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Hermite-Hadamard-Norm Type Inequalities for Fractional Integrals

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ABSTRACT: Hermite-Hadamard norm inequality plays a crucial role in solving problems involving inequalities in different branches of science and engineering. After the introduction of fractional calculus, the scope for solving problems have been expanded. It has maintained the greater accuracy. It has been applied for solving different unsolved problems. Motivated by these concepts, we in this paper by using the Jensen-Norm inequality, we proved Hermite-Hadamard's Inequalities.

Key Words: Hermite-Hadamard inequality, norm inequality, Jensen-Norm inequality, fractional integration, gamma function, convex function, Reimann-Liouville integral.

Contents

1. Introduction

The idea of convexity is thought to be improved upon by the Hermite-Hadamard inequality. This idea has been thoroughly studied by many researchers since it was separately discovered by Hermite 1883 and Hadamard in 1896. Specifically, much work has been done over the last 20 years to establish new boundaries for the Hermite-Hadamard inequality's left and right sides. Numerous research works have suggested innovative methods to strengthen, expand, and enhance this disparity. One may refer to the articles [1], [3], [5], [6], [7], [8], [9], [10], and [11] to have the knowledge, uses and applications of the norm-type inequalities. For matric inequalities one may refer to [2] and [4].

Convex functions can be used to derive some inequalities, according to numerous researchers. Among the most well-known inequalities relating to a convex function's integral mean. Hermite-Hadamard inequality, the statement of this inequality is found in Dragomir and Pearce [3].

Hermite-Hadamard norm inequality plays a crucial role in solving problems involving inequalities in different branches of science and engineering. After the introduction of fractional calculus, the scope for solving problems have been expanded. It has maintained the greater accuracy. It has been applied for solving different unsolved problems. Motivated by these concepts, we in this paper by using the Jensen-Norm inequality, we have proved Hermite-Hadamard's Inequalities for fractional integrals in this article.

Let $0 \leq y_1 \leq y_2 \leq \cdots \leq y_n$ and let $\Omega = (\Omega_1, \Omega_2, \ldots \Omega_n)$ non-negative weights such that $\sum_{j=1}^{n} \Omega_j = 1$. The well-known Jensen inequality in literature states that "If g is a convex function on an interval containing y_n then,

$$\left\| g\left(\sum_{j=1}^{n} \Omega_j y_j\right) \right\| \leq \sum_{j=1}^{n} |\Omega_j| \left\| g\left(y_j\right) \right\|.$$

The inequalities discovered by Hermite-Hadamard for convex functions state that "If $g : [\hat{c}, \hat{d}] \sqsubseteq R \rightarrow R$ be an integrable normed linear space convex function. Then

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$$\left\|g\left(\frac{\hat{c}+\hat{d}}{2}\right)\right\| \leq \frac{1}{|\hat{d}-\hat{c}|}\int_{\hat{c}}^{\hat{d}}\|g(w)\|dw \leq \frac{\|g(\hat{c})\|+\|g(\hat{d})\|}{2}." \tag{1}$$

Both the inequalities hold in the reversed direction if g is concave [7].
It is proved that the norm variant of Jensen inequality known as Jensen-Norm Inequality.

Theorem 1.1 *If g is a convex function on $[\hat{c},\hat{d}]$, then*

$$\left\|g\left(\hat{c}+\hat{d}-\sum_{j=1}^{n}\Omega_j y_j\right)\right\| \leq \|g(\hat{c})\|+\|g(\hat{d})\|-\sum_{j=1}^{n}\mid \Omega_j \left\|\boldsymbol{g}\left(\boldsymbol{y}_j\right)\right\| \tag{2}$$

Definition 1.1 Let the function $g \in L(\hat{c},\hat{d})$ then Reimann-Liouville integrals $\mathrm{I}_{\hat{c}+}^{\alpha} g$ and $\mathrm{I}_{\hat{d}-}^{\alpha} g$ with condition $\alpha > 0$ and $\hat{c} \geq 0$ are defined as

$$\gamma(\alpha)I_{\hat{c}+}^{\alpha}g((\hat{x})) = \int_{\hat{c}}^{\hat{x}}(\hat{x}-\mathrm{w})^{(\alpha-1)}g(\mathrm{w})\mathrm{dw}, \hat{x}>\hat{c}$$

$$\gamma(\alpha)I_{\hat{d}-}^{\alpha}g((\hat{x})) = \int_{\hat{x}}^{\hat{d}}(\mathrm{w}-\hat{x})^{\alpha-1}g(\mathrm{w})\mathrm{dw}, \hat{x}<\hat{d}.$$

Here, $I_{\hat{c}+}^{0}g(\hat{x}) = I_{\hat{d}-}^{0}g(\hat{x}) = g(\hat{x})$. and $\gamma(\alpha) =$ Gamma function.

2. Hermite-Hadamard-Norm's Inequalities for Fractional Integrals

By using the Jensen-Norm inequality, Hermite-Hadamard's inequalities can be represented in fractional integral forms as follows.

Theorem 2.1 *Suppose that* $\mathrm{g}:[\hat{c},\hat{d}] \rightarrow R$ *is a convex function. Then we have*

$$\begin{aligned}\left\|g\left(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}\right)\right\| &\leq \|g(\widehat{c})\|+\|g(\hat{d})\|-\frac{\gamma(\alpha+1)}{2(\hat{y}-\hat{x})^{\alpha}}\left[I_{\hat{x}+}^{\alpha}g(\hat{y})+I_{\hat{y}-}^{\alpha}g(\hat{x})\right]\\ &\leq \|g(\widehat{c})\|+\|g(\hat{d})\|-\left\|g\left(\frac{\hat{x}+\hat{y}}{2}\right)\right\|\end{aligned} \tag{3}$$

and

$$\begin{aligned}\left\|g\left(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}\right)\right\| &\leq \frac{\gamma(\alpha+1)}{2(\hat{y}-\hat{x})^{\alpha}}\left[I_{(\hat{c}+\hat{d}-\hat{y})^{+}}^{\alpha}+g(\hat{c}+\hat{d}-\hat{x})+I_{(\hat{c}+\hat{d}-\hat{x})^{-}}^{\alpha}g(\hat{c}+\hat{d}-\hat{y})\right]\\ &\leq \frac{g(\hat{c}+\hat{d}-\hat{x})+g(\hat{c}+\hat{d}-\hat{y})}{2}\\ &\leq \|g(\widehat{c})\|+\|g(\hat{d})\|-\frac{\|g(\hat{x})\|+\|g(\hat{y})\|}{2}\end{aligned} \tag{4}$$

For all $\hat{x},\hat{y} \in [\widehat{c},\hat{d}]$ and $\alpha > 0$.

Proof: Using the Jensen-Norm type inequality, we have

$$\left\|g\left(\hat{c}+\hat{d}-\frac{\widehat{x_1}+\widehat{y_1}}{2}\right)\right\| \leq \|g(\widehat{c})\|+\|g(\hat{d})\|-\frac{\|g\left(\widehat{x_1}\right)\|+\|g\left(\widehat{y_1}\right)\|}{2} \tag{5}$$

For all $\widehat{x_1}, \hat{y_1} \in [\ \hat{c},\ \ \hat{d}]$. By changing of the variables

$$\widehat{x_1} = w\hat{x} + (1-w)\hat{y}, \widehat{y_1} = (1-w)\hat{x} + w\hat{y}, \text{ for } \widehat{x}, \hat{y} \in [\widehat{c}, \hat{d}] \text{ and } w \in [0,1].$$

We obtain

$$\left\| g\left(\hat{c} + \hat{d} - \frac{\hat{x}+\hat{y}}{2}\right)\right\| \leq \|g(\widehat{c})\| + \|g(\hat{d})\| - \frac{\|g(w\hat{x} + (1-w)\hat{y})\| + \|g((1-w)\hat{x} + w\hat{y})\|}{2}. \tag{6}$$

Multiplying $w^{\alpha-1}$ both sides of the Eq. (6), then integrating the resulting inequality with respect to w over $[0,1]$ we obtain

$$\begin{aligned}
\frac{1}{\alpha}\left\| g\left(\hat{c} + \hat{d} - \frac{\hat{x}+\hat{y}}{2}\right)\right\| &\leq \frac{1}{\alpha}[\|g(\widehat{c})\| + \|g(\hat{d})\|] - \frac{1}{2}\int_0^1 \mathrm{w}^{\alpha-1}[\|g(w\hat{x} + (1-w)\hat{y})\| \\
&+ \|g((1-w)\hat{x} + w\hat{y})\|]\mathrm{d}w \\
&= \frac{1}{\alpha}[\|g(\widehat{c})\| + \|g(\hat{d})\|] - \frac{1}{2(\hat{y}-\hat{x})^{\alpha}}\left[\int_x^y (y-u)^{\alpha-1}g(u)du + \int_x^y (u-x)^{\alpha-1}g(u)du\right] \\
&= \frac{1}{\alpha}[\|g(\widehat{c})\| + \|g(\hat{d})\|] - \frac{\gamma(\alpha)}{2(\hat{y}-\hat{x})^{\alpha}}\left[I^{\alpha}_{\hat{x}+}g(\hat{y}) + I^{\alpha}_{\hat{y}-}g(\hat{x})\right].
\end{aligned}$$

That is

$$\left\| g\left(\hat{c} + \hat{d} - \frac{\hat{x}+\hat{y}}{2}\right)\right\| \leq \|g(\widehat{c})\| + \|g(\hat{d})\| - \frac{\gamma(\alpha+1)}{2(\hat{y}-\hat{x})^{\alpha}}\left[I^{\alpha}_{\hat{x}+}g(\hat{y}) + I^{\alpha}_{\hat{y}} - g(\hat{x})\right]. \tag{7}$$

and so the first inequality of Eq. (3) proved. For the proof of the second inequality in Eq. (3), we first note that if g is a convex function, then for $w \in [0,1]$, it yields

$$\begin{aligned}
\left\| g\left(\frac{\hat{x}+\hat{y}}{2}\right)\right\| &= \left\| g\left(\frac{w\hat{x} + (1-w)\hat{y} + (1-w)\hat{x} + w\hat{y}}{2}\right)\right\| \\
&\leq \frac{\|g(w\hat{x} + (1-w)\hat{y})\| + \|g((1-w)\hat{x} + w\hat{y})\|}{2}.
\end{aligned} \tag{8}$$

On multiplying by $w^{\alpha-1}$ to both sides of the Eq.(8), then integrating the resulting inequality with respect to w over [0,1]. we obtain

$$\begin{aligned}
\frac{1}{\alpha}\left\| g\left(\frac{\hat{x}+\hat{y}}{2}\right)\right\| &\leq \frac{1}{2}\int_0^1 \mathrm{w}^{\alpha-1}[\|g(w\hat{x} + (1-w)\hat{y})\| + \|g((1-w)\hat{x} + w\hat{y})\|]\mathrm{d}w \\
&= \frac{\gamma(\alpha)}{2(\hat{y}-\hat{x})^{\alpha}}\left[I^{\alpha}_{\hat{x}+}g(\hat{y}) + I^{\alpha}_{\hat{y}-}g(\hat{x})\right].
\end{aligned}$$

Then we have,

$$-\left\| g\left(\frac{\hat{x}+\hat{y}}{2}\right)\right\| \geq -\frac{\gamma(\alpha+1)}{2(\hat{y}-\hat{x})^{\alpha}}\left[I^{\alpha}_{\hat{x}+}g(\hat{y}) + I^{\alpha}_{\hat{y}} - g(\hat{x})\right]. \tag{9}$$

Adding $\|g(\widehat{c})\| + \|g(\hat{d})\|$ to both sides of Eq.(9), we find the second inequality of Eq. (3). Now we prove the inequality Eq (4). From the convexity of g we have

$$\begin{aligned}
\left\| g\left(\hat{c} + \hat{d} - \frac{\widehat{x_1}+\widehat{y_1}}{2}\right)\right\| &= \left\| g\left(\frac{\hat{c} + \hat{d} - \widehat{x_1} + \hat{c} + \hat{d} - \widehat{y_1}}{2}\right)\right\| \\
&\leq \frac{1}{2}\left[\left\| g\left(\hat{c} + \hat{d} - \widehat{x_1}\right)\right\| + \left\| g\left(\hat{c} + \hat{d} - \widehat{y_1}\right)\right\|\right].
\end{aligned} \tag{10}$$

For all $\widehat{x_1}, \widehat{y_1} \in [\widehat{c}, \hat{d}]$. By changing of the variables $\hat{c}+\hat{d}-\widehat{x_1} = w(\hat{c}+\hat{d}-\hat{x})+ (1-w)(\hat{c}+\hat{d}-\hat{y})$ and $\hat{c}+\hat{d}-\widehat{y_1} = (1-w)(\hat{c}+\hat{d}-\hat{x}) + w(\hat{c}+\hat{d}-\hat{y})$ for $\hat{x}, \hat{y} \in [\ \hat{c},\ \ \hat{d}]$ and $w \in [0,1]$ in Eq. (10) we find that

$$\left\| g\left(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}\right)\right\| \le \frac{1}{2}\left[\begin{array}{c} \|g(w(\hat{c}+\hat{d}-\hat{x})+(1-w))(\hat{c}+\hat{d}-\hat{y})\| \\ +\|g((1-w)(\hat{c}+\hat{d}-\hat{x})+w(\hat{c}+\hat{d}-\hat{y})\| \end{array}\right]. \tag{11}$$

On multiplying by $w^{\alpha-1}$ to both sides of the Eq. (11) then integrating the resulting inequality with respect to w over [0,1], we obtain

$$\begin{aligned}
\frac{1}{\alpha}\left\| g\left(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}\right)\right\| &\le \frac{1}{2}\left[\int_0^1 w^{\alpha-1}\left\|g(w(\hat{c}+\hat{d}-\hat{x})+(1-w)(\hat{c}+\hat{d}-\hat{x}))\right\| dw\right. \\
&\left. + \int_0^1 w^{\alpha-1}\left\|g((1-w)(\hat{c}+\hat{d}-\hat{x})+w(\hat{c}+\hat{d}-\hat{y}))\right\| dw\right] \\
&= \frac{1}{2(\hat{y}-\hat{x})^{\alpha}}\left[\int_{\hat{c}+\hat{d}-\hat{y}}^{\hat{c}+\hat{d}-\hat{x}} (u-(\hat{c}+\hat{d}-\hat{y}))^{\alpha-1} g(u)du\right. \\
&\left. + \int_{\hat{c}+\hat{d}-\hat{y}}^{\hat{c}+\hat{d}-\hat{x}} ((\hat{c}+\hat{d}-\hat{x})-u)^{\alpha-1} g(u)du\right] \\
&= \frac{\gamma(\alpha)}{2(\hat{y}-\hat{x})^{\alpha}}\left[I^{\alpha}_{(\hat{c}+\hat{d}-\hat{y})^+} g(\hat{c}+\hat{d}-\hat{x}) + I^{\alpha}_{(\hat{c}+\hat{d}-\hat{x})^-} g(\hat{c}+\hat{d}-\hat{y})\right].
\end{aligned}$$

Thus, we get

$$\left\| g\left(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}\right)\right\| \le \frac{\gamma(\alpha+1)}{2(\hat{y}-\hat{x})^{\alpha}}\left[I^{\alpha}_{(\hat{c}+\hat{d}-\hat{y})^+} g(\hat{c}+\hat{d}-\hat{x}) + I^{\alpha}_{(\hat{c}+\hat{d}-\hat{x})^-} g(\hat{c}+\hat{d}-\hat{y})\right].$$

The proof of first inequality of Eq. (4) is completed. On the other hand, using the convexity of g we can write

$$\begin{aligned}
g(w(\hat{c}+\hat{d}-\hat{x})+(1-w)(\hat{c}+\hat{d}-\hat{y})) &\le wg((\hat{c}+\hat{d}-\hat{x})+(1-w)g(\hat{c}+\hat{d}-\hat{y}), \\
g((1-w)(\hat{c}+\hat{d}-\hat{x})+w(\hat{c}+\hat{d}-\hat{y})) &\le (1-w)g(\hat{c}+\hat{d}-\hat{x})+wg(\hat{c}+\hat{d}-\hat{y}).
\end{aligned}$$

By adding these inequalities and using Jensen-Norm inequality, we have

$$\begin{aligned}
& g(w(\hat{c}+\hat{d}-\hat{x})+(1-w)(\hat{c}+\hat{d}-\hat{y})) + g((1-w)(\hat{c}+\hat{d}-\hat{x})+w(\hat{c}+\hat{d}-\hat{y})) \\
& \le g(\hat{c}+\hat{d}-\hat{x}) + g(\hat{c}+\hat{d}-\hat{y}) \\
& \le 2[\|g(\widehat{c})\| + \|g(\hat{d})\|] - [\|g(\hat{x})\| + \|g(\hat{y})\|].
\end{aligned} \tag{12}$$

Multiplying both sides of Eq. (12) by $w^{\alpha-1}$ and then integrating the resulting inequlity with respect to w over [0,1], we obtain second and third inequality of Eq. (4).

Remark 1 *Under the assumption of Theorem 2.1 with $\alpha = 1$, we have*

$$\begin{aligned}
\left\| g\left(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}\right)\right\| &\le \|g(\widehat{c})\| + \|g(\hat{d})\| - \int_0^1 g(w\hat{x}+(1-w)\hat{y})dw \\
&\le \|g(\widehat{c})\| + \|g(\hat{d})\| - \left\| g\left(\frac{\hat{x}+\hat{y}}{2}\right)\right\|.
\end{aligned}$$

and

$$\left\|g\left(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}\right)\right\| \leq \frac{1}{(\hat{y}-\hat{x})}\int_x^y g(\hat{c}+\hat{d}-w)dw$$
$$\leq \|g(\widehat{c})\|+\|g(\hat{d})\|-\frac{\|g(\hat{x})\|+\|g(\hat{y})\|}{2}, \tag{13}$$

for all $\widehat{x}, \hat{y} \in [\ \widehat{c},\quad \hat{d}]$.
The proof of Remark 1. is proved by Kian and Mosleyhian [5, Theorem 2.1]

Similarly, we obtain the following Hermite-Hadamard-Norm inequalities for fractional integrals.

Theorem 2.2 *Let* g : $[\hat{c}, \hat{d}] \to R$ *be a convex function. Then we have*

$$\left\|g\left(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}\right)\right\| \leq \frac{2^{\alpha-1}\gamma(\alpha+1)}{(\hat{y}-\hat{x})^\alpha}\left[I^\alpha_{(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2})^+}g(\hat{c}+\hat{d}-\hat{x})+I^\alpha_{(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2})^-}g(\hat{c}+\hat{d}-\hat{y})\right] \tag{14}$$
$$\leq \|g(\hat{c})\|+\|g(\hat{d})\|-\frac{\|g(\hat{x})\|+\|g(\hat{y})\|}{2}, \tag{2.1}$$

for all $\widehat{x}, \hat{y} \in [\widehat{c}, \hat{d}]$ *and* $\alpha > 0$.

Proof: To prove the first inequality of Eq. (14), by writing $\widehat{x_1} = \frac{w}{2}\hat{x} + \frac{2-w}{2}\widehat{y}$ and $\widehat{y_1} = \frac{2-w}{2}\hat{x} + \frac{w}{2}\hat{y}$ for $\widehat{x}, \hat{y} \in [\widehat{c}, \hat{d}]$ and $w \in [0, 1]$ in the inequality Eq. (10), we get

$$2\left\|g\left(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}\right)\right\|$$
$$\leq \left[\left\|g\left(\hat{c}+\hat{d}-\left(\frac{w}{2}\hat{x}+\frac{2-w}{2}\hat{y}\right)\right)\right\|+\left\|g\left(\hat{c}+\hat{d}-\left(\frac{2-w}{2}\hat{x}+\frac{w}{2}\hat{y}\right)\right)\right\|\right]. \tag{15}$$

Then, multiplying both sides of Eq. (12) by $w^{\alpha-1}$ and then integrating the resulting inequlity with respect to w over [0,1], we have

$$\frac{2}{\alpha}\left\|g\left(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}\right)\right\|$$
$$\leq \int_0^1 w^{\alpha-1}g\left(\hat{c}+\hat{d}-\left(\frac{w}{2}\hat{x}+\frac{2-w}{2}\hat{y}\right)\right)dw+\int_0^1 w^{\alpha-1}g\left(\hat{c}+\hat{d}-\left(\frac{2-w}{2}\hat{x}+\frac{w}{2}\hat{y}\right)\right)dw$$
$$=\frac{2^\alpha}{(\hat{y}-\hat{x})^\alpha}\left[\int_{\hat{c}+\hat{d}-\hat{y}}^{\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}}(u-(\hat{c}+\hat{d}-\hat{y}))^{\alpha-1}g(u)du+\int_{\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}}^{\hat{c}+\hat{d}-\hat{c}}((\hat{c}+\hat{d}-\hat{x})-u)^{\alpha-1}g(u)du\right]$$
$$=\frac{2^\alpha\gamma(\alpha)}{(\hat{y}-\hat{x})^\alpha}\left[I^\alpha_{(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2})+}g(\hat{c}+\hat{d}-\hat{y})+I^\alpha_{(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2})+}g(\hat{c}+\hat{d}-\hat{x})\right]$$

Hence, we have

$$\left\|g\left(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2}\right)\right\| \leq \frac{2^{\alpha-1}\gamma(\alpha+1)}{(\hat{y}-\hat{x})^\alpha}\left[I^\alpha_{(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2})+}g(\hat{c}+\hat{d}-\hat{y})-I^\alpha_{(\hat{c}+\hat{d}-\frac{\hat{x}+\hat{y}}{2})+}g(\hat{c}+\hat{d}-\hat{x})\right].$$

The first inequality of Eq.(14) is proved.
For the proof of the second inequality of Eq. (14), by using Jensen-norm inequality, we obtain

$$g\left(\hat{c}+\hat{d}-\left(\frac{w}{2}\hat{x}+\frac{2-w}{2}\hat{y}\right)\right) \leq g(\hat{c})+g(\hat{d})-\left[\frac{w}{2}g(\hat{x})+\frac{2-w}{2}g(\hat{y})\right],$$
$$g\left(\hat{c}+\hat{d}-\left(\frac{2-w}{2}\hat{x}+\frac{w}{2}\hat{y}\right)\right) \leq g(\hat{c})+g(\hat{d})-\left[\frac{2-w}{2}g(\hat{x})+\frac{w}{2}g(\hat{y})\right].$$

By adding these inequalities, we have

$$g\left(\hat{c}+\hat{d}-\left(\frac{w}{2}\hat{x}+\frac{2-w}{2}\hat{y}\right)\right)+g\left(\hat{c}+\hat{d}-\left(\frac{2-w}{2}\hat{x}+\frac{w}{2}\hat{y}\right)\right)$$
$$\leq 2[\|g(\widehat{c})\|+\|g(\hat{d})\|]-\frac{\|g(\hat{x})\|+\|g(\hat{y})\|}{2}. \tag{16}$$

Multiplying both sides of Eq. (16) by $w^{\alpha-1}$ and then integrating the resulting inequality with respect to w over [0,1], we find second inequality of Eq. (14).

Remark 2 *If we have $\alpha = 1$ in the Theorem 2.3, then the inequality of Eq. (14) reduces inequality of Eq. (13).*

3. Conclusion

This paper focused on examining the notion of Norm-type convex functions of Hermite-Hadamard inequalities of fractional order. The research also explored applications derived from these findings. The outcome of the findings contributes for different applications. The upcoming researchers can establish similar inequalities different types of convexities in their future works.

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