neq p$ of f and g.Then,

$$\begin{aligned}\overline{S}(gr, gp, gp) &= \overline{S}(fr, fp, fp)\\ &\leq \left[\overline{S}(gr, fr, fr)\ \left(\overline{S}(gp, fp, fp)\right)^2\right]^b\\ &= \left[\overline{S}(gr, gr, gr)\ \left(\overline{S}(gp, gp, gp)\right)^2\right]^b.\end{aligned}$$

It implies that $\overline{S}(gr, gp, gp) \leq 1$. But as $\overline{S}(gr, gp, gp) \geq 1$, implies that $\overline{S}(gr, gp, gp) = 1$. Hence, $gr = gp$. This shows that f and g have unique coincidence point. If f and g are weakly compatible, then by Proposition 2.1, f and g have unique common fixed point in X. □

Theorem 3.3 *Let $(X, \overline{S})$ be S-multiplicative metric space and $f, g : X \to X$ be the mappings for which there is a real number $0 \leq b < \frac{1}{4}$ such that*

$$\overline{S}(fx, fy, fz) \leq \left[\overline{S}(gx, fx, fy)\ \overline{S}(gy, fy, fz)\ \overline{S}(gz, fz, fx)\right]^b. \tag{3.5}$$

If

(i) $f(X) \subseteq g(X)$;

(ii) $g(X)$ is complete,

then f and g have unique coincidence point in X. Moreover, if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$ there exists $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing in the same way, for any $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. Then,

$$\begin{aligned}&\overline{S}(gx_n, gx_n, gx_{n+1}) = \overline{S}(fx_{n-1}, fx_{n-1}, fx_n)\\ &\leq \left[\overline{S}(gx_{n-1}, fx_{n-1}, fx_{n-1})\ \overline{S}(gx_{n-1}, fx_{n-1}, fx_n)\ \overline{S}(gx_n, fx_n, fx_{n-1})\right]^b\\ &= \left[\overline{S}(gx_{n-1}, gx_n, gx_n)\ \overline{S}(gx_{n-1}, gx_n, gx_{n+1})\ \overline{S}(gx_n, gx_{n+1}, gx_n)\right]^b\\ &\leq \left[\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\ \overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\ \overline{S}(gx_{n+1}, gx_{n+1}, gx_n)\ \overline{S}(gx_{n+1}, gx_{n+1}, gx_n)\right]^b.\end{aligned}$$

Therefore,

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \leq \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\right)^{2b} \left(\overline{S}(gx_n, gx_n, gx_{n+1})\right)^{2b},$$

which implies that

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \leq \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\right)^{\frac{2b}{1-2b}}.$$

Let $k = \dfrac{2b}{1-2b} < 1$. Then,

$$\begin{aligned}\overline{S}(gx_n, gx_n, gx_{n+1}) &\leq \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\right)^{k} \\ &\leq \left(\overline{S}(gx_{n-2}, gx_{n-2}, gx_{n-1})\right)^{k^2}\end{aligned}$$

Repeating the same process, we obtain,

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \leq \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n}. \tag{3.6}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have,

$$\begin{aligned}\overline{S}(gx_n, gx_m, gx_m) &\leq \overline{S}(gx_n, gx_n, gx_{n+1}) \left(\overline{S}(gx_m, gx_m, gx_{n+1})\right)^2 \\ &\leq \overline{S}(gx_n, gx_n, gx_{n+1}) \left(\left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^2 \overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2}\right)^2 \\ &= \overline{S}(gx_n, gx_n, gx_{n+1}) \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^{2^2}.\end{aligned}$$

Continuing in same way, we obtain,

$$\begin{aligned}\overline{S}(gx_n, gx_m, gx_m) &\leq \overline{S}(gx_n, gx_n, gx_{n+1}) \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \cdots \left(\overline{S}(gx_{m-1}, gx_{m-1}, gx_m)\right)^{2^{n-m}} \\ &\leq \overline{S}(gx_n, gx_n, gx_{n+1}) \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \left(\overline{S}(gx_{n+2}, gx_{n+2}, gx_{n+3})\right)^{2^2} \cdots.\end{aligned}$$

Using inequality (3.6), we obtain,

$$\begin{aligned}\overline{S}(gx_n, gx_m, gx_m) &\leq \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n} \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^{n+1}}\right)^2 \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^{n+2}}\right)^{2^2} \cdots \\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n + 2k^{n+1} + 2^2 k^{n+2} + 2^3 k^{n+3} \cdots} \\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n\left(1 + (2k) + (2k)^2 + (2k)^3 + \cdots\right)} \\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{k^n}{1-2k}}. \qquad (\because \quad 0 \leq k < 1)\end{aligned}$$

Therefore,

$$\overline{S}(gx_n, gx_m, gx_m) \leq \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{k^n}{1-2k}}.$$

Since, $0 \leq k < 1$, letting $n, m \to \infty$, we have, $\dfrac{k^n}{1-2k} \to 0$ and hence,

$$\lim_{n,m \to \infty} \overline{S}(gx_n, gx_m, gx_m) = 1.$$

Now, for $n, m, l \in \mathbb{N}$ with $n > m > l$, we have,

$$\overline{S}(gx_n, gx_m, gx_l) \leq \overline{S}(gx_n, gx_n, gx_{n-1}) \overline{S}(gx_m, gx_m, gx_{n-1}) \overline{S}(gx_l, gx_l, gx_{n-1}).$$

Letting $n, m, l \to \infty$, we obtain,

$$\lim_{n,m,l \to \infty} \overline{S}(gx_n, gx_m, gx_l) = 1.$$

This shows that the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists a point $q \in g(X)$ such that $gx_n \to q$. That is,

$$\lim_{n\to\infty} gx_n = q = \lim_{n\to\infty} fx_{n-1}.$$

Since, $q \in g(X)$, there exists $p \in X$ such that $q = g(p)$. Then,

$$\begin{aligned}\overline{S}(gx_{n+1}, gx_{n+1}, fp) &= \overline{S}(fx_n, fx_n, fp)\\ &\le \left[\overline{S}(gx_n, fx_n, fx_n)\ \overline{S}(gx_n, fx_n, fp)\ \overline{S}(gp, fp, fx_n)\right]^b.\end{aligned}$$

Letting $n \to \infty$, we obtain,

$$\overline{S}(q, q, fp) \le \left[\overline{S}(q, q, q)\ \overline{S}(q, q, fp)\ \overline{S}(q, fp, q)\right]^b.$$

Therefore, using Proposition 2.4, we get,

$$\overline{S}(q, q, fp) \le \left[\overline{S}(q, q, q)\ \overline{S}(q, q, fp)\ \overline{S}(q, q, fp)\right]^b.$$

It implies that $\overline{S}(q, q, fp) \le \left(\overline{S}(q, q, fp)\right)^{2b}$, from which it follows that $\left(\overline{S}(q, q, fp)\right)^{1-2b} \le 1$. Therefore, $\left(\overline{S}(q, q, fp)\right) \le 1$. But since $\left(\overline{S}(q, q, fp)\right) \ge 1$, we conclude that $\left(\overline{S}(q, q, fp)\right) = 1$. Hence, $fp = q = gp$. Thus, p is coincidence point of f and g. Now, we claim that the f and g have unique coincidence point. Let us suppose that there is another coincidence point r of f and g. Then,

$$\begin{aligned}\overline{S}(gr, gr, gp) &= \overline{S}(fr, fr, fp)\\ &\le \left[\overline{S}(gr, fr, fr)\ \overline{S}(gr, fr, fp)\ \overline{S}(gp, fp, fr)\right]^b\\ &= \left[\overline{S}(gr, gr, gr)\ \overline{S}(gr, gr, gp)\ \overline{S}(gp, gp, gr)\right]^b.\end{aligned}$$

Therefore, using Proposition 2.2 , we get,

$$\overline{S}(gr, gr, gp) \le \left[\overline{S}(gr, gr, gr)\ \overline{S}(gr, gr, gp)\ \overline{S}(gr, gr, gp)\right]^b.$$

It implies that,

$$\left[\overline{S}(gr, gr, gp)\right]^{1-2b} \le 1.$$

Therefore, $S(gr, gr, gp) \le 1$. But since $S(gr, gr, gp) \ge 1$, we conclude that $S(gr, gr, gp) = 1$ Hence, $gr = gp$. Thus, f and g have unique coincidence point. If f and g are weakly compatible, then by Proposition 2.1, f and g have unique common fixed point in X. □

Theorem 3.4 *Let $(X, \overline{S})$ be S-multiplicative metric space and $f, g : X \to X$ be the mappings for which there is a real number $0 \le b < \frac{1}{2}$ such that*

$$\overline{S}(fx, fy, fz) \le \left(\max\left\{\overline{S}(gx, fx, fx),\ \overline{S}(gy, fy, fy),\ \overline{S}(gz, fz, fz)\right\}\right)^b. \tag{3.7}$$

If

(i) $f(X) \subseteq g(X)$;

(ii) $g(X)$ is complete,

then f and g have unique coincidence point in X. Moreover, if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$ there exists $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing in the same way, for any $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. Then,

$$\begin{aligned}\overline{S}(gx_n, gx_n, gx_{n+1}) &= \overline{S}(fx_{n-1}, fx_{n-1}, fx_n)\\ &\le \left(\max\left\{\overline{S}(gx_{n-1}, fx_{n-1}, fx_{n-1}),\ \overline{S}(gx_{n-1}, fx_{n-1}, fx_{n-1}),\ \overline{S}(gx_n, fx_n, fx_n)\right\}\right)^b\\ &= \left(\max\left\{\overline{S}(gx_{n-1}, fx_{n-1}, fx_{n-1}),\ \overline{S}(gx_n, fx_n, fx_n)\right\}\right)^b\\ &= \left(\max\left\{\overline{S}(gx_{n-1}, gx_n, gx_n),\ \overline{S}(gx_n, gx_{n+1}, gx_{n+1})\right\}\right)^b\\ &\le \left(\max\left\{\overline{S}(gx_{n-1}, gx_{n-1}, gx_n),\ \overline{S}(gx_n, gx_n, gx_{n+1})\right\}\right)^b \quad \left(\because \overline{S}(x,y,y) \le \overline{S}(x,x,y)\right).\end{aligned}$$

Therefore,

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \le \left(\max\left\{\overline{S}(gx_{n-1}, gx_{n-1}, gx_n),\ \overline{S}(gx_n, gx_n, gx_{n+1})\right\}\right)^b. \tag{3.8}$$

Case I: If $\max\left\{\overline{S}(gx_{n-1}, gx_{n-1}, gx_n),\ \overline{S}(gx_n, gx_n, gx_{n+1})\right\} = \overline{S}(gx_n, gx_n, gx_{n+1})$, then (3.8) implies that $\overline{S}(gx_n, gx_n, gx_{n+1}) \le \left(\overline{S}(gx_n, gx_n, gx_{n+1})\right)^b$, which is a contradiction, since $0 \le b < \frac{1}{2}$.

Case II: If

$$\max\left\{\overline{S}(gx_{n-1}, gx_{n-1}, gx_n),\ \overline{S}(gx_n, gx_n, gx_{n+1})\right\} = \overline{S}(gx_{n-1}, gx_{n-1}, gx_n),$$

then (3.8) implies that

$$\begin{aligned}\overline{S}(gx_n, gx_n, gx_{n+1}) &\le \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\right)^b\\ &\le \left(\overline{S}(gx_{n-2}, gx_{n-2}, gx_{n-1})\right)^{b^2}.\end{aligned}$$

Continuing this, we obtain,

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \le \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^n}. \tag{3.9}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have,

$$\begin{aligned}\overline{S}(gx_n, gx_m, gx_m) &\le \overline{S}(gx_n, gx_n, gx_{n+1})\ \left(\overline{S}(gx_m, gx_m, gx_{n+1})\right)^2\\ &\le \overline{S}(gx_n, gx_n, gx_{n+1})\ \left(\left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^2\ \overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2\\ &= \overline{S}(gx_n, gx_n, gx_{n+1})\ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2\ \left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^{2^2}.\end{aligned}$$

Continuing in the same way, we obtain,

$$\begin{aligned}&\overline{S}(gx_n, gx_m, gx_m)\\ &\le \overline{S}(gx_n, gx_n, gx_{n+1})\ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \cdots \left(\overline{S}(gx_{m-1}, gx_{m-1}, gx_m)\right)^{2^{n-m}}\\ &\le \overline{S}(gx_n, gx_n, gx_{n+1})\ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2\ \left(\overline{S}(gx_{n+2}, gx_{n+2}, gx_{n+3})\right)^{2^2} \cdots.\end{aligned}$$

Using inequality (3.9), we obtain,

$$\begin{aligned}&\overline{S}(gx_n, gx_m, gx_m)\\ &\le \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^n}\ \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^{n+1}}\right)^2\ \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^{n+2}}\right)^{2^2} \cdots\\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^n + 2b^{n+1} + 2^2 b^{n+2} + 2^3 k^{n+3} + \cdots}\\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^n\left(1 + (2b) + (2b)^2 + (2b)^3 + \cdots\right)}\\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{b^n}{1-2b}}. \qquad \left(\because \quad 0 \le b < \frac{1}{2}\right)\end{aligned}$$

Therefore,

$$\overline{S}(gx_n, gx_m, gx_m) \leq \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{b^n}{1-2b}}.$$

Since, $0 \leq b < \dfrac{1}{2} < 1$, letting $n, m \to \infty$, we have, $\dfrac{b^n}{1-2b} \to 0$ and hence,

$$\lim_{n,m\to\infty} \overline{S}(gx_n, gx_m, gx_m) = 1.$$

Now, for $n, m, l \in \mathbb{N}$ with $n > m > l$, we have,

$$\overline{S}(gx_n, gx_m, gx_l) \leq \overline{S}(gx_n, gx_n, gx_{n-1})\ \overline{S}(gx_m, gx_m, gx_{n-1})\ \overline{S}(gx_l, gx_l, gx_{n-1}).$$

Letting $n, m, l \to \infty$, we obtain,

$$\lim_{n,m,l\to\infty} \overline{S}(gx_n, gx_m, gx_l) = 1.$$

This shows that the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$.

Since $g(X)$ is complete, there exists a point $q \in g(X)$ such that $gx_n \to q$. That is,

$$\lim_{n\to\infty} gx_n = q = \lim_{n\to\infty} fx_{n-1}.$$

Now, as $q \in g(X)$, there exists $p \in X$ such that $q = g(p)$. Then,

$$\begin{aligned}\overline{S}(gx_{n+1}, fp, fp) &= \overline{S}(fx_n, fp, fp)\\ &\leq \left(\max\left\{\overline{S}(gx_n, fx_n, fx_n),\ \overline{S}(gp, fp, fp),\ \overline{S}(gp, fp, fp)\right\}\right)^b\\ &= \left(\max\left\{\overline{S}(gx_n, fx_n, fx_n),\ \overline{S}(q, fp, fp)\right\}\right)^b.\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\overline{S}(q, fp, fp) \leq \left(\max\left\{\overline{S}(q, q, q),\ \overline{S}(q, fp, fp)\right\}\right)^b.$$

It implies that $\left(\overline{S}(q, fp, fp)\right)^{1-b} \leq 1$. Therefore, $\overline{S}(q, fp, fp) \leq 1$. But $\overline{S}(q, fp, fp) \geq 1$, gives us that $\overline{S}(q, fp, fp) = 1$. Therefore, $q = fp = gp$. Thus, p is a coincidence point of f and g. We now claim that f and g have a unique coincidence point. Suppose, for the sake of contradiction, that there exists another coincidence point r of f and g. Then,

$$\begin{aligned}\overline{S}(gr, gp, gp) &= \overline{S}(fr, fp, fp)\\ &\leq \left(\max\left\{\overline{S}(gr, fr, fr),\ \overline{S}(gp, fp, fp),\ \overline{S}(gp, fp, fp)\right\}\right)^b.\end{aligned}$$

Since, $gr = fr$ and $gp = fp$, therefore, it follows that, $S(gr, gp, gp) \leq 1$. But since $S(gr, gp, gp) \geq 1$, we conclude that $S(gr, gr, gp) = 1$. Hence, $gr = gp$. Thus, f and g have unique coincidence point. If f and g are weakly compatible, then by Proposition 2.1, f and g have unique common fixed point in X. □

Theorem 3.5 *Let $(X, \overline{S})$ be S-multiplicative metric space and $f, g : X \to X$ be the mappings for which there is a real number $0 \leq b < \dfrac{1}{3}$ such that*

$$\overline{S}(fx, fy, fz) \leq \left(\max\left\{\overline{S}(gx, fx, fy),\ \overline{S}(gy, fy, fz),\ \overline{S}(gz, fz, fx)\right\}\right)^b. \tag{3.10}$$

If

(i) $f(X) \subseteq g(X)$;

(ii) $g(X)$ is complete,

then f and g have unique coincidence point in X. Moreover, if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing this process, for each $x_n \in X$, we can find $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. Then,

$$\begin{aligned}\overline{S}(gx_n, gx_n, gx_{n+1}) &= \overline{S}(fx_{n-1}, fx_{n-1}, fx_n) \\ &\leq \left(\max\left\{\overline{S}(gx_{n-1}, fx_{n-1}, fx_{n-1}),\ \overline{S}(gx_{n-1}, fx_{n-1}, fx_n),\ \overline{S}(gx_n, fx_n, fx_{n-1})\right\}\right)^b.\end{aligned}$$

Therefore,

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \leq \left(\max\left\{\overline{S}(gx_{n-1}, gx_n, gx_n),\ \overline{S}(gx_{n-1}, gx_n, gx_{n+1}),\ \overline{S}(gx_n, gx_{n+1}, gx_n)\right\}\right)^b. \tag{3.11}$$

Case I: If

$$\max\left\{\overline{S}(gx_{n-1}, gx_n, gx_n),\ \overline{S}(gx_{n-1}, gx_n, gx_{n+1}),\ \overline{S}(gx_n, gx_{n+1}, gx_n)\right\} = \overline{S}(gx_n, gx_{n+1}, gx_n),$$

then inequality (3.11) implies that

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \leq \left(\overline{S}(gx_n, gx_{n+1}, gx_n)\right)^b.$$

Using Proposition 2.4, we obtain,

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \leq \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_n)\right)^b.$$

Therefore, using Proposition 2.3, we get

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \leq \left(\overline{S}(gx_n, gx_n, gx_{n+1})\right)^b,$$

which is a contradiction as $0 \leq b < \dfrac{1}{3}$.

Case II: If

$$\max\left\{\overline{S}(gx_{n-1}, gx_n, gx_n),\ \overline{S}(gx_{n-1}, gx_n, gx_{n+1}),\ \overline{S}(gx_n, gx_{n+1}, gx_n)\right\} = \overline{S}(gx_{n-1}, gx_n, gx_n),$$

then inequality (3.11) implies that

$$\begin{aligned}\overline{S}(gx_n, gx_n, gx_{n+1}) &\leq \left(\overline{S}(gx_{n-1}, gx_n, gx_n)\right)^b \\ &\leq \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\right)^b \qquad \left(\because\ \overline{S}(x,y,y) \leq \overline{S}(x,x,y)\right) \\ &\leq \left(\overline{S}(gx_{n-2}, gx_{n-2}, gx_{n-1})\right)^{b^2}.\end{aligned}$$

Continuing this process, we obtain,

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \leq \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^n}. \tag{3.12}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have,

$$\begin{aligned}\overline{S}(gx_n, gx_m, gx_m) &\leq \overline{S}(gx_n, gx_n, gx_{n+1})\ \left(\overline{S}(gx_m, gx_m, gx_{n+1})\right)^2 \\ &\leq \overline{S}(gx_n, gx_n, gx_{n+1})\ \left(\left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^2\ \overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \\ &= \overline{S}(gx_n, gx_n, gx_{n+1})\ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2\ \left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^{2^2}.\end{aligned}$$

Continuing in the same way, we obtain,

$$\begin{aligned}
&\overline{S}(gx_n, gx_m, gx_m)\\
&\quad\le \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \cdots \left(\overline{S}(gx_{m-1}, gx_{m-1}, gx_m)\right)^{2^{n-m}}\\
&\quad\le \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \ \left(\overline{S}(gx_{n+2}, gx_{n+2}, gx_{n+3})\right)^{2^2} \cdots .
\end{aligned}$$

Using inequality (3.12), we get,

$$\begin{aligned}
&\overline{S}(gx_n, gx_m, gx_m)\\
&\quad\le \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^n} \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^{n+1}}\right)^2 \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^{n+2}}\right)^{2^2} \cdots\\
&\quad= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^n+2b^{n+1}+2^2b^{n+2}+2^3k^{n+3}+\cdots}\\
&\quad= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{b^n\left(1+(2b)+(2b)^2+(2b)^3+\cdots\right)}\\
&\quad= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{b^n}{1-2b}} . \qquad \left(\because \ \ 0 \le b < \frac{1}{2}\right)
\end{aligned}$$

Therefore,

$$\overline{S}(gx_n, gx_m, gx_m) \le \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{b^n}{1-2b}} .$$

Since, $0 \le b < \dfrac{1}{3} < 1$, letting $n, m \to \infty$, we have, $\dfrac{k^n}{1-2b} \to 0$ and hence,

$$\lim_{n,m\to\infty} \overline{S}(gx_n, gx_m, gx_m) = 1.$$

Now, for $n, m, l \in \mathbb{N}$ with $n > m > l$, we have,

$$\overline{S}(gx_n, gx_m, gx_l) \le \overline{S}(gx_n, gx_n, gx_{n-1}) \ \overline{S}(gx_m, gx_m, gx_{n-1}) \ \overline{S}(gx_l, gx_l, gx_{n-1}).$$

Letting $n, m, l \to \infty$, we obtain,

$$\lim_{n,m,l\to\infty} \overline{S}(gx_n, gx_m, gx_l) = 1.$$

This shows that the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$.

Case III: If

$$\max\left\{\overline{S}(gx_{n-1}, gx_n, gx_n), \ \overline{S}(gx_{n-1}, gx_n, gx_{n+1}), \ \overline{S}(gx_n, gx_{n+1}, gx_n)\right\} = \overline{S}(gx_{n-1}, gx_n, gx_{n+1}),$$

then inequality (3.11) implies that

$$\begin{aligned}
\overline{S}(gx_n, gx_n, gx_{n+1}) &\le \left(\overline{S}(gx_{n-1}, gx_n, gx_{n+1})\right)^b\\
&\le \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n) \ \overline{S}(gx_n, gx_n, gx_n) \ \overline{S}(gx_{n+1}, gx_{n+1}, gx_n)\right)^b\\
&= \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n) \ \overline{S}(gx_n, gx_n, gx_{n+1})\right)^b .
\end{aligned}$$

It implies that

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \le \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\right)^{\frac{b}{1-b}} .$$

Let $k = \dfrac{b}{1-b} < 1$. Then,

$$\begin{aligned}
\overline{S}(gx_n, gx_n, gx_{n+1}) &\le \left(\overline{S}(gx_{n-1}, gx_{n-1}, gx_n)\right)^k\\
&\le \left(\overline{S}(gx_{n-2}, gx_{n-2}, gx_{n-1})\right)^{k^2} .
\end{aligned}$$

Continuing with a similar argument, we get

$$\overline{S}(gx_n, gx_n, gx_{n+1}) \le \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n}. \tag{3.13}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have,

$$\begin{aligned}\overline{S}(gx_n, gx_m, gx_m) &\le \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_m, gx_m, gx_{n+1})\right)^2 \\ &\le \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^2 \ \overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \\ &= \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \ \left(\overline{S}(gx_m, gx_m, gx_{n+2})\right)^{2^2}.\end{aligned}$$

Continuing in the same way, we get,

$$\begin{aligned}&\overline{S}(gx_n, gx_m, gx_m) \\ &\le \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \cdots \left(\overline{S}(gx_{m-1}, gx_{m-1}, gx_m)\right)^{2^{n-m}} \\ &\le \overline{S}(gx_n, gx_n, gx_{n+1}) \ \left(\overline{S}(gx_{n+1}, gx_{n+1}, gx_{n+2})\right)^2 \ \left(\overline{S}(gx_{n+2}, gx_{n+2}, gx_{n+3})\right)^{2^2} \cdots.\end{aligned}$$

Using inequality (3.13), we get,

$$\begin{aligned}&\overline{S}(gx_n, gx_m, gx_m) \\ &\le \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n} \ \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^{n+1}}\right)^2 \ \left(\left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^{n+2}}\right)^{2^2} \cdots \\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n + 2k^{n+1} + 2^2 k^{n+2} + 2^3 k^{n+3} \cdots} \\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{k^n \left(1 + (2k) + (2k)^2 + (2k)^3 + \cdots\right)} \\ &= \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{k^n}{1-2k}}. \qquad (\because \ \ 0 \le 2k < 1)\end{aligned}$$

Therefore,

$$\overline{S}(gx_n, gx_m, gx_m) \le \left(\overline{S}(gx_0, gx_0, gx_1)\right)^{\frac{k^n}{1-2k}}.$$

Since, $0 \le k < 1$, letting $n, m \to \infty$, we have, $\dfrac{k^n}{1-2k} \to 0$ and hence,

$$\lim_{n,m \to \infty} \overline{S}(gx_n, gx_m, gx_m) = 1.$$

Now, for $n, m, l \in \mathbb{N}$ with $n > m > l$, we have,

$$\overline{S}(gx_n, gx_m, gx_l) \le \overline{S}(gx_n, gx_n, gx_{n-1}) \ \overline{S}(gx_m, gx_m, gx_{n-1}) \ \overline{S}(gx_l, gx_l, gx_{n-1}).$$

Letting $n, m, l \to \infty$, we obtain,

$$\lim_{n,m,l \to \infty} \overline{S}(gx_n, gx_m, gx_l) = 1.$$

This shows that the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$.

Thus, in all possible cases the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists a point $q \in g(X)$ such that $gx_n \to q$. That is,

$$\lim_{n \to \infty} gx_n = q = \lim_{n \to \infty} fx_{n-1}.$$

Now, as $q \in g(X)$, there exists $p \in X$ such that $q = g(p)$. Then,

$$\begin{aligned}\overline{S}(gx_{n+1}, gx_{n+1}, fp) &= \overline{S}(fx_n, fx_n, fp) \\ &\le \left(\max\left\{\overline{S}(gx_n, fx_n, fx_n), \ \overline{S}(gx_n, fx_n, fp), \ \overline{S}(gp, fp, fx_n)\right\}\right)^b.\end{aligned}$$

As $n \to \infty$, we obtain

$$\overline{S}(q, q, fp) \leq \left(\max\left\{\overline{S}(q, q, q),\ \overline{S}(q, q, fp),\ \overline{S}(q, fp, q)\right\}\right)^{b}$$

Therefore, using Proposition 2.4, we get,

$$\overline{S}(q, q, fp) \leq \left(\max\left\{1,\ \overline{S}(q, q, fp),\ \overline{S}(q, q, fp)\right\}\right)^{b}.$$

It implies that $\overline{S}(q, q, fp) \leq \left(\overline{S}(q, q, fp)\right)^{b}$. Therefore, $\left(\overline{S}(q, q, fp)\right)^{1-b} \leq 1$. Thus, it follows that $\overline{S}(q, q, fp) \leq 1$. But $\overline{S}(q, q, fp) \geq 1$. Therefore, we must have, $\overline{S}(q, q, fp) = 1$. Hence, $q = fp = gq$. Hence, p is a coincidence point of f and g. Now, we show that this point is unique. Suppose, to the contrary, that there exists another coincidence point r of f and g. Then,

$$\begin{aligned}
\overline{S}(gr, gr, gp) &= S(fr, fr, fp) \\
&\leq \left(\max\left\{\overline{S}(gr, fr, fr),\ \overline{S}(gr, fr, fp),\ \overline{S}(gp, fp, fr)\right\}\right)^{b} \\
&= \left(\max\left\{\overline{S}(gr, gr, gr),\ \overline{S}(gr, gr, gp),\ \overline{S}(gp, gp, gr)\right\}\right)^{b}.
\end{aligned}$$

By using Proposition 2.2, we get,

$$\overline{S}(gr, gr, gp) \leq \left(\max\left\{\overline{S}(gr, gr, gr),\ \overline{S}(gr, gr, gp),\ \overline{S}(gr, gr, gp)\right\}\right)^{b}.$$

Therefore,

$$\overline{S}(gr, gr, gp \leq \left(\max\left\{1,\ \overline{S}(gr, gr, gp)\right\}\right)^{b}.$$

It implies that $\left(\overline{S}(gr, gr, gp)\right)^{1-b} \leq 1$. Therefore, $\overline{S}(gr, gr, gp) \leq 1$. But since $\overline{S}(gr, gr, gp) \geq 1$, we conclude that $S(gr, gr, gp) = 1$ Hence, $gr = gp$. Thus, f and g have unique coincidence point. If f and g are weakly compatible, then by Proposition 2.1, f and g have unique common fixed point in X. □

Example 3.1 Let $X = [1, \infty)$ and consider S-multiplicative metric defined in the Example 2.4. Define $f, g : X \to X$ by $f(x) = 1$ and $g(x) = x^{\frac{1}{4}}$. Then $f(X) = \{1\} \subseteq [1, \infty) = g(X)$. That is, $f(X) \subseteq g(X)$ and $g(X) = [1, \infty)$ is complete. Now,

$$\overline{S}(fx, fy, fz) = \max\{1, 1, 1\} = 1. \tag{3.14}$$

Also,

$$\begin{aligned}
\overline{S}(gx, gy, gz) &= \overline{S}\left(x^{\frac{1}{4}}, y^{\frac{1}{4}}, z^{\frac{1}{4}}\right) \\
&= \max\left\{\left(\frac{x}{y}\right)^{\frac{1}{4}}, \left(\frac{y}{x}\right)^{\frac{1}{4}}, \left(\frac{y}{z}\right)^{\frac{1}{4}}, \left(\frac{z}{y}\right)^{\frac{1}{4}}, \left(\frac{x}{z}\right)^{\frac{1}{4}}, \left(\frac{z}{x}\right)^{\frac{1}{4}}\right\}.
\end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we obtain,

$$\overline{S}(fx, fy, fz) \leq \left(\overline{S}(gx, gy, gz)\right)^{k}, \quad \text{any} \quad 0 \leq k < \frac{1}{4}.$$

Now, $f(x) = g(x)$, implies that $1 = x^{\frac{1}{4}}$. Therefore, $x = 1$ is the coincidence point of f and g. Moreover, $fg(1) = f(1) = 1$ and $gf(1) = g(1) = 1$. Thus, f and g are weakly compatible mappings. Therefore, by Theorem 3.1, f and g have unique common fixed point $x = 1 \in X$.

Example 3.2 Let $X = [1, \infty)$ and consider S-multiplicative metric defined in the Example 2.4. Define $f, g : X \to X$ by $f(x) = x^{\frac{1}{4}}$ and $g(x) = x$. Then $f(X) = [1, \infty) = g(X)$. That is, $f(X) \subseteq g(X)$ and $g(X) = [1, \infty)$ is complete. Now,

$$\begin{aligned}
\overline{S}(fx, fy, fz) &= \overline{S}\left(x^{\frac{1}{4}}, y^{\frac{1}{4}}, z^{\frac{1}{4}}\right) \\
&= \max\left\{\left(\frac{x}{y}\right)^{\frac{1}{4}}, \left(\frac{y}{x}\right)^{\frac{1}{4}}, \left(\frac{y}{z}\right)^{\frac{1}{4}}, \left(\frac{z}{y}\right)^{\frac{1}{4}}, \left(\frac{x}{z}\right)^{\frac{1}{4}}, \left(\frac{z}{x}\right)^{\frac{1}{4}}\right\}.
\end{aligned} \tag{3.16}$$

Also,

$$\overline{S}(gx, gy, gz) = \overline{S}(x, y, z) = \max\left\{\frac{x}{y}, \frac{y}{x}, \frac{y}{z}, \frac{z}{y}, \frac{x}{z}, \frac{z}{x}\right\}. \tag{3.17}$$

From (3.16) and (3.17), we obtain,

$$\overline{S}(fx, fy, fz) \leq \left(\overline{S}(gx, gy, gz)\right)^k, \text{ for } k = \frac{1}{4}.$$

Now, $f(x) = g(x)$, implies that $x^{\frac{1}{4}} = x$. It implies that $x = 1$ is the coincidence point of f and g. Moreover, $fg(1) = f(1) = 1$ and $gf(1) = g(1) = 1$. Therefore, f and g are weakly compatible mappings. Therefore, by Theorem 3.1, f and g have unique common fixed point $x = 1 \in X$.

4. Conclusion

In this paper, we have investigated the existence and uniqueness of a common fixed point for two weakly compatible self-maps defined on S-multiplicative metric spaces under various contractive conditions. By extending and generalizing existing results, we have demonstrated that such mappings admit a unique common fixed point under appropriate assumptions. Furthermore, an illustrative example has been provided to validate the theoretical findings and to show the applicability of the established result.

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