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Entropy Solutions for Anisotropic Neumann Problems with Variable Exponents and Hardy Potential

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ABSTRACT: In this work, we study an anisotropic obstacle problem driven by a Leray-Lions-type operator with a Hardy-type singular potential, defined in anisotropic weighted Sobolev spaces with variable exponents. The problem involves a nonlinear lower-order term depending on the gradient, under homogeneous Neumann boundary conditions. We establish the existence of entropy solutions using truncation techniques combined with the monotonicity method.

Key Words: Anisotropic weighted Sobolev spaces, Neumann elliptic problem, entropy solutions, hardy potential, variable exponent, obstacle problems.

Contents

1. Introduction

Our research is situated at the confluence of two pivotal domains within the theory of partial differential equations (PDEs): the theory of anisotropic Sobolev spaces, as elaborated in [18,19], and the theory of variable exponent Sobolev spaces, introduced in [11,12,16]. It is worth noting that the mathematical literature contains relatively few results concerning weighted anisotropic problems. In contrast, the theory of variable exponent Sobolev spaces has attracted considerable attention over the past few decades [21,22], leading to a wealth of studies that explore their diverse applications.

Variable exponent Sobolev spaces have played a particularly important role in the study of materials, especially in the field of electro-rheological fluids, commonly known as smart fluids. Revolutionary contributions in this field were made by Winslow, who obtained a US patent on this phenomenon in 1947 [24] and published a seminal paper in 1949 [25]. Electro-rheological fluids have the remarkable property that their viscosity changes considerably when they are subjected to an electromagnetic field. Winslow observed that the viscosity of these fluids is inversely proportional to the field strength, with the electromagnetic field inducing string-like structures aligned with its direction. These structures can lead to an increase in viscosity of up to five orders of magnitude, a phenomenon now known as the Winslow effect.

For a comprehensive discussion on the properties, modeling, and applications of variable exponent spaces in the study of electro-rheological fluids, we refer the reader to [13,14,20]. Furthermore, variable exponent Sobolev spaces have found substantial applications in other areas, including elasticity theory [28] and image processing [9].

In sequel of this paper, let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded open subset with a smooth boundary $\partial\Omega$. Our objective is to investigate the existence of entropy solutions for the following Neumann problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \kappa_i(z, w, \nabla w) + \mathcal{H}(z, w, \nabla w) + |w|^{r(z)-1} w = f + \mu \dfrac{|w|^{p_0(z)-2} w}{|z|^{p_0(z)}} & \text{in } \Omega, \\ \sum_{i=1}^{N} \kappa_i(z, w, \nabla w) \cdot \eta_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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in the convex class $\mathcal{D}_\Delta := \{w \in W^{1,\vec{p}(z)}(\Omega,\vec{\varrho}), w \geq \Delta \quad \text{a.e in } \Omega\}$, where Δ is a fixed obstacle function, such that

$$(\mathcal{H}_1) \qquad \Delta^+ \in W^{1,\vec{p}(z)}(\Omega,\vec{\varrho}) \cap L^\infty(\Omega).$$

In the following pages, we will consider Ω is an open bounded set of $\mathbb{R}^N (N \geq 2)$. The functions $\kappa_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathódory functions, which satisfy the following conditions, for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $\xi' \in \mathbb{R}^N$ and a. e. in $z \in \Omega$,

$$\kappa_i(z,s,\xi)\xi_i \geq \alpha\, \varrho_i\, |\xi_i|^{p_i(z)}, \tag{1.2}$$

$$|\kappa_i(z,s,\xi)| \leq \beta \varrho_i^{\frac{1}{p_i(z)}} \Big(R_i(z) + \varrho_i^{\frac{1}{p_i'(z)}} |s|^{\frac{p_i(z)}{p_i'(z)}} + \varrho_i^{\frac{1}{p_i'(z)}} |\xi_i|^{p_i(z)-1} \Big), \tag{1.3}$$

$$\Big(\kappa_i(z,s,\xi) - \kappa_i(z,s,\xi')\Big)\Big(\xi_i - \xi_i'\Big) > 0 \quad \text{for} \quad \xi_i \neq \xi_i', \tag{1.4}$$

where $R_i(z)$ is a nonnegative function lying in $L^{p_i'(z)}(\Omega)$ and $\alpha,\ \beta > 0$. Moreover, we suppose that

$$f \in L^1(\Omega) \quad \textit{and} \quad \frac{|w|^{p_0(z)-2}w}{|z|^{p_0(z)}} \in L^1(\Omega). \tag{1.5}$$

The nonlinear term $\mathcal{H}(z,s,\xi)$ is a Carathéodory function which satisfies only the growth condition

$$\big|\mathcal{H}(z,s,\xi)\big| \leq b(z) + g(|s|) \sum_{i=1}^N \varrho_i \big|\xi_i\big|^{p_i(z)} \tag{1.6}$$

where $g : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$ and $b(z) \in L^1(\Omega)$. In addition to addressing the main problem, we will also discuss relevant previous studies. Several authors, including Hjiej et al. [3], Benkirane and Elmahi [5], and Chrif [10], have provided proofs for the existence of solutions to certain nonlinear elliptic obstacle problems. These solutions were established in the classical anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$ and the weighted anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega,\vec{\varrho})$.

In [29,30,31], Zineddaine et al. studied an anisotropic elliptic problem involving the $p(z)$-Laplacian operator or the Leray-Lions operator. Using the monotonicity method, they established the existence of a weak approximate solution to the problem under consideration. Subsequently, they proved the existence of an entropy solution by leveraging the strong convergence of the truncated sequence of approximate solutions.

On the other hand, in the most general setting, the problems were addressed in [2], where the authors considered the framework of anisotropic Sobolev spaces and imposed only mild assumptions on the coefficients a_i. Their objective was to establish the uniqueness of weak solutions to the considered problems.

The main point in our study is to consider separately some class of anisotropic obstacle nonlinear elliptic problems of kind (1.1), and prove only existence results, the uniqueness problem being a rather delicate one, this kind of problems still attracting the interest of the researchers (see [23,7] for a survey). This paper is to extend the results in [3], One of the motivations for studying (1.1) comes from applications to elasticity as the equations that models the shape of an elastic membrane which is pushed by an obstacle from one side affecting its shape.

The remaining part of this paper is organized as follows : Section 2 contains a brief discussion of the weighted space with variable exponent Lebesgue and the weighted anisotropic variable exponent Sobolev space, moreover we give some useful technical lemmas, The main existence results are stated and proved in Section 3.

2. Preliminaries and interesting properties

This section aims to provide a general overview of the aforementioned spaces. We set $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{z\in\overline{\Omega}} p(z) > 1\}$ and denote, for all $p \in C_+(\overline{\Omega})$,

$$p^+ = \sup_{z\in\Omega} p(z) \quad \text{ and } \quad p^- = \inf_{z\in\Omega} p(z).$$

Let $p_i \in \mathcal{C}_+(\Omega)$ for any $1 \leq i \leq N$, we pose

$$(\mathcal{H}_2) \qquad p_0(z) \geq \max\{p_i(z),\ i = 1, 2,\ \dots,\ N\},$$

and $\vec{\varrho} = \{\varrho_0, \dots, \varrho_N\}$ be a vector of weight functions; i.e., every component ϱ_i is a measurable function which is strictly positive a.e. in Ω.
We define the weighted Lebesgue space with variable exponent $L^{p_i(z)}(\Omega, \varrho_i)$ as follows

$$L^{p_i(z)}(\Omega, \varrho_i) = \big\{w \text{ is a measurable real-valued function } : \int_\Omega |w|^{p_i(z)} \varrho_i dz < \infty\big\}$$

endowed with the so-called Luxemburg norm

$$\|w\|_{L^{p_i(z)}(\Omega,\varrho_i)} = \|w\|_{p_i(z),\varrho_i} = \inf\Big\{\mu > 0 : \int_\Omega \Big|\frac{w}{\mu}\Big|^{p_i(z)} \varrho_i dx \leq 1\Big\}.$$

Throughout this paper, we assume there exists a weight function ϱ_i, for any $i = 1, \dots, N$, such that

$$(\mathcal{H}_3) \qquad \varrho_i \in L^1_{loc}(\Omega); \quad \varrho_i^{\frac{-1}{p_i(z)-1}} \in L^1_{loc}(\Omega).$$

$$(\mathcal{H}_4) \qquad \varrho_i^{-s(z)} \in L^1(\Omega) \text{ with } s(z) \in \Big(\frac{N}{p_i(z)}, \infty\Big) \cap \Big[\frac{1}{p_i(z)-1}, \infty\Big).$$

If $(\mathcal{H}_3)$ holds, then $\big(L^{p_i(z)}(\Omega, \varrho_i), \|\cdot\|_{p_i(z),\varrho_i}\big)$ is a Banach, separable and reflexive space for each $0 \leq i \leq N$ (see for example [13,27]).

Next, we introduce the anisotropic weighted Sobolev space with variable exponents, employed in the analysis of our obstacle elliptic Neumann problem (1.1). We denote

$$\vec{p}(z) = \{p_0(z), p_1(z), \dots, p_N(z)\},\ \partial_0 w = w \text{ and } \partial_i w = \frac{\partial w}{\partial z_i} \text{ for } i = 1, \dots, N,$$

and if we set

$$\underline{p} = \min\{p_0^-, p_1^-, \dots, p_N^-\}, \text{ then } \underline{p} > 1. \tag{2.1}$$

The anisotropic weighted Sobolev space with variable exponents $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$ is defined as follow

$$W^{1,\vec{p}(z)}(\Omega, \vec{\varrho}) = \Big\{w \in L^{p_0(z)}(\Omega, \varrho_0) \quad \text{and} \quad \partial_i w \in L^{p_i(z)}(\Omega, \varrho_i),\ i = 1, \cdots, N\Big\},$$

is a Banach space with respect to norm (cf. [10])

$$\|w\| := \|w\|_{1,\vec{p}(z),\vec{\varrho}} = \|w\|_{p_0(z)} + \sum_{i=1}^N \|\partial_i w\|_{p_i(z),\varrho_i}. \tag{2.2}$$

Let $\mathcal{V} = L^{p_0(z)}(\Omega) \times \prod_{i=1}^N L^{p_i(z)}(\Omega)$, and consider the operator $\mathcal{T} : W^{1,\vec{p}(z)}(\Omega, \vec{\varrho}) \to \mathcal{V}$, defined by $\mathcal{T}(w) = \big(w, w\varrho_i^{\frac{1}{p_i(z)}}\big)$. It is evident that $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$ and $\mathcal{V}$ are isometric via $\mathcal{T}$, as

$$\|\mathcal{T}w\|_\mathcal{V} = \|w\|_{p_0(z)} + \sum_{i=1}^N |\partial_i w|_{p_i(z),\varrho_i} = \|w\|.$$

Thus, $\mathcal{T}\big(W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})\big)$ is a closed subspace of $\mathcal{V}$, which is a reflexive Banach space. By [8, Proposition III.17], it follows that $\mathcal{T}\big(W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})\big)$ is reflexive, and consequently, $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$ is itself a reflexive Banach space.

Lemma 2.1 *Let Ω be a smooth bounded open subset of $\mathbb{R}^N (N \geq 2)$. Under the hypothesis $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ and $\inf \varrho_i(z) > 0$ a.e. in Ω for each $1 \leq i \leq N$, we have the following continuous and compact embedding*

1. *If* $\underline{p} < N$ *then* $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho}) \hookrightarrow\hookrightarrow L^{q(z)}(\Omega)$ *for any* $q \in [\underline{p}, p^*[$ *where* $p^* = \frac{Np}{N-\underline{p}}$
2. *If* $\underline{p} = N$*, then* $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho}) \hookrightarrow\hookrightarrow L^{q(z)}(\Omega)$ *for all* $q \in [\underline{p}, +\infty[$,
3. *If* $\underline{p} > N$*, then* $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho}) \hookrightarrow\hookrightarrow L^{\infty}(\Omega) \cap C^0(\overline{\Omega})$.

Proof: See [4, Proposition 2.1] □

Moreover, we consider

$$\mathcal{T}^{1,\vec{p}(z)}(\Omega, \vec{\varrho}) := \{w : \Omega \to \mathbb{R}, \text{ measurable, such that } T_\ell(w) \in W^{1,\vec{p}(z)}(\Omega, \vec{\varrho}), \text{ for any } \ell > 0 \},$$

where $T_\ell(s)$ is the truncation function setting by

$$T_\ell(s) = \begin{cases} s & \text{if} \quad |s| \le \ell, \\ \ell\dfrac{s}{|s|} & \text{if} \quad |s| > \ell. \end{cases}$$

In sequel, we provide some preliminary Lemmas that are crucial to prove our main result.

Lemma 2.2 *[4] Let* ϱ_i *be a function weight in* Ω*,* $r_i \in \mathcal{C}^+(\Omega)$*,* $g \in L^{r_i(z)}(\Omega, \varrho_i)$ *and* $(g_\varepsilon)_\varepsilon \subset L^{r_i(z)}(\Omega, \varrho_i)$ *such that* $\|g_n\|_{r_i(z),\varrho_i} \le C$*, for any* $i \in \{1, \dots, N\}$.
If $g_\varepsilon \to g$ *a.e. in* Ω*, then* $g_\varepsilon \rightharpoonup g$ *weakly in* $L^{r_i(z)}(\Omega, \varrho_i)$.

Lemma 2.3 *[4] Assume that* (1.2)-(1.4) *are true, and let* $(w_\varepsilon)_\varepsilon$ *be a sequence in* $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$ *and* $w \in W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$*, if*

$$w_\varepsilon \rightharpoonup w \text{ weakly in } W^{1,\vec{p}(z)}(\Omega, \vec{\varrho}),$$

and

$$\sum_{i=1}^{N} \int_\Omega (\kappa_i(z, w_\varepsilon, \nabla w_\varepsilon) - \kappa_i(z, w_\varepsilon, \nabla w))(\partial_i w_\varepsilon - \partial_i w)dz + \int_\Omega \left(|w_\varepsilon|^{p_0(z)-2} w_\varepsilon - |w|^{p_0(z)-2} w\right)(w_\varepsilon - w)dz \to 0,$$

then, $w_\varepsilon \longrightarrow w$ *strongly in* $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$.

Lemma 2.4 *[1] Let* $(w_\varepsilon)_\varepsilon$ *be a sequence of* $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$ *such that* $w_\varepsilon \rightharpoonup w$ *weakly in* $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$. *Then* $T_\ell(w_\varepsilon) \rightharpoonup T_\ell(w)$ *weakly in* $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$.

3. Existence of entropy solutions

In this section, we define entropy solutions to the obstacle elliptic problem (1.1). Furthermore, we demonstrate the primary outcome of this paper.

Definition 3.1 *A measurable function* w *is said to be an entropy solution for the obstacle problem* (1.1)*, if*

$$w \in \mathcal{T}^{1,\vec{p}(z)}(\Omega, \vec{\varrho}), \ |w|^{r(z)} \in L^1(\Omega), \ \frac{|w|^{p_0(z)-2}w}{|z|^{p_0(z)}} \in L^1(\Omega)$$

and

$$\sum_{i=1}^{N} \int_\Omega \kappa_i(z, w, \nabla w)\partial_i T_\ell(w - \psi)dz + \int_\Omega \mathcal{H}(z, w, \nabla w)T_\ell(w - \psi)dz + \int_\Omega |w|^{r(z)-1} w T_\ell(w - \psi)dz \le \int_\Omega f T_\ell(w - \psi)dz + \mu \int_{\partial\Omega} \frac{|w|^{p_0(z)-2}w}{|z|^{p_0(z)}} T_\ell(w - \psi)dz, \quad (3.1)$$

for all $\psi \in \mathcal{D}_\Delta \cap L^\infty(\Omega)$.

Theorem 3.1 *Let* $f \in L^1(\Omega)$*, supposing that* $(\mathcal{H}_1)$*–* $(\mathcal{H}_2)$ *and* (1.2) – (1.6) *hold. Then there exists at least one entropy solution for the problem* (1.1).

Proof of the Theorem 3.1

To prove the result of Theorem 3.1, we divide the proof into several steps.

Step 1: Approximate problems Let $(f_\varepsilon)_{\varepsilon\in\mathbb{N}}$ be a sequence of smooth functions such that $f_\varepsilon \to f$ in $L^1(\Omega)$ and $|f_\varepsilon| \le |f|$. We consider the approximate problem

$$\begin{cases} w_\varepsilon \in \mathcal{D}_\Delta \\ \displaystyle\sum_{i=1}^{N}\int_\Omega \kappa_i\big(z, T_\varepsilon(w_\varepsilon), \nabla w_\varepsilon\big)\big(\partial_i w_\varepsilon - \partial_i\varphi\big)dz + \frac{1}{\varepsilon}\int_\Omega |w_\varepsilon|^{p_0(z)-2}w_\varepsilon(w_\varepsilon-\varphi)dz \\ \displaystyle + \int_\Omega |T_\varepsilon(w_\varepsilon)|^{r(z)-1}T_\varepsilon(w_\varepsilon)(w_\varepsilon-\varphi)dz + \int_\Omega \mathcal{H}_\varepsilon\big(z, T_\varepsilon(w_\varepsilon), \nabla w_\varepsilon\big)(w_\varepsilon-\varphi)dz \\ \displaystyle \le \int_\Omega f_\varepsilon\big(w_\varepsilon-\varphi\big)dz + \mu\int_\Omega \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-2}T_\varepsilon(w_\varepsilon)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}} \qquad \forall\varphi\in\mathcal{D}_\Delta, \end{cases} \tag{3.2}$$

where $\mathcal{H}_\varepsilon(z,s,\xi) = \dfrac{\mathcal{H}(z,s,\xi)}{1+\frac{1}{\varepsilon}|\mathcal{H}(z,s,\xi)|}$.
Note that $|\mathcal{H}_\varepsilon(z,s,\xi)| \le |\mathcal{H}(z,s,\xi)|$ and $|\mathcal{H}_\varepsilon(z,s,\xi)| \le \varepsilon$ for any $\varepsilon\in\mathbb{N}^*$
Let's consider the operator $\mathcal{G}_\varepsilon : \mathcal{D}_\Delta \to \mathcal{D}_\Delta^*$ by

$$\langle \mathcal{G}_\varepsilon w, v\rangle = \int_\Omega |T_\varepsilon(w)|^{r(z)-1}T_\varepsilon(w)v dz - \mu\int_\Omega \frac{|T_\varepsilon(w)|^{p_0(z)-2}T_\varepsilon(w)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}v dz + \int_\Omega \mathcal{H}_\varepsilon(z,T_\varepsilon(w_\varepsilon),\nabla w_\varepsilon)v dz, \qquad \text{for any } w,\, v\in\mathcal{D}_\Delta.$$

Thanks to the Hölder's inequality and by using (2.2), we have

$$\begin{aligned} |\langle G_\varepsilon w, v\rangle| &\le \int_\Omega |T_\varepsilon(w)|^{r(z)}|v|dz + \mu\int_\Omega \frac{|T_\varepsilon(w)|^{p_0(z)-1}}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}|v|dz \\ &+ \int_\Omega |\mathcal{H}_\varepsilon(z,T_\varepsilon(w_\varepsilon),\nabla w_\varepsilon)||v|dz \le \Big(\int_\Omega |T_\varepsilon(w)|^{r(z)(p_0'(z)}dz\Big)^{\frac{1}{p_0'(z)}}\|v\|_{p_0(z)} \\ &+ \mu\Big(\int_\Omega \Big(\frac{|T_\varepsilon(w)|^{p_0(z)-1}}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}\Big)^{p_0'(z)}dz\Big)^{\frac{1}{(p_0')^-}}\|v\|_{p_0(z)} \\ &+ \Big(\frac{1}{p_0^-}+\frac{1}{(p_0^-)'}\Big)\Big(\int_\Omega \varepsilon^{(p_0^-)'}dz + 1\Big)^{\frac{1}{(p_0^-)'}}\|v\|_{p_0(z)} \\ &\le (\varepsilon^{r^-}+\mu\varepsilon^{p_0^-}+\varepsilon^{(p_0^-)'}+1)(meas(\Omega))^{\frac{1}{(p_0(z))'}}\|v\|_{p_0(z)} \\ &\le C'\|v\|. \end{aligned} \tag{3.3}$$

Lemma 3.1 *We consider the operator $\mathcal{L}_\varepsilon : \mathcal{D}_\Delta \to \mathcal{D}_\Delta^*$ defined by*

$$\langle \mathcal{L}_\varepsilon u, v\rangle = \sum_{i=1}^{N}\int_\Omega \kappa_i\big(z, T_\varepsilon(w_\varepsilon), \nabla w_\varepsilon\big)\partial_i v dz + \frac{1}{\varepsilon}\int_\Omega |w_\varepsilon|^{p_0(z)-2}w_\varepsilon v dz.$$

The operator $\mathcal{B}_\varepsilon = \mathcal{L}_\varepsilon + \mathcal{G}_\varepsilon$ acted from $W^{1,\vec{p}(z)}(\Omega,\vec{\varrho})$ into $\big(W^{1,\vec{p}(z)}(\Omega,\vec{\varrho})\big)^*$ is bounded, pseudo-monotone and coercive in the following sense

$$\frac{\langle \mathcal{B}_\varepsilon v, v-v_0\rangle}{\|v\|} \to +\infty \text{ as } \|v\|\to\infty \text{ for } v\in\mathcal{D}_\Delta.$$

Proof: Let us first show that $\mathcal{L}_\varepsilon$ is bounded, thus by applying Hölder's inequality, and combining it with the growth condition (1.3), let w_0 belongs to $\mathcal{D}_\Delta$, for any w in $\mathcal{D}_\Delta$, we obtain

$$\begin{aligned}
\Big|\langle \mathcal{L}_\varepsilon w, w_0\rangle\Big| &= \Big|\sum_{i=1}^{N}\int_\Omega \kappa_i(w, T_\varepsilon(w), \nabla w)\partial_i w_0 dz + \frac{1}{\varepsilon}\int_\Omega |w|^{p_0(z)-2} w w_0 dz\Big| \\
&\leq \sum_{i=1}^{N}\Big(\int_\Omega |\kappa_i(z, T_\varepsilon(w), \nabla w)|^{p_i'(z)} \varrho_i^{1-p_i'(z)} dz\Big)^{\frac{1}{(p_i')^-}} \|\varrho_i^{\frac{1}{p_i(z)}}\partial_i w_0\|_{L^{p_i(z)}(\Omega)} \\
&+ \Big(\int_\Omega |w|^{(p_0(z)-1)p_0'(z)} dz\Big)^{\frac{1}{(p_0^-)'}} \|w_0\|_{p_0(z)} \\
&\leq \beta\sum_{i=1}^{N}\Big(\int_\Omega \big(R_i^{p_i'(z)} + |T_\varepsilon(w)|^{p_i(z)} + \sum_{i=1}^{N}\varrho_i|\partial_i w|^{p_i(z)}\big)\Big)^{\frac{1}{(p_i')^-}} \|\partial_i w_0\|_{L^{p_i(z)}(\Omega,\varrho_i)} \\
&+ \Big(\int_\Omega |w|^{p_0(z)} dz\Big)^{\frac{1}{(p_0)^-}} \|w_0\|_{p_0(z)} \leq C_0\|w_0\|,
\end{aligned} \tag{3.4}$$

then, in view of (3.4) and (3.3) we can conclude that $\mathcal{B}_\varepsilon$ is bounded.
Thereafter, to establish the coercivity, let w_0 belongs to $\mathcal{D}_\Delta$. Then, for any w in $\mathcal{D}_\Delta$, According to (1.2) we obtain

$$\begin{aligned}
\Big|\langle \mathcal{L}_\varepsilon w, w\rangle\Big| &\geq \alpha\sum_{i=1}^{N}\int_\Omega \big|\partial_i w\big|^{p_i(z)}\varrho_i(z)dz + \frac{1}{m}\int_\Omega |w|^{p_0(x)}dz\Big| \\
&\geq \underline{\alpha}\|w\|^{\underline{p}},
\end{aligned} \tag{3.5}$$

with $\underline{\alpha} = \min(\alpha, \dfrac{1}{m})$.
Combining (3.4) and (3.5), we have

$$\begin{aligned}
\langle \mathcal{L}_\varepsilon w, w - w_0\rangle &= \langle \mathcal{L}_\varepsilon w, w\rangle - \langle \mathcal{L}_\varepsilon w, w_0\rangle \\
&\geq \underline{\alpha}\sum_{i=1}^{N}\int_\Omega \varrho_i|\partial_i w|^{p_i(z)}dz - C_0\|w_0\| \\
&\geq \underline{\alpha}\|w\|^{\underline{p}} - C_0\|w_0\|,
\end{aligned}$$

it follows that

$$\frac{\langle \mathcal{L}_\varepsilon w, w - w_0\rangle}{\|w\|} \geq \frac{\underline{\alpha}}{\|w\|}\|w\|^{\underline{p}} - \frac{C_0}{\|w\|}\|w_0\| \longrightarrow +\infty \text{ as } \|w\| \to \infty.$$

Witch implies that

$$\frac{\langle \mathcal{B}_\varepsilon w, w - w_0\rangle}{\|w\|} = \frac{\langle \mathcal{L}_\varepsilon w, w - w_0\rangle}{\|w\|} + \frac{\langle \mathcal{G}_\varepsilon w, w - w_0\rangle}{\|w\|} \longrightarrow +\infty \text{ as } \|w\| \to \infty.$$

We still need to prove that $\mathcal{B}_\varepsilon$ is pseudo-monotone. Let $(w_\varepsilon)_{\varepsilon\in\mathbb{N}}$ be a sequence in $W^{1,\vec{p_i}(z)}(\Omega, \vec{\varrho})$ satisfying the following condition

$$\begin{cases} w_n \rightharpoonup w & \text{in} \quad W^{1,\vec{p}(z)}(\Omega, \vec{\varrho}) \\ \mathcal{B}_\varepsilon w_\varepsilon \rightharpoonup \chi_\varepsilon & \text{in} \quad (W^{1,\vec{p}(z)})(\Omega, \vec{\varrho}))^* \\ \limsup\limits_{n\to\infty}\langle \mathcal{B}_\varepsilon w_n, w_n\rangle \leq \langle \chi_\varepsilon, w\rangle \end{cases} \tag{3.6}$$

We will show that $\chi_\varepsilon = \mathcal{B}_\varepsilon w$ and $\langle \mathcal{B}_\varepsilon w_n, w_n\rangle \to \langle \chi_\varepsilon, w\rangle$ as $n \to +\infty$.
With the help of the compact embedding $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho}) \hookrightarrow L^{\underline{p}}(\Omega)$, we obtain w_n converges to w in $L^{\underline{p}}(\Omega)$ for a subsequence noted again $(w_n)_{n\in\mathbb{N}}$.
Since $(w_n)_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$. By (1.3) it is obvious that the sequence

$(\kappa_i(z,T_m(w_n),\nabla w_n))_{n\in\mathbb{N}}$ is bounded in $L^{p_i'(z)}(\Omega,\varrho_i^*)$, which implies the existence of a measurable function $\pi_i^\varepsilon \in L^{p_i'(z)}(\Omega,\varrho_i^*)$ such that

$$\kappa_i(z,T_\varepsilon(w),\nabla w) \rightharpoonup \pi_i^\varepsilon \qquad in \quad L^{p_i'(z)}(\Omega,\varrho_i^*) \ as\ n\to\infty. \tag{3.7}$$

We apply the Lebesgue dominated convergence theorem to obtain

$$\frac{|T_\varepsilon(w_n)|^{p_0(z)-2}T_\varepsilon(w_n)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}} \to \frac{|T_\varepsilon(w)|^{p_0(z)-2}T_\varepsilon(w)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}} \quad in \quad L^{p_0'(z)}(\Omega,\varrho_0^*). \tag{3.8}$$

$$|T_\varepsilon(w_n)|^{r(z)-1}T_\varepsilon(w_n) \to |T_\varepsilon(w)|^{r(z)-1}T_\varepsilon(w) \quad in \quad L^{p_0'(z)}(\Omega,\varrho_0^*). \tag{3.9}$$

Also, we have

$$|w_n|^{p_0(z)-2}w_n \rightharpoonup |w|^{p_0(z)-2}w \ in\ L^{p_0'(z)}(\Omega,\omega^*). \tag{3.10}$$

Similarly, we have $(\mathcal{H}_\varepsilon(z,w_n,\nabla w_n))_{\varepsilon\in\mathbb{N}}$ is bounded in $L^{\underline{p}'}(\Omega,\gamma^*)$ (i.e γ^* is the conjugate of $\gamma := \inf_{i\in\mathbb{N}}\varrho_i$) , then there exists a function $\sigma_\varepsilon \in L^{\underline{p}'}(\Omega,\gamma^*)$ such that

$$\mathcal{H}_\varepsilon(z,w_n,\nabla w_n) \to \sigma_\varepsilon \text{ in } \quad L^{\underline{p}'}(\Omega,\gamma^*) \text{ as } n\to\infty. \tag{3.11}$$

For all $\omega \in W^{1,\vec{p}(z)}(\Omega,\vec{\varrho})$, we obtain

$$\begin{aligned}
&\langle\chi_\varepsilon,\varphi\rangle = \lim_{n\to\infty}\langle B_\varepsilon w_n,\varphi\rangle = \lim_{n\to\infty}\sum_{i=1}^N\int_\Omega \kappa_i(z,T_m(w_n),\nabla w_n)\partial_i\varphi dx\\
&+\lim_{n\to\infty}\frac{1}{\varepsilon}\int_\Omega |w_n|^{p_0(z)-2}w_n\varphi dz - \lim_{n\to\infty}\mu\int_\Omega \frac{|T_\varepsilon(w_n)|^{p_0(z)-2}T_\varepsilon(w_n)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}\varphi dz\\
&+\lim_{n\to\infty}\int_\Omega \mathcal{H}_\varepsilon\Big(z,T_\varepsilon(w_n),\nabla w_n\Big)\varphi dz + \int_\Omega |T_\varepsilon(w_n)|^{r(z)-1}T_\varepsilon(w_n)\varphi dz\\
&=\sum_{i=1}^N\int_\Omega \pi_i^\varepsilon\partial_i\varphi dz + \int_\Omega |w|^{p_0(z)-2}w\varphi dz - \mu\int_\Omega \frac{|T_\varepsilon(w)|^{p_0-2}T_\varepsilon(w)}{|z|^{p_0}+\frac{1}{\varepsilon}}\varphi dz\\
&+\sum_{i=1}^N\int_\Omega \sigma_\varepsilon\partial_i\varphi dz + \int_\Omega |T_\varepsilon(w)|^{r(z)-1}T_\varepsilon(w)\varphi dz.
\end{aligned} \tag{3.12}$$

From (3.6) and (3.12), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle\mathcal{B}_\varepsilon w_n,w_n\rangle = \limsup_{n\to\infty}\Big\{\sum_{i=1}^N\int_\Omega \kappa_i(z,T_\varepsilon(w_n),\nabla w_n)\partial_i w_n dz\\
&+\frac{1}{\varepsilon}\int_\Omega |w_n|^{p_0(z)}dz - \mu\int_\Omega \frac{|T_\varepsilon(w_n)|^{p_0(z)-2}T_m(w_n)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}w_n dz\\
&+\int_\Omega \mathcal{H}_\varepsilon\Big(z,T_\varepsilon(w_n),\nabla w_n\Big)w_n dz + \int_\Omega |T_\varepsilon(w_n)|^{r(z)-1}T_\varepsilon(w_n)\varphi dz\Big\}\\
&\le\sum_{i=1}^N\int_\Omega \pi_i^\varepsilon\partial_i w dz + \frac{1}{\varepsilon}\int_\Omega |w|^{p_0(z)}dz - \mu\int_\Omega \frac{|T_\varepsilon(w)|^{p_0(z)-2}T_\varepsilon(w)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}w dz\\
&+\int_\Omega \sigma_\varepsilon w dz + \int_\Omega |T_\varepsilon(w)|^{r(z)-1}T_\varepsilon(w)w dz.
\end{aligned}$$

By using (3.8)-(3.11), we obtain

$$\int_\Omega \frac{|T_\varepsilon(w_n)|^{p_0(z)-2}T_\varepsilon(w_n)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}w_n dz \to \int_\Omega \frac{|T_\varepsilon(w)|^{p_0(z)-2}T_\varepsilon(w)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}w dz, \tag{3.13}$$

$$\int_\Omega |T_\varepsilon(w_n)|^{r(z)-1}T_\varepsilon(w_n)w_n dz \to \int_\Omega |T_\varepsilon(w)|^{r(z)-1}T_\varepsilon(w)w dz \tag{3.14}$$

and

$$\int_\Omega \mathcal{H}_\varepsilon\Big(z, T_\varepsilon(w_n), \nabla w_n\Big)w_n dz \to \int_\Omega \sigma_\varepsilon w dz. \tag{3.15}$$

Which implies that

$$\limsup_{n\to\infty}\Big(\sum_{i=1}^N\int_\Omega \kappa_i(z, T_\varepsilon(w_n), \nabla w_n)\partial_i w_n dz + \frac{1}{\varepsilon}\int_\Omega |w_n|^{p_0(z)}dz\Big) \leq \sum_{i=1}^N\int_\Omega \pi_i^\varepsilon \partial_i w dz + \frac{1}{\varepsilon}\int_\Omega |w|^{p_0(z)}dz. \tag{3.16}$$

On the other side, taking into account (1.4), we get

$$\sum_{i=1}^N\int_\Omega(\kappa_i(z, T_\varepsilon(w_n), \nabla w_n) - \kappa_i(x, T_\varepsilon(w_n), \nabla w))(\partial_i w_n - \partial_i w)dz + \frac{1}{\varepsilon}\int_\Omega(|w_n|^{p_0(z)-2}w_n - |w|^{p_0(z)-2}w)(w_n - w)dz \geq 0,$$

hence

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega \kappa_i(z, T_\varepsilon(w_n), \nabla w_n)\partial_i w_n dz + \frac{1}{\varepsilon}\int_\Omega |w_n|^{p_0(z)}dz\\ &\geq \sum_{i=1}^N\int_\Omega \kappa_i(z, T_\varepsilon(w_n), \nabla w_n)\partial_i w_n dz + \frac{1}{\varepsilon}\int_\Omega |w|^{p_0(z)-2}w_n w dz\\ &+\sum_{i=1}^N\int_\Omega \kappa_i(z, T_\varepsilon(w_k), \nabla w)(\partial_i w_n - \partial_i w)dz + \int_\Omega |w|^{p_0(z)-2}w(w_n - w)dz.\end{aligned}$$

The Lebesgue dominated convergence theorem implies $T_\varepsilon(w_n) \longrightarrow T_\varepsilon(w)$ in $L^{p_i(z)}(\Omega, \varrho_i)$, hence $\kappa_i(z, T_\varepsilon(w_n), \nabla w)$ converges to $\kappa_i(z, T_\varepsilon(w), \nabla w)$ in $L^{p'_i(z)}(\Omega, \varrho_i^*)$, by employing (3.6) we infer

$$\liminf_{n\to\infty}\Big(\sum_{i=1}^N\int_\Omega \kappa_i(z, T_\varepsilon(w_n), \nabla w_n)\partial_i w_n dz + \frac{1}{\varepsilon}\int_\Omega |w_n|^{p_0(z)}dz\Big) \geq \sum_{i=1}^N\int_\Omega \pi_i \partial_i w dz + \frac{1}{\varepsilon}\int_\Omega |w|^{p_0(z)}dz.$$

According to (3.16), we deduce that

$$\lim_{n\to\infty}\Big(\sum_{i=1}^N\int_\Omega \kappa_i(z, T_\varepsilon(w_n), \nabla w_n)\partial_i w_n dz + \frac{1}{\varepsilon}\int_\Omega |w_n|^{p_0(z)}dz\Big) = \sum_{i=1}^N\int_\Omega \pi_i^\varepsilon \partial_i w dz + \frac{1}{\varepsilon}\int_\Omega |w|^{p_0(z)}dz. \tag{3.17}$$

Hence, from (3.10)-(3.13), it follows that

$$\langle \mathcal{B}_\varepsilon w_n, w_n\rangle \to \langle \chi_\varepsilon, w\rangle \ as\ n \to \infty.$$

In the sequel, by means (3.17) we can establish that

$$\lim_{n\to+\infty}\Big(\sum_{i=1}^{N}\int_{\Omega}(\kappa_i(z,T_\varepsilon(w_n),\nabla w_n)-\kappa_i(z,T_\varepsilon(w_n),\nabla w))(\partial_i w_n-\partial_i w)dz + \frac{1}{\varepsilon}\int_{\Omega}\Big(|w_n|^{p_0(z)-2}w_n-|w|^{p_0(z)-2}w)(w_n-w)dz\Big)=0.$$

Once again, by Lemma 2.3, we obtain

$$w_n\to w \quad \text{in } W^{1,\vec{p}(z)}(\Omega,\vec{\varrho}) \qquad \text{and} \qquad \partial_i w_n\to\partial_i w \quad \text{a.e. in } \Omega,$$

which means that

$$\kappa_i(z,T_\varepsilon(w_n),\nabla w_n)\rightharpoonup\kappa_i(z,T_\varepsilon(w),\nabla w) \quad \text{in } L^{p_i'(z)}(\Omega,\varrho_i^*) \quad \text{for } i=1,\ldots,N,$$

and

$$\mathcal{H}_\varepsilon(z,T_\varepsilon(w_n),\nabla w_n)\rightharpoonup\mathcal{H}_\varepsilon(z,T_\varepsilon(w),\nabla w) \quad \text{in } L^{\underline{p}'(z)}(\Omega,\gamma^*),$$

and it follows from (3.7)-(3.10) that $\chi_\varepsilon=\mathcal{B}_\varepsilon w$, which conclude the proof of Lemma 3.1. □

According to Lemma 3.1, there exists at least one weak solution $w_n\in W^{1,\vec{p}(z)}(\Omega,\vec{\varrho})$ of the problem (3.2).

Step 2 : A priori estimates

Lemma 3.2 *Let us suppose that w_ε is a weak solution of the problem (3.2). In this case, the regularity results stated below hold*

$$w\in W^{1,q(z)}(\Omega,\omega),\ \textit{where}\quad q(z)=\Big(r(z),q_1(z),\cdots,q_N(z)\Big), \tag{3.18}$$

such that $r(z)>\frac{N(p_0(z)-1)}{N-p_0(z)}$, $1\le q_i(z)<\frac{p_i(z)r(z)}{r(z)-p_i(z)}$ *and* $\omega^{-\frac{p_i(z)}{1-p_i(z)}}\in L^1(\Omega)$,

$$\sum_{i=1}^{N}\int_{\Omega}\frac{\varrho_i|\partial_i w_\varepsilon|^{p_i(z)}}{(1+|w_\varepsilon|)^{\tau(z)}}dz\le C \textit{ for each } \ 1<\tau(z)<\frac{r(z)(p_i(z)-q_i(z))}{q_i(z)}, \tag{3.19}$$

$$\sum_{i=1}^{N}\int_{\Omega}\varrho_i|\partial_i T_\ell(w_\varepsilon)|^{p(z)}dz\le C(1+\ell)^{\tau^+} \textit{ for all } \ \ell>0, \tag{3.20}$$

where C is a positive constant independent of ε and ℓ.

Proof: In this step we will use some methods of [26]. We choose $\tau(z)>1$ and define the function $v(t)$, which defines from $\mathbb{R}$ to $\mathbb{R}$ as follows

$$\vartheta(t)=\Big(1-\frac{1}{(1+|t|)^{\tau(z)-1}}\Big)sign(t) \text{ and } G(b)=\frac{1}{\alpha}\int_0^b g(|\beta|)d\beta.$$

Note that, as the function $g(\cdot)$ is integrable on $\mathbb{R}$, then $0\le G(\infty)=\frac{1}{\alpha}\int_0^\infty g(|\beta|)d\beta<\infty$.
Let us consider the function $\varphi=w_\varepsilon-\xi\,\vartheta(w_\varepsilon)exp(G(|w_\varepsilon|))$, where $\xi>0$. It's obvious that $\varphi\in W^{1,\vec{p}(z)}(\Omega,\vec{\varrho})\cap L^\infty(\Omega)$ and for all ξ small enough, we deduce that $\varphi\ge\Delta$. Then φ is an admissible

test function in (3.2), this allows to write

$$
\begin{aligned}
&(\tau^- -1)\sum_{i=1}^N\int_\Omega \frac{\kappa_i(z,T_\varepsilon(w_\varepsilon),\nabla w_\varepsilon)}{(1+|w_\varepsilon|)^{\tau(z)}}\partial_i w_\varepsilon e^{G(|w_\varepsilon|)}dz+\frac{1}{\alpha}\sum_{i=1}^N\int_\Omega \kappa_i(z,T_\varepsilon(w_\varepsilon),\nabla w_\varepsilon)\\
&\times \partial_i w_\varepsilon g(|w_\varepsilon|)\vartheta(w_\varepsilon)e^{G(|w_\varepsilon|)}dz+\frac{1}{\varepsilon}\int_\Omega |w_\varepsilon|^{p_0(x)-2}w_\varepsilon\vartheta(w_\varepsilon)e^{G(|w_\varepsilon|)}dz\\
&+\int_\Omega \mathcal{H}_\varepsilon(z,T_\varepsilon(w_\varepsilon),\nabla w_\varepsilon)\vartheta(w_\varepsilon)e^{G(|w_\varepsilon|)})dz+\int_\Omega |T_\varepsilon(w_\varepsilon)|^{r(z)-1}T_\varepsilon(w_\varepsilon)\vartheta(w_\varepsilon)e^{G(|w_\varepsilon|)}dz\\
&\le \int_\Omega f_\varepsilon\vartheta(w_\varepsilon)e^{G(|w_\varepsilon|)}dz+\mu\int_\Omega \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-2}T_\varepsilon(w_\varepsilon)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}\vartheta(w_\varepsilon)e^{G(|w_\varepsilon|)}dz
\end{aligned}
$$

Additionally, the sign of $\vartheta(w_\varepsilon)$ is the same as that of w_ε, which makes the third term of the previous inequality positive. Furthermore, based on the condition (1.2) and $|\vartheta(\cdot)|\le 1$ we conclude that

$$
\begin{aligned}
&(\tau^- -1)\alpha\sum_{i=1}^N\int_\Omega \frac{\varrho_i|\partial_i w_\varepsilon|^{p_i(z)}}{(1+|w_\varepsilon|)^{\tau(z)}}e^{G(|w_\varepsilon|)}dz+\sum_{i=1}^N\int_\Omega \varrho_i|\partial_i w_\varepsilon|^{p_i(z)}g(|w_\varepsilon|)\\
&\times\vartheta(w_\varepsilon)e^{G(|w_\varepsilon|)}dz+\int_\Omega |T_\varepsilon(w_\varepsilon)|^{r(z)}|\vartheta(w_\varepsilon)|e^{G(|w_\varepsilon|)}dz\\
&\le \int_\Omega (|f|+|b|)e^{G(|w_\varepsilon|)}dz+\mu\int_\Omega \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-1}}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}e^{G(|w_\varepsilon|)}dz\\
&\quad+\sum_{i=1}^N\int_\Omega \varrho_i|\partial_i w_\varepsilon|^{p_i(z)}g(|w_\varepsilon|)\vartheta(w_\varepsilon)e^{G(|w_\varepsilon|)}dz,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(\tau^- -1)\alpha\sum_{i=1}^N\int_\Omega \frac{\varrho_i|\partial_i w_\varepsilon|^{p_i(z)}}{(1+|w_\varepsilon|)^{\tau(z)}}e^{G(|w_\varepsilon|)}dz+\int_\Omega |T_\varepsilon(w_\varepsilon)|^{r(z)}|\vartheta(w_\varepsilon)|e^{G(|w_\varepsilon|)}dz\\
&\qquad\le 2e^{G(\infty)}\big(\|f\|_{L^1(\Omega)}+\|b\|_{L^1(\Omega)}\big)+\mu\int_\Omega \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-1}}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}e^{G(|w_\varepsilon|)}dz \quad (3.21)
\end{aligned}
$$

It is easy to see that

$$
\frac{1}{2}\le 1-\frac{1}{(1+|w_\varepsilon|)^{\tau(z)-1}}\quad \textit{for } |w_\varepsilon|\ge R=\max\big(2^{\frac{1}{\tau^- -1}}-1,1\big).
$$

then, we infer

$$
\begin{aligned}
\frac{1}{2}\int_{\{|w_\varepsilon|\ge R\}}\big|T_\varepsilon(w_\varepsilon)\big|^{r(z)}dz&\le \int_{\{|w_\varepsilon|\ge R\}}\big|T_\varepsilon(w_\varepsilon)\big|^{r(z)}\Big(1-\frac{1}{(1+|w_\varepsilon|)^{\tau(z)-1}}\Big)dz\\
&\le \int_\Omega \big|T_\varepsilon(w_\varepsilon)\big|^{r(z)}\Big(1-\frac{1}{(1+|w_\varepsilon|)^{\tau(z)-1}}\Big)dz,
\end{aligned}
$$

which means

$$
\begin{aligned}
\frac{1}{2}\int_\Omega \big|T_\varepsilon(w_\varepsilon)\big|^{r(z)}dz&=\frac{1}{2}\int_{\{|w_n|<R\}}\big|T_\varepsilon(w_\varepsilon)\big|^{r(z)}dz+\frac{1}{2}\int_{\{|w_\varepsilon|\ge R\}}\big|T_\varepsilon(w_\varepsilon)\big|^{r(z)}dz\\
&\le \frac{1}{2}max(R^{r^-},R^{r^+})|\Omega|+\int_\Omega \big|T_\varepsilon(w_\varepsilon)\big|^{r(z)}\Big(1-\frac{1}{(1+|w_\varepsilon|)^{\tau(z)-1}}\Big)dz.
\end{aligned}
$$

Using (3.21), we obtain

$$(\tau^- - 1)\alpha \sum_{i=1}^{N} \int_\Omega \frac{\varrho_i|\partial_i w_\varepsilon|^{p_i(z)}}{(1+|w_\varepsilon|)^{\tau(z)}}dz + \frac{1}{2}\int_\Omega |T_\varepsilon(w_\varepsilon)|^{r(z)}dz \le \frac{1}{2} max(R^{r^-}, R^{r^+})|\Omega|$$
$$+ 2e^{G(\infty)}\big(\|f\|_{L^1(\Omega)} + \|b\|_{L^1(\Omega)}\big) + \mu \int_\Omega \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-1}}{|z|^{p_0(z)}} e^{G(\infty)}dz. \quad (3.22)$$

As $r(z) > p_0(z) - 1$, according to Young's inequality, we obtain

$$\mu \int_\Omega \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-1}}{|z|^{p_0(z)}}dz \le \frac{1}{4}\int_\Omega |T_\varepsilon(w_\varepsilon)|^{r(z)}dz + C_1 \int_\Omega \frac{dz}{|z|^{\frac{r(z)p_0(z)}{r(z)-p_0(z)+1}}},$$

with C_1 is a positive constant depending only on $r(z)$, $p_0(z)$ and μ. Thus, we obtain

$$(\tau^- - 1)\alpha \sum_{i=1}^{N} \int_\Omega \frac{\varrho_i|\partial_i w_\varepsilon|^{p_i(z)}}{(1+|w_\varepsilon|)^{\tau(z)}}dz + \frac{1}{4}\int_\Omega |T_\varepsilon(w_\varepsilon)|^{r(z)}dz \quad (3.23)$$
$$\le \frac{1}{2} max(R^{r^-}, R^{r^+})|\Omega| + 2e^{G(\infty)}\big(\|f\|_{L^1(\Omega)} + \|b\|_{L^1(\Omega)}\big) + C_1 \int_\Omega \frac{dz}{|z|^{\frac{r(z)p_0(z)}{r(z)-p_0(z)+1}}}. \quad (3.24)$$

Under the assumption $r(z) > \frac{N(p_0(z)-1)}{N-p_0(z)}$, the integral $\displaystyle\int_\Omega \frac{dz}{|z|^{\frac{r(z)p_0(z)}{r(z)-p_0(z)+1}}}$ is finite. Consequently, (3.19) is valid. Furthermore, we have

$$\int_\Omega |T_\varepsilon(w_\varepsilon)|^{r(z)}dz \le C. \quad (3.25)$$

If we take $q_i(z)$ such that $1 \le q_i(z) < p_i(z)$ for $i = 1, \dots, N$. By means of Hölder's generalized inequality, we derive

$$\sum_{i=1}^{N} \int_\Omega \varrho_i |\partial_i w_\varepsilon|^{q_i(z)}dz \le \sum_{i=1}^{N} \Big(\int_\Omega \frac{\varrho_i|\partial_i w_\varepsilon|^{p_i(z)}}{(1+|w_\varepsilon|)^{\tau(z)}}dz\Big)^{\frac{q_i^-}{p_i^-}} \|\varrho_i\|_{p_i'(z)}^{1-\frac{q_i^+}{p_i^+}} \quad (3.26)$$
$$\times \Big\|(1+|w_\varepsilon|)^{\frac{q_i(z)\tau(z)}{p_i(z)-q_i(z)}}\Big\|_{p_i(z)}^{1-\frac{q_i^+}{p_i^+}} \le \Big(\sum_{i=1}^{N} \int_\Omega \frac{\varrho_i|\partial_i w_\varepsilon|^{p_i(z)}}{(1+|w_\varepsilon|)^{\tau(z)}}dz\Big)^{\frac{q_i^-}{p_i^-}}$$
$$\times \Big(\int_\Omega \varrho_i^{-\frac{p_i(z)}{1-p_i(z)}}dz\Big)^{(1-\frac{1}{p_i^+})(1-\frac{q_i^+}{p_i^+})} \Big(\int_\Omega (1+|w_\varepsilon|)^{\frac{q_i(z)\tau(z)p_i(z)}{(p_i(z)-q_i(z))}}dz\Big)^{\frac{1}{p_i^-}(1-\frac{q_i^+}{p_i^+})}$$
$$\le C_2 \Big(\sum_{i=1}^{N} \int_\Omega \frac{\varrho_i|\partial_i w_\varepsilon|^{p_i(z)}}{(1+|w_\varepsilon|)^{\tau(z)}}dz\Big)^{\frac{q_i^-}{p_i^-}} \Big(\int_\Omega (1+|w_\varepsilon|)^{\frac{p_i(z)\tau(z)q_i(z)}{p_i(z)-q_i(z)}}dz\Big)^{\frac{1}{p_i^-}(1-\frac{q_i^+}{p_i^+})}.$$

We now choose $\tau(z) > 1$ such that $\frac{q_i(z)\tau(z)p_i(z)}{p_i(z)-q_i(z)} < r(z)$, such a real number $\tau(z)$ exists if

$$1 < \frac{r(z)(p_i(z)-q_i(z))}{p_i(z)q_i(z)} \quad \text{that is} \quad q_i(z) < \frac{p_i(z)r(z)}{r(z)+p_i(z)}.$$

By combining equations (3.23)-(3.26), we obtain the desired estimates expressed by (3.18). To derive (3.20), we use (3.19), which allows us to conclude that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_\ell(w_\varepsilon)|^{p_i(z)}\varrho_i dz = \sum_{i=1}^{N} \int_{\{|w_\varepsilon|<\ell\}} \varrho_i |\partial_i w_\varepsilon|^{p_i(z)}dz \le (1+\ell)^{\tau^+} \sum_{i=1}^{N} \int_\Omega \frac{\varrho_i|\partial_i w_\varepsilon|^{p_i(z)}}{(1+|w_\varepsilon|)^{\tau(z)}}dz.$$

□

Step 3: Weak convergence of truncations To demonstrate the weak convergence of $(T_\ell(w_\varepsilon))_\varepsilon$ in $W^{1,\vec{p}(z)}(\Omega,\vec{\varrho})$, we first show that $(w_\varepsilon)_\varepsilon$ is a Cauchy sequence. This is possible because of the equation (3.20).

$$\sum_{i=1}^{N}\int_\Omega |\partial_i T_\ell(w_\varepsilon)|^{p_i(z)}\varrho_i dz \leq C(1+\ell)^{\tau^+} + \ell^{p_0^-}|\Omega| \quad for\ \ell \geq 1,$$

Consequently, if the sequence $(T_\ell(w_\varepsilon))_\varepsilon$ is bounded in $W^{1,\vec{p}(z)}(\Omega,\vec{\varrho})$, then it is possible to identify a specific subsequence denoted by $(T_\ell(w_\varepsilon))_\varepsilon$ such that

$$\begin{cases} T_\ell(w_\varepsilon) \rightharpoonup \delta_\ell \ in\ W^{1,\vec{p}(z)}(\Omega,\vec{\varrho}), \\ T_\ell(w_\varepsilon) \to \delta_\ell \ in\ L^{\underline{p}}(\Omega,\gamma) \text{ and a.e. in } \Omega. \end{cases} \tag{3.27}$$

With the help of equation (3.20), we can conclude that there exists a constant C_4 that is independent of both ℓ and ε, implying that

$$\|\nabla T_\ell(u_\varepsilon)\|_{L^{\underline{p}}(\Omega,\gamma)} \leq C_4 \ell^{\frac{\tau^+}{\underline{p}}} \ for\ \ell \geq 1 \tag{3.28}$$

Given a ball B_R in Ω, if ℓ is taken to be sufficiently large, by utilizing equation (3.28) and invoking the Poincaré type inequality and Lemma 2.1, we arrive at the conclusion that

$$\begin{aligned} \ell\ meas\big(\{|w_\varepsilon|>\ell\}\cap B_R\big) &= \int_{\{|w_\varepsilon|>\ell\}\cap B_R} |T_\ell(w_\varepsilon)| dz \\ &\leq C_5 \|\nabla T_\ell(w_\varepsilon)\|_{L^{\underline{p}}(\Omega,\gamma)} \\ &\leq C_6 \ell^{\frac{\tau^+}{\underline{p}}}. \end{aligned} \tag{3.29}$$

Taking $\tau(z)$ such that $(1<\tau(z)<\underline{p})$, we infer

$$meas\big(\{|w_\varepsilon|>\ell\}\cap B_R\big) \leq C_6 \frac{1}{\ell^{1-\frac{\tau^+}{\underline{p}}}} \to 0 \ as\ \ell \to +\infty. \tag{3.30}$$

For each $\zeta>0$, we obtain

$$\begin{aligned} meas\big(\{|w_\varepsilon - w_\eta|>\zeta\}\cap B_R\big) &\leq \text{meas}\big(\{|w_\varepsilon|>\ell\}\cap B_R\big) \\ &\quad + meas\big(\{|w_\eta|>\ell\}\cap B_R\big) + meas\big(\{|T_\ell(w_\varepsilon)-T_\ell(w_\eta)|>\zeta\}\big). \end{aligned}$$

By the equation (3.30) we can take a sufficiently large value of $\ell=\ell(m)$ where $m>0$.

$$meas\big(\{|w_\varepsilon|>\ell\}\cap B_R\big) \leq \frac{m}{3} \text{ and } meas\big(\{|w_\eta|>\ell\}\cap B_R\big) \leq \frac{m}{3}. \tag{3.31}$$

In other words, from the equation (3.27), let $(T_\ell(w_\varepsilon))_{\varepsilon\in\mathbb{N}}$ is a Cauchy sequence in measure. Consequently, for every positive value of ℓ and ζ, and for every positive value of m, there exists a specific value $m_0 = m_0(\ell,\zeta,m)$ such that

$$meas\big\{|T_\ell(w_\varepsilon)-T_\ell(w_\eta)|>\zeta\big\} \leq \frac{m}{3} \text{ for all } \varepsilon,\ \eta \geq m_0(\ell,\zeta,m). \tag{3.32}$$

From the equations (3.31) and (3.32), we conclude that for all positive values of ζ and m there exists a value $m_0 = m_0(\ell(m),\zeta,R)$ such that

$$meas\big(\{|w_\varepsilon - w_\eta|>\zeta\}\cap B_R\big) \leq m \quad \forall \varepsilon,\eta \geq m_0(\ell(m),\zeta,R).$$

This demonstrates that the sequence $(w_\varepsilon)_\varepsilon$ converges in measure and therefore converges a.e. to a measurable function w. As a result, we can state that

$$T_\ell(w_\varepsilon) \rightharpoonup T_\ell(w) \ in\ W^{1,\vec{p}(z)}(\Omega,\vec{\varrho}), \tag{3.33}$$

and by means of the dominated convergence theorem of Lebesgue we arrive at

$$T_\ell(w_\varepsilon) \to T_\ell(w) \ in\ L^{p_0(z)}(\Omega,\varrho_0) \text{ and a.e in } \Omega. \tag{3.34}$$

Step 4 : Strong convergence of truncations In the next section, we use the notation $n_i(\varepsilon)$, where $i = 1, 2, \ldots$, to represent various real-valued functions with respect to real variables. These functions converge to 0 as ε approaches infinity. Let $g_\ell = \max\{g(t) : |t| \leq \ell\}$, where $\ell \geq 0$, and define $\varphi_\lambda(t) = te^{\lambda t^2}$, with $\lambda = (\frac{g_\ell}{2\alpha})^2$. It can be readily verified that

$$\varphi'_\lambda(t) - \frac{g_\ell}{\alpha}|\varphi_\lambda(t)| \geq \frac{1}{2}, \ \forall t \in \mathbb{R}.$$

Let $t > \ell > 0$ and define $\mathcal{N} = 4\ell + s$. Introduce $\sigma_\varepsilon := w_\varepsilon - T_s(w_\varepsilon) + T_\ell(w_\varepsilon) - T_\ell(w)$ and $\varpi_\varepsilon := T_{2\ell}(\sigma_\varepsilon)$. On the set $\{|w_\varepsilon| \geq \mathcal{N}\}$, it is simple to check that the function ϖ_ε, proposed in [17], is constant and so $\partial_i \varpi_\varepsilon = 0$. Taking $\varphi = w_\varepsilon - \xi e^{G(|w_\varepsilon|)}\varphi_\lambda(\varpi_\varepsilon)$, we infer $\varphi \in W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$ and let ξ small enough such that $\varphi \geq \Delta$, then φ is an admissible test function in (3.2), and by (1.2) we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \kappa_i\Big(z, T_\varepsilon(w_\varepsilon), \nabla w_\varepsilon\Big)\partial_i\big(e^{G(|w_\varepsilon|)}\varphi_\lambda(\varpi_\varepsilon^+)\big)dz \\
&+ \int_\Omega \mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)e^{G(|w_\varepsilon|)}\varphi_\lambda(\varpi_\varepsilon^+)dz + \frac{1}{\varepsilon}\int_\Omega |w_\varepsilon|^{p_0(z)-2}w_\varepsilon e^{G(|w_\varepsilon|)}\varphi_\lambda(\varpi_\varepsilon^+)dz \\
&+ \int_\Omega |T_\varepsilon(w_\varepsilon)|^{r(z)-1}T_\varepsilon(w_\varepsilon)e^{G(|w_\varepsilon|)}\varphi_\lambda(\varpi +_\varepsilon)dz \\
&\leq \mu \int_\Omega \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-2}T_\varepsilon(w_\varepsilon)}{|z|^{p_0(z)} + \frac{1}{\varepsilon}}e^{G(|w_\varepsilon|)}\varphi_\lambda(\varpi_\varepsilon^+)dz + \int_\Omega f_\varepsilon e^{G(|w_\varepsilon|)}\varphi_\lambda(\varpi_\varepsilon^+)dz.
\end{aligned}$$

Given that w_ε and ϖ_ε exhibit identical signs within the set $\{|w_\varepsilon| > \ell\}$, we are able to write that

$$\begin{aligned}
\int_\Omega \mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)\varphi_\gamma(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dx = &\int_{\{|w_\varepsilon| \leq \ell\}} \mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)\varphi_\gamma(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz \\
&+ \int_{\{w_\varepsilon > \ell\}} \mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)\varphi_\gamma(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz.
\end{aligned}$$

Combined with (1.6) and Young's inequality this leads to

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{\varpi_\varepsilon \geq 0\}} \kappa_i(z, w_\varepsilon, \nabla w_\varepsilon)\partial_i \varpi_\varepsilon \varphi'_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz + \int_{\{|w_\varepsilon| \leq \ell\}} \mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz \\
&+ \int_{\{|w_\varepsilon| \leq \ell\}} |w_\varepsilon|^{r(z)-1}w_\varepsilon \varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz + \frac{1}{\varepsilon}\int_{\{|w_\varepsilon| \leq \ell\}} |w_\varepsilon|^{p_0(z)-1}w_\varepsilon \varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz \\
&\leq \mu \int_{\{|w_\varepsilon| \leq \ell\}} \frac{|w_\varepsilon|^{p_0(z)-1}}{|z|^{p_0(z)} + \frac{1}{\varepsilon}}\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz + C_3 \int_{\{w_\varepsilon > \ell\}} \frac{\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}}{|z|^{\frac{p_0(z)r(z)}{r(z)-p_0(z)+1}}}dz \\
&+ \int_\Omega (|f| + |b|)\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz.
\end{aligned} \tag{3.35}$$

where $C_3 = \zeta\lambda^\zeta\big(\frac{p_0(z)-1}{r(z)}\big)^\zeta$, such that $\zeta = \frac{r^- - p^+ + 1}{r^+}$. It is straightforward to verify that

$$\begin{aligned}
\sum_{i=1}^{N} \kappa_i(z, w_\varepsilon, \nabla w_\varepsilon)D^i \varpi_\varepsilon \geq &\sum_{i=1}^{N} \kappa_i(z, T_\ell \ \nabla T_\ell(w_\varepsilon))\partial_i(T_\ell(w_\varepsilon) - T_\ell(w)) \\
&- \Big|\sum_{i=1}^{N} \kappa_i(z, T_\mathcal{N}(w_\varepsilon), \nabla T_\mathcal{N}(w_\varepsilon))\Big||\partial_i T_\ell(w)|\chi_{\{|w_\varepsilon| > \ell\}},
\end{aligned}$$

which means that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{\varpi_\varepsilon\geq 0\}}\big(\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))-\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))\big)\big(\partial_i T_\ell(w_\varepsilon)-\partial_i T_\ell(w)\big)\varphi'_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\\
&\leq \sum_{i=1}^{N}\int_{\{|w_\varepsilon|>\ell\}}\big|\kappa_i(z,T_{\mathcal{N}}(w_\varepsilon),\nabla T_{\mathcal{N}}(w_\varepsilon))\big||\partial_i T_\ell(w)|\varphi'_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\\
&+\sum_{i=1}^{N}\int_{\{\varpi_\varepsilon\geq 0\}}\kappa_i(z,w_\varepsilon,\nabla w_\varepsilon)\partial_i\varpi_\varepsilon\varphi'_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\\
&-\sum_{i=1}^{N}\int_{\{\varpi_\varepsilon\geq 0\}}\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))\big(\partial_i T_\ell(w_\varepsilon)-\partial_i T_\ell(w)\big)\varphi'_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz.\end{aligned}$$

Then, we get

$$\begin{aligned}&\Big|\sum_{i=1}^{N}\int_{\{|w_\varepsilon|>\ell\}}|\kappa_i(z,T_{\mathcal{N}}(w_\varepsilon),\nabla T_{\mathcal{N}}(w_\varepsilon))||\partial_i T_\ell(w)|\varphi'_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\Big|\\
&\qquad\qquad\leq \varphi'_\lambda(2\ell)e^{G(\infty)}\sum_{i=1}^{N}\int_{\{|w_\varepsilon|>\ell\}}|\kappa_i(z,T_{\mathcal{N}}(w_\varepsilon),\nabla T_{\mathcal{N}}(w_\varepsilon))||\partial_i T_\ell(w)|dz.\end{aligned}$$

The integral on the right tends to zero as ε approaches infinity. This is guaranteed by (3.20), since it preserves the boundedness of the sequence $\{\kappa_i(z,T_X(w_\varepsilon),\nabla T_X(w_\varepsilon))\}_\varepsilon$ in the space $(L^{p'_i(z)}(\Omega,\varrho_i^*))^N$. Furthermore, the Lebesgue dominated convergence theorem gives, for all $i=1,\dots,N$, that

$$|\partial_i T_\ell(w)|\chi_{\{|w_\varepsilon|>\ell\}}\to 0 \text{ strongly in } L^{p_i(z)}(\Omega,\varrho_i) \ as\ \varepsilon\to\infty.$$

This implies that

$$\sum_{i=1}^{N}\int_{\{|w_\varepsilon|>\ell\}}|\kappa_i(z,T_{\mathcal{N}}(w_\varepsilon),\nabla T_{\mathcal{N}}(w_\varepsilon))||\partial_i T_\ell(w)|\varphi'_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz=n_1(\varepsilon).$$

By employing (3.20), (3.33), and (3.34), it is straightforward to verify that

$$\sum_{i=1}^{N}\int_{\{\varpi_\varepsilon\geq 0\}}\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))\big(\partial_i T_\ell(w_\varepsilon)-\partial_i T_\ell(w)\big)\varphi'_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz=n_2(\varepsilon).$$

As a result, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{\varpi_\varepsilon\geq 0\}}(\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))-\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w)))(\partial_i T_\ell(w_\varepsilon)-\partial_i T_\ell(w))\varphi'_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\\
&\qquad\qquad\leq\sum_{i=1}^{N}\int_{\{\varpi_\varepsilon\geq 0\}}\kappa_i(z,w_\varepsilon,\nabla w_\varepsilon)\partial_i\varpi_\varepsilon\varphi'_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz+n_3(\varepsilon).\qquad(3.36)\end{aligned}$$

In view of (1.2) and (1.6) we are able to express the following

$$
\begin{aligned}
&\Big|\int_{\{|w_\varepsilon|\le \ell\}} \mathcal{H}_\varepsilon(z,w_\varepsilon,\nabla w_\varepsilon)\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\Big| \\
&\le \int_{\{|w_\varepsilon|\le \ell\}} b\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz + \sum_{i=1}^N \int_{\{|w_\varepsilon|\le \ell\}} \varrho_i|\partial_i T_\ell(w_\varepsilon)|^{p_i(z)})\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz \\
&\le \int_{\{|w_\varepsilon|\le \ell\}} b\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz + \frac{g_\ell}{\alpha}\sum_{i=1}^N \int_\Omega \kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))\partial_i T_\ell(w_\varepsilon)\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz \\
&\le \int_{\{|w_\varepsilon|\le \ell\}} b\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz + \frac{g_\ell}{\alpha}\sum_{i=1}^N \int_\Omega \kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))\partial_i T_\ell(w)\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz \\
&+ \frac{g_\ell}{\alpha}\sum_{i=1}^N \int_\Omega (\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon)) - \kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))(\partial_i T_\ell(w_\varepsilon) - \partial_i T_\ell(w))\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz \\
&+ \frac{g_\ell}{\alpha}\sum_{i=1}^N \int_\Omega \kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))(\partial_i T_\ell(w_\varepsilon) - \partial_i T_\ell(w))\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz.
\end{aligned} \tag{3.37}
$$

According to the Lebesgue Dominated Convergence Theorem, we have

$$
\lim_{\varepsilon\to\infty} \int_{\{|w_\varepsilon|\le \ell\}} b\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz = \int_{\{|w|\le \ell\}} b\varphi_\lambda(T_{2\ell}(w-T_s(w))^+)e^{G(w)}dz = 0.
$$

Regarding the last term on the right-hand side of (3.37), analogous reasoning as previously discussed leads to

$$
\sum_{i=1}^N \int_\Omega \kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))(\partial_i T_\ell(w_\varepsilon) - \partial_i T_\ell(w))\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz \to 0,
$$

as $\varepsilon \to \infty$. Since $\{\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))\}_\varepsilon$ bounded in $(L^{p_i'(z)}(\Omega,\varrho_i^*))^N$, there is a vector function $\pi_{i,\ell} \in (L^{p_i'(z)}(\Omega,\varrho_i^*))^N$ such that

$$
\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon)) \rightharpoonup \pi_{i,\ell} \quad \text{in } (L^{p_i'(z)}(\Omega,\varrho^*))^N, \ \textit{as } \varepsilon\to\infty, \tag{3.38}
$$

and due to

$$
\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)} \rightharpoonup \varphi_\lambda(T_{2\ell}(w-T_s(w))^+)e^{G(w)} \quad \text{in } \ L^\infty(\Omega) \quad \text{for } \ \sigma^*(L^\infty,L^1),
$$

as $\varepsilon\to\infty$, the last term on the right-hand side of (3.37) can be simplified as follows

$$
\lim_{\varepsilon\to\infty}\sum_{i=1}^N \int_\Omega \kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))\partial_i T_\ell(w)\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz = \int_\Omega \pi_{i,\ell}\partial_i T_\ell(w)\varphi_\lambda(T_{2\ell}(w-T_s(w))^+)e^{G(w)}dz = 0.
$$

Hence, we can conclude that

$$
\begin{aligned}
&\Big|\int_{\{|w_\varepsilon|\le \ell\}} \mathcal{H}_\varepsilon(z,w_\varepsilon,\nabla w_\varepsilon)\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\Big| \\
&\le \frac{g_\ell}{\alpha}\sum_{i=1}^N \int_\Omega (\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon)) - \kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))(\partial_i T_\ell(w_\varepsilon) - \partial_i T_\ell(w))\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz + n_4(\varepsilon).
\end{aligned} \tag{3.39}
$$

Combining (3.35), (3.36), and (3.39), we can derive that

$$\sum_{i=1}^{N}\int_{\{\varpi_\varepsilon\geq 0\}}(\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))-\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))$$
$$\times(\partial_i T_\ell(w_\varepsilon)-\partial_i T_\ell(w))(\varphi'_\lambda(\varpi_\varepsilon^+)-\frac{g_\ell}{\alpha}\varphi_\lambda(\varpi_\varepsilon^+))e^{G(w_\varepsilon)}dz$$
$$\leq\sum_{i=1}^{N}\int_{\{\varpi_\varepsilon\geq 0\}}\kappa_i(z,w_\varepsilon,\nabla w_\varepsilon)\partial_i\varpi_\varepsilon\varphi'_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz+\int_{\{|w_\varepsilon|\leq\ell\}}\mathcal{H}_\varepsilon(z,w_\varepsilon,\nabla w_\varepsilon)\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz+n_5(\varepsilon)$$
$$\leq\Big|\int_{\{|w_\varepsilon|\leq\ell\}}|w_\varepsilon|^{r(z)-1}w_\varepsilon\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\Big|+\frac{1}{\varepsilon}\Big|\int_{\{|w_\varepsilon|\leq\ell\}}|w_\varepsilon|^{p_0(z)-1}w_\varepsilon\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\Big|$$
$$+\Big|\mu\int_{\{|w_\varepsilon|\leq\ell\}}\frac{|w_\varepsilon|^{p_0(z)-1}}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\Big|$$
$$+C_1\Big|\int_{\{w_\varepsilon>\ell\}}\frac{\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}}{|z|^{\frac{p_0(z)r(z)}{r(z)-p_0(z)+1}}}dz\Big|+\int_\Omega(|f|+|b|)\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz+n_5(\varepsilon).$$

It can be observed that

$$\Big|\int_{\{|w_\varepsilon|\leq\ell\}}|w_\varepsilon|^{\kappa(z)-1}w_\varepsilon\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\Big|$$
$$\leq\max\left(\ell^{\kappa^+},\ell^{\kappa^-}\right)\int_{\{|w_\varepsilon|\leq\ell\}}\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\qquad\text{with }\kappa(z)=r(z)\text{ or }p_0(z).$$

By applying the Lebesgue Dominated Convergence Theorem, we find that

$$\lim_{\varepsilon\to\infty}\int_{\{|w_\varepsilon|\leq\ell\}}\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz=\int_{\{|w|\leq\ell\}}\varphi_\lambda(T_{2\ell}(w-T_\ell(w))^+)e^{G(w)}dz=0,$$

which means that

$$\lim_{\varepsilon\to\infty}\int_{\{|w_\varepsilon|\leq\ell\}}|w_\varepsilon|^{r(z)-1}w_\varepsilon\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz=0,$$
$$\lim_{\varepsilon\to\infty}\frac{1}{\varepsilon}\int_{\{|w_\varepsilon|\leq\ell\}}|w_\varepsilon|^{p_0(z)-1}w_\varepsilon\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz=0.$$

Applying the Hölder inequality, we can deduce that

$$\Big|\mu\int_{\{|w_\varepsilon|\leq\ell\}}\frac{|w_\varepsilon|^{p_0(z)-1}}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\Big|$$
$$\leq\mu\max(\ell^{r^+},\ell^{r^-})e^{G(\infty)}\Big(\int_\Omega\frac{dz}{|z|^{\frac{p_0(z)r(z)}{r(z)-p_0(z)+1}}}\Big)^{\frac{r^--p_0^++1}{r^-}}\Big(\int_{\{|w_\varepsilon|\leq\ell\}}\varphi_\lambda\left(\varpi_\varepsilon^+\right)^{\frac{r(z)}{p_0(z)-1}}dz\Big)^{\frac{p_0^--1}{r^-}}$$

After the previous steps, we arrive at

$$\lim_{\varepsilon\to\infty}\Big|\mu\int_{\{|w_\varepsilon|\leq\ell\}}\frac{|w_\varepsilon|^{p_0(z)-1}}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz\Big|=0.$$

With the application of the Lebesgue dominated convergence theorem, we can conclude that

$$\lim_{\varepsilon\to\infty}\int_{\{w_\varepsilon>\ell\}}\frac{\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}}{|z|^{\frac{p_0(z)r(z)}{r(z)-p_0(z)+1}}}dz=\int_{\{w>\ell\}}\frac{\varphi_\lambda(T_{2\ell}(w-T_s(w))^+)e^{G(w)}}{|z|^{\frac{p_0(z)r(z)}{r(z)-p_0(z)+1}}}dz.$$

Regarding the last term, and considering that $\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)} \rightharpoonup \varphi_\lambda(T_{2\ell}(w-T_s(w))^+)e^{G(w)}$ in $L^\infty(\Omega)$ for $\sigma^*(L^\infty, L^1)$, we obtain

$$\lim_{\varepsilon\to\infty}\int_\Omega (|f|+|b|)\varphi_\lambda(\varpi_\varepsilon^+)e^{G(w_\varepsilon)}dz = \int_\Omega (|f|+|b|)\varphi_\lambda(T_{2\ell}(w-T_s(w))^+)e^{G(w)}dz.$$

With these considerations, we can rewrite (3) as

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^N\int_{\{\varpi_\varepsilon\geq 0\}}\big(\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))-\kappa_i(z,T_\ell(w_\varepsilon),\ \nabla T_\ell(w))\big)(\partial_i T_\ell(w_\varepsilon)-\partial_i T_\ell(w))e^{G(w_\varepsilon)}dz\\&\leq \int_{\{w>\ell\}}\frac{\varphi_\lambda(T_{2\ell}(w-T_s(w))^+)e^{G(w)}}{|z|^{\frac{p_0(z)r(z)}{r(z)-p_0(z)+1}}}dz+\int_\Omega(|f|+|b|)\varphi_\lambda(T_{2\ell}(w-T_s(w))^+)e^{G(w)}dz+n_6(\varepsilon).\end{aligned}$$

Taking the limit as s tends to infinity, considering (1.4), we arrive at

$$\lim_{\varepsilon\to\infty}\sum_{i=1}^N\int_{\{\varpi_\varepsilon\geq 0\}}(\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))-\Theta(z,T_\ell(w_\varepsilon),\nabla T_\ell(w)))(\partial_i T_\ell(w_\varepsilon)-\partial_i T_\ell(w))e^{G(w_{n\varepsilon})}dz\leq 0. \tag{3.40}$$

Next, by taking $\varphi = w_\varepsilon + \xi\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}$, we have $\varphi \in W^{1,\vec{p}(z)}(\Omega,\vec{\varrho})$, let ξ small enough such that $\varphi \geq \Delta$, then φ is an admissible test function in (3.2), and a similar approach, we obtain

$$\begin{aligned}&\sum_{i=1}^N\int_{\{\varpi_\varepsilon\leq 0\}}\kappa_i(z,w_\varepsilon,\nabla w_\varepsilon)\partial_i\varpi_\varepsilon\varphi'_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz-\int_{\{|w_\varepsilon|\leq\ell\}}\mathcal{H}_\varepsilon(z,w_\varepsilon,\nabla w_\varepsilon)\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz\\&-\int_{\{|w_\varepsilon|\leq\ell\}}|w_\varepsilon|^{r(z)-1}w_\varepsilon\varphi_\gamma(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz-\frac{1}{\varepsilon}\int_{\{|w_\varepsilon|\leq\ell\}}|w_\varepsilon|^{p_0(z)-1}w_\varepsilon\varphi_\gamma(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz\\&\leq\mu\int_{\{|w_\varepsilon|\leq\ell\}}\frac{|w_\varepsilon|^{p_0(z)-1}}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz+C_3\int_{\{w_\varepsilon>\ell\}}\frac{\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}}{|z|^{\frac{p_0(z)r(z)}{r(z)-p_0(z)+1}}}dz\\&+\int_\Omega(|f|+|b|)\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz.\end{aligned}\tag{3.41}$$

As in the process used to derive equation (3.36), we also establish that

$$\begin{aligned}&\sum_{i=1}^N\int_{\{\varpi_\varepsilon\leq 0\}}(\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))-\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))(\partial_i T_\ell(w_\varepsilon)-\partial_i T_\ell(w))\varphi'_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz\\&\qquad\leq\sum_{i=1}^N\int_{\{\varpi_\varepsilon\leq 0\}}\kappa_i(z,w_\varepsilon,\nabla w_\varepsilon)D^i\varpi_\varepsilon\varphi'_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz+n_7(\varepsilon).\end{aligned}\tag{3.42}$$

By estimating the term $\Big|\int_{\{|w_\varepsilon|\leq\ell\}}\mathcal{H}_\varepsilon(z,w_\varepsilon,\nabla w_\varepsilon)\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz\Big|$ in the manner demonstrated in equation (3.37), we arrive at

$$\begin{aligned}&\Big|\int_{\{|w_\varepsilon|\leq\ell\}}\mathcal{H}_\varepsilon(z,w_\varepsilon,\nabla w_\varepsilon)\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz\Big|\\&\leq\frac{g_\ell}{\alpha}\sum_{i=1}^N\int_\Omega\big(\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))-\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))\big)(\partial_i T_\ell(w_\varepsilon)-\partial_i T_\ell(w))\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz+n_8(\varepsilon).\end{aligned}\tag{3.43}$$

According to (3.41), (3.42) and (3.43) we get

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{\varpi_\varepsilon\geq 0\}}\big(\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))-\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))\big)\\
&\times\big(\partial_i T_k(w_\varepsilon)-\partial_i T_\ell(w)\big)(\varphi_\lambda'(\varpi_\varepsilon^+)-\frac{g_{ell}}{\alpha}\varphi_\lambda(\varpi_\varepsilon^+))e^{-G(w_\varepsilon)}dz\\
&\leq\sum_{i=1}^{N}\int_{\{\varpi_\varepsilon\leq 0\}}\kappa_i(z,w_\varepsilon,\nabla w_\varepsilon)\partial_i\varpi_\varepsilon\varphi_\lambda'(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz-\int_{\{|w_\varepsilon|\leq\ell\}}\mathcal{H}_\varepsilon(z,w_\varepsilon,\nabla w_\varepsilon)\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz+n_9(\varepsilon)\\
&\leq\Big|\int_{\{|w_\varepsilon|\leq\ell\}}|w_\varepsilon|^{r(z)-1}w_\varepsilon\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz\Big|+\frac{1}{\varepsilon}\Big|\int_{\{|w_\varepsilon|\leq\ell\}}|w_\varepsilon|^{p_0(z)-1}w_\varepsilon\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz\Big| \qquad (3.44)\\
&+\Big|\mu\int_{\{|w_\varepsilon|\leq\ell\}}\frac{|w_\varepsilon|^{p_0(z)-1}}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz\Big|+C_3\Big|\int_{\{w_\varepsilon<-\ell\}}\frac{\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}}{|z|^{\frac{p_0(z)r(z)}{r(z)-p_0(z)+1}}}dz\Big|\\
&+\int_\Omega(|f|+|b|)\varphi_\lambda(\varpi_\varepsilon^-)e^{-G(w_\varepsilon)}dz+n_9(\varepsilon).
\end{aligned}$$

As above, going to the limit as ε and then as s tends to 0 on both sides of (3.44), we get

$$\lim_{\varepsilon\to\infty}\sum_{i=1}^{N}\int_{\{\varpi_\varepsilon\leq 0\}}(\kappa_i(z,T_\ell,\nabla T_\ell(w_\varepsilon))-\kappa_i(z,T_\ell,\nabla T_\ell(w))(\partial_i T_\ell(w_\varepsilon)-\partial_i T_\ell(w))e^{-G(w_\varepsilon)}dz\leq 0. \qquad (3.45)$$

Continuing with the analysis, we sum up the two inequalities (3.40) and (3.45), leading us to

$$\lim_{\varepsilon\to\infty}\sum_{i=1}^{N}\int_\Omega\big(\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w_\varepsilon))-\kappa_i(z,T_\ell(w_\varepsilon),\nabla T_\ell(w))\big)(\partial_i T_\ell(w_\varepsilon)-\partial_i T_\ell(w))dz=0. \qquad (3.46)$$

Hence, by applying Lemma 5 from [6], we arrive at

$$T_\ell(w_\varepsilon)\to T_\ell(w)\text{ strongly in }\quad W^{1,\vec{p}(z)}(\Omega,\vec{\varrho}). \qquad (3.47)$$

Then we can infer, up to a subsequence still indexed by ε, that

$$\partial_i w_\varepsilon\to\partial_i w\quad\text{a.e. in }\Omega\quad\text{for all }i=1,\dots,N. \qquad (3.48)$$

Step 4: The equi-integrability of the nonlinear terms In this part, we will prove that

$$\mathcal{H}_\varepsilon(z,w_\varepsilon,\nabla w_\varepsilon)\to\mathcal{H}(z,w,\nabla w)\text{ strongly in }L^1(\Omega), \qquad (3.49)$$

$$|T_\varepsilon(w_\varepsilon)|^{r(z)-1}T_\varepsilon(w_\varepsilon)\to|w|^{r(z)-1}w\text{ strongly in }L^1(\Omega), \qquad (3.50)$$

$$\frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-2}T_\varepsilon(w_\varepsilon)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}\to\frac{|w|^{p_0(\varepsilon)-2}w}{|z|^{p_0(z)}}\text{ strongly in }L^1(\Omega), \qquad (3.51)$$

and

$$\frac{1}{\varepsilon}|w_\varepsilon|^{p_0(z)-2}w_\varepsilon\to 0\text{ strongly in }L^1(\Omega). \qquad (3.52)$$

By combining (3.47) and (3.48), we obtain

$$\mathcal{H}_\varepsilon(x,w_\varepsilon,\nabla w_\varepsilon)\to\mathcal{H}(z,w,\nabla w)\quad\text{a.e. in }\Omega, \qquad (3.53)$$

$$|T_\varepsilon(w_\varepsilon)|^{r(z)-1}T_\varepsilon(w_\varepsilon)\to|w|^{r(z)-1}w\quad\text{a.e. in }\Omega, \qquad (3.54)$$

$$\frac{|T_\varepsilon(w_\varepsilon)|^{p_0(w)-2}T_\varepsilon(w_\varepsilon)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}\to\frac{|w|^{p_0(z)-2}w}{|z|^{p_0(z)}}\quad\text{a.e. in }\Omega, \qquad (3.55)$$

and

$$\frac{1}{\varepsilon}|w_\varepsilon|^{p_0(z)-2}w_\varepsilon \to 0 \quad \text{a.e. in } \Omega. \tag{3.56}$$

To establish the uniform equi-integrability of these functions. Taking $\varphi = w_\varepsilon - \xi T_1(w_\varepsilon - T_\ell(w))$, we infer $\varphi \in W^{1,\vec{p}(z)}(\Omega, \vec{\varrho})$ and let ξ small enough such that $\varphi \geq \Delta$, then φ is an admissible test function in (3.2), and by (1.2) we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{\ell<|w_\varepsilon|\leq \ell+1\}} |\partial_i w_\varepsilon|^{p_i(z)} \varrho_i dz + \int_{\{|w_\varepsilon|\geq \ell\}} \mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon) T_1(w_\varepsilon - T_\ell(w_\varepsilon)) dz \\
&+ \int_{\{|w_\varepsilon|\geq \ell\}} |T_\varepsilon(w_\varepsilon)|^{r(z)} |T_1(w_\varepsilon - T_\ell(w_\varepsilon))| dz + \frac{1}{\varepsilon} \int_{\{|u_n|\geq \ell+1\}} |w_\varepsilon|^{p_0(z)-1} dz \\
&\leq \int_{\{|w_\varepsilon|\geq \ell\}} |f_\varepsilon| dz + \mu \int_{\{|w_\varepsilon|\geq \ell\}} \frac{|T_1(w_\varepsilon|^{p_0(z)-1}}{|z|^{p_0(z)} + \frac{1}{\varepsilon}} |T_1(w_\varepsilon - T_\ell(w)| dz.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\{|w_\varepsilon|\geq \ell\}} \mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon) T_1(w_\varepsilon - T_\ell(w_\varepsilon)) dz \\
&\qquad \geq \int_{\{|w_\varepsilon|\geq \ell+1\}} \mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon) T_1(w_\varepsilon - T_\ell(w_\varepsilon)) dz = \int_{\{|w_\varepsilon|\geq \ell+1\}} |\mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)| dz.
\end{aligned}$$

Given Young's inequality, we obtain

$$\begin{aligned}
\mu \int_{\{|w_\varepsilon|\geq \ell\}} \frac{|T_1(w_\varepsilon|^{p_0(z)-1}}{|z|^{p_0(z)} + \frac{1}{\varepsilon}} |T_1(w_\varepsilon - T_\ell(w)| dz &\leq \frac{1}{3} \int_{\{|w_\varepsilon|\geq \ell\}} |T_\varepsilon(w_\varepsilon)|^{r(x)} |T_1(w_\varepsilon - T_\ell(w_\varepsilon))| dz \\
&\quad + C_3 \int_{\{|w_\varepsilon|\geq \ell\}} \frac{|T_1(w_\varepsilon - T_\varepsilon(w_\varepsilon))|}{|z|^{\frac{r(z)p_0(z)}{r(z)-p_0(z)+1}}} dz,
\end{aligned}$$

As a result

$$\begin{aligned}
&\int_{\{|w_\varepsilon|\geq \ell+1\}} |\mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)| dz + \frac{1}{3} \int_{\{|w_\varepsilon|\geq \ell+1\}} |T_\varepsilon(w_\varepsilon)|^{r(z)} dz \\
&\qquad + \mu \int_{\{|w_\varepsilon|\geq \ell+1\}} \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-1}}{|z|^{p_0(z)} + \frac{1}{\varepsilon}} dz + \frac{1}{\varepsilon} \int_{\{|u_n|\geq \ell+1\}} |w_\varepsilon|^{p_0(z)-1} dz \\
&\qquad\qquad \leq 2C_3 \int_{\{|w_\varepsilon|\geq \ell\}} \frac{|T_1(w_\varepsilon - T_\ell(w_\varepsilon))|}{|z|^{\frac{r(z)p_0(z)}{r(z)-p_0(z)+1}}} dz + \int_{\{|w_\varepsilon|\geq \ell\}} |f_\varepsilon| dz.
\end{aligned}$$

Therefore, for each $\delta > 0$, there exists $\ell(\delta) > 0$ such that

$$\begin{aligned}
&\int_{\{|w_\varepsilon|\geq \ell(\delta)\}} |\mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)| dz + \int_{\{|w_\varepsilon|\geq \ell(\delta)\}} |T_\varepsilon(w_\varepsilon)|^{r(z)} dz \\
&\qquad + \int_{\{|w_\varepsilon|\geq \ell(\delta)\}} \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-1}}{|z|^{p_0(z)} + \frac{1}{\varepsilon}} dz + \frac{1}{\varepsilon} \int_{\{|u_n|\geq \ell(\delta)\}} |w_\varepsilon|^{p_0(z)-1} dz \leq \frac{\delta}{2}.
\end{aligned} \tag{3.57}$$

Conversely, for every measurable subset $\mathcal{F} \subseteq \Omega$, we have

$$\begin{aligned}
&\int_{\mathcal{F}} |\mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)| dz + \int_{\mathcal{F}} |T_\varepsilon(w_\varepsilon)|^{r(z)} dz + \int_{\mathcal{F}} \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-1}}{|z|^{p_0(z)} + \frac{1}{\varepsilon}} dz + \frac{1}{\varepsilon} \int_{\mathcal{F}} |w_\varepsilon|^{p_0(z)-1} dz \\
&\leq \int_{\mathcal{F}\cap\{|w_\varepsilon|<\ell(\delta)\}} |\mathcal{H}_\varepsilon(z, T_{\ell(\delta)}(w_\varepsilon), \nabla T_{\ell(\delta)}(w_\varepsilon)| dz + \int_{\mathcal{F}\cap\{|w_\varepsilon|<\ell(\delta)\}} |T_{\ell(\delta)}(w_\varepsilon)|^{r(z)} dz \qquad (3.58)\\
&+ \frac{1}{\varepsilon} \int_{\mathcal{F}\cap\{|w_\varepsilon|<\ell(\delta)\}} |w_\varepsilon|^{p_0(z)-1} dz + \int_{\mathcal{F}\cap\{|w_\varepsilon|<\ell(\delta)\}} \frac{|T_{\ell(\delta)}(w_\varepsilon)|^{p_0(z)-1}}{|z|^{p_0(z)} + \frac{1}{\varepsilon}} dz + \int_{\{|w_\varepsilon|\geq\ell(\delta)\}} |\mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)| dz \\
&+ \int_{\{|w_\varepsilon|\geq\ell(\delta)\}} |T_\varepsilon(w_\varepsilon)|^{r(z)} dz + \int_{\{|w_\varepsilon|\geq\ell(\delta)\}} \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-1}}{|z|^{p_0(z)} + \frac{1}{\varepsilon}} dz + \frac{1}{\varepsilon} \int_{\{|w_\varepsilon|\geq\ell(\delta)\}} |w_\varepsilon|^{p_0(z)-1} dz
\end{aligned}$$

In the sequel, by (1.6) we have

$$\int_{\mathcal{F}\cap\{|w_\varepsilon|<\ell(\delta)\}} |\mathcal{H}_\varepsilon(z, T_{\ell(\delta)}(w_\varepsilon), \nabla T_{\ell(\delta)}(w_\varepsilon)| dz \leq \int_{\mathcal{F}\cap\{|w_\varepsilon|<\ell(\delta)\}} \Big(b(z) + g(|\ell(\delta)|) \sum_{i=1}^{N} \varrho_i |\partial_i T_{\ell(\delta)}(w_\varepsilon)|^{p_i(z)}\Big) dz.$$

Hence, from (3.47) and (3.48), there exists $\gamma(\delta) > 0$ such that : for each $\mathcal{F} \subseteq \Omega$ with $meas(\mathcal{F}) \leq \gamma(\delta)$

$$\begin{aligned}
&\int_{\mathcal{F}\cap\{|w_\varepsilon|<\ell(\delta)\}} |\mathcal{H}_\varepsilon(z, T_{\ell(\delta)}(w_\varepsilon), \nabla T_{\ell(\delta)}(w_\varepsilon)| dz + \int_{\mathcal{F}\cap\{|w_\varepsilon|<\ell(\delta)\}} |T_{\ell(\delta)}(w_\varepsilon)|^{r(z)} dz \\
&\qquad + \int_{\mathcal{F}\cap\{|w_\varepsilon|<\ell(\delta)\}} \frac{|T_{\ell(\delta)}(w_\varepsilon)|^{p_0(z)-1}}{|z|^{p_0(z)} + \frac{1}{\varepsilon}} dz + \frac{1}{\varepsilon} \int_{\mathcal{F}\cap\{|w_\varepsilon|<\ell(\delta)\}} |w_\varepsilon|^{p_0(z)-1} dz \leq \frac{\delta}{2}. \qquad (3.59)
\end{aligned}$$

Finally, according to (3.57)-(3.59), we infer

$$\begin{aligned}
&\int_{\mathcal{F}} |\mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)| dz + \int_{\mathcal{F}} |T_\varepsilon(w_\varepsilon)|^{r(z)} dz + \int_{\mathcal{F}} \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-1}}{|z|^{p_0(z)} + \frac{1}{\varepsilon}} dz \\
&\qquad + \frac{1}{\varepsilon} \int_{\mathcal{F}} |w_\varepsilon|^{p_0(z)-1} dz \leq \delta, \qquad \text{with } meas(\mathcal{F}) \leq \beta(\delta).
\end{aligned}$$

It follows that $\big(\mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon)\big)_\varepsilon$, $\big(|T_\varepsilon(w_\varepsilon)|^{r(z)-1} T_\varepsilon(w_\varepsilon)\big)_\varepsilon$, $\big(|w_\varepsilon|^{p_0(z)-1} w_\varepsilon\big)_\varepsilon$ and $\Big(\frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-2} T_\varepsilon(w_\varepsilon)}{|z|^{p_0(z)} + \frac{1}{\varepsilon}}\Big)_\varepsilon$ are equi-integrable. In view of (3.53)-(3.59) and Vitali's theorem, the convergences (3.49)-(3.56) are established

Step 6: Passage to the limit Let $\psi \in \mathcal{D}_\Delta \cap L^\infty(\Omega)$ and $\mathcal{K} = \ell + \|\psi\|_\infty$, with $\ell > 0$. By taking $\varphi = w_\varepsilon - \xi T_\ell(w_\varepsilon - \psi)$ as a test function in (3.2), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \kappa_i(z, T_\varepsilon(w_\varepsilon), \nabla w_\varepsilon) \partial_i T_\ell(w_\varepsilon - \psi) dz + \int_\Omega \mathcal{H}_\varepsilon(z, w_\varepsilon, \nabla w_\varepsilon) T_\ell(w - \psi) dz + \frac{1}{\varepsilon} \int_\Omega |w_\varepsilon|^{p_0(z)-2} w_\varepsilon T_\ell(w_\varepsilon - \psi) dz \\
&+ \int_\Omega |T_\varepsilon(w_\varepsilon)|^{r(z)-1} T_\varepsilon(w_\varepsilon) T_\ell(w_\varepsilon - \psi) dz = \mu \int_\Omega \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-2} T_\varepsilon(w_\varepsilon)}{|z|^{p_0(z)} + \frac{1}{\varepsilon}} T_\ell(w_\varepsilon - \psi) dz + \int_\Omega f_\varepsilon T_\ell(w_\varepsilon - \psi) dz.
\end{aligned}$$

One the one side, when $|w_\varepsilon| > \mathcal{K}$ we infer $|w_\varepsilon - \psi| \geq |w_\varepsilon| - \|\psi\|_\infty > \ell$, then $\{|w_\varepsilon - \psi| \leq \ell\} \subseteq \{|w_\varepsilon| \leq \mathcal{K}\}$, this means that

$$\begin{aligned}
&\int_\Omega \kappa_i(z, T_\varepsilon(w_\varepsilon), \nabla w_\varepsilon) \partial_i T_\ell(w_\varepsilon - \psi) dz = \int_\Omega \kappa_i(z, T_\mathcal{K}(w_\varepsilon), \nabla T_\mathcal{K}(w_\varepsilon))(\partial_i T_\mathcal{K}(w_\varepsilon) - \partial_i \psi) \chi_{\{|w_\varepsilon - \psi| \leq \ell\}} dz \\
&\quad = \int_\Omega (\kappa_i(z, T_\mathcal{K}(w_\varepsilon), \nabla T_\mathcal{K}(w_\varepsilon)) - \kappa_i(z, T_\mathcal{K}(w_\varepsilon), \nabla \psi))(\partial_i T_\mathcal{K}(w_\varepsilon) - \partial_i \psi) \chi_{\{|w_\varepsilon - \psi| \leq \ell\}} dz \\
&\qquad + \int_\Omega \kappa_i(z, T_\mathcal{K}(w_\varepsilon), \nabla \psi)(\partial_i T_\mathcal{K}(w_\varepsilon) - \partial_i \psi) \chi_{\{|w_\varepsilon - \psi| \leq \ell\}} dz.
\end{aligned}$$

It's obvious that

$$\lim_{\varepsilon\to\infty}\int_\Omega \kappa_i(z,T_{\mathcal{K}}(w_\varepsilon),\nabla\psi)(\partial_i T_\varepsilon(w_\varepsilon)-\partial_i\psi)\chi_{\{|w_\varepsilon-\psi|\le\ell\}}dz=\int_\Omega \kappa_i(z,T_{\mathcal{K}}(w),\nabla\psi)(\partial_i T_{\mathcal{K}}(w)-\partial_i\psi)\chi_{\{|w-\psi|\le\ell\}}dz.$$

In view of Fatou's Lemma, we get

$$\begin{aligned}
&\liminf_{\varepsilon\to\infty}\sum_{i=1}^N\int_\Omega \kappa_i(z,T_\varepsilon(w_\varepsilon),\nabla w_\varepsilon)\partial_i T_\ell(w_\varepsilon-\psi)dz\ge\sum_{i=1}^N\int_\Omega(\kappa_i(z,T_{\mathcal{K}}(w),\nabla T_{\mathcal{K}}(w))-\kappa_i(z,T_{\mathcal{K}}(w),\nabla\psi))\\
&\times(\partial_i T_{\mathcal{K}}(w)-\partial_i\psi)\chi_{\{|w-\psi|\le\ell\}}dz+\sum_{i=1}^N\int_\Omega \kappa_i(z,T_{\mathcal{K}}(w),\nabla\psi)(\partial_i T_{\mathcal{K}}(w)-\partial_i\psi)\chi_{\{|w-\psi|\le\ell\}}dz\\
&=\sum_{i=1}^N\int_\Omega \kappa_i(z,T_{\mathcal{K}}(w),\nabla T_{\mathcal{K}}(w))(\partial_i T_{\mathcal{K}}(w)-\partial_i\psi)\chi_{\{|w-\psi|\le\ell\}}dz\\
&=\sum_{i=1}^N\int_\Omega \kappa_i(z,w,\nabla w)\partial_i T_\ell(w-\psi)dz.
\end{aligned}$$

Conversely, we can observe that $T_\ell(w_\varepsilon-\psi)\rightharpoonup T_\ell(w-\psi)$ weak-* in $L^\infty(\Omega)$ and by using (3.49)-(3.52), it follows that

$$\begin{aligned}
&\int_\Omega \mathcal{H}_\varepsilon(z,w_\varepsilon,\nabla w_\varepsilon)T_\ell(w-\psi)dz\to\int_\Omega \mathcal{H}(z,w,\nabla w)T_\ell(w-\psi)dz,\\
&\int_\Omega |T_\varepsilon(w_\varepsilon)|^{r(z)-1}T_\varepsilon(w_\varepsilon)T_\ell(w_\varepsilon-\psi)dz\to\int_\Omega |w|^{r(z)-1}wT_\ell(w-\psi)dz,\\
&\int_\Omega \frac{|T_\varepsilon(w_\varepsilon)|^{p_0(z)-2}T_\varepsilon(w_\varepsilon)}{|z|^{p_0(z)}+\frac{1}{\varepsilon}}T_\ell(w_\varepsilon-\psi)dz\to\int_\Omega \frac{|w|^{p_0(z)-2}w}{|z|p_0(z)}T_\ell(w-\psi)dz,\\
&\frac{1}{\varepsilon}\int_\Omega |w_\varepsilon|^{p_0(z)-2}w_\varepsilon T_\ell(w_\varepsilon-\psi)dz\to 0,\ \text{and}\ \int_\Omega f_\varepsilon T_\ell(w_\varepsilon-\psi)dz\to\int_\Omega fT_\ell(w-\psi)dz.
\end{aligned}$$

Finally, combining all these components, we have now successfully concluded the proof of Theorem 3.1.

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