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Generalization of Titchmarsh's Theorem and Dini Lipschitz Theorem for the Kontorovich-Lebedev-Clifford Transform

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ABSTRACT: By employing a generalized dual translation operator, we establish an analog of Titchmarsh's theorem and the Dini-Lipschitz theorem for the Kontorovich-Lebedev-Clifford transform. These results are derived for functions $f \in L^1\left(\mathbb{R}^+; x^{-1}\,dx\right) \cap L^2\left(\mathbb{R}^+; x^{-1}\,dx\right)$.

Keywords: Kontorovich-Lebedev-Clifford transform, generalized translation operator, Titchmarsh's theorem.

Contents

1. Introduction

Integral transforms and their corresponding inversion formulae such as the classical Bessel, Fourier, and Kontorovich-Lebedev transforms play an indispensable role in the analytical and numerical treatment of a broad spectrum of problems arising in pure and applied mathematics. These include, but are not limited to, problems in differential and integral equations, theoretical mechanics, mathematical physics, signal processing, and computational mathematics (see [1]–[[6]).

In recent decades, the theory of integral transforms has been significantly enriched by the introduction of wavelet analysis and the study of generalized transforms within various algebraic and geometric settings. Notable contributions in this direction include the development of wavelet transforms associated with the Kontorovich-Lebedev-Clifford transform [[24], [25]], the analysis of Weyl-type operators and windowed transforms [25], and the establishment of Jackson-type approximation theorems in non-standard function spaces [26]. Further extensions involve Hausdorff-type operators linked to the modified Whittaker transform [27], wavelet packet decompositions within the Heckman-Opdam framework [28], and the study of localization operators and scalograms for the Mehler–Fock wavelet transform [29]. Very recently, analogs of classical theorems have also been established for the Mehler–Fock–Clifford transform [30], demonstrating the continued vitality of this research area.

A cornerstone in the harmonic analysis of function spaces is the work of Titchmarsh [7], who provided a profound characterization of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition. This characterization is formulated in terms of the asymptotic decay of the norm of their Fourier transform outside compact intervals. Specifically, Titchmarsh proved the following fundamental result:

Theorem 1.1 *Let* $\gamma \in]0,1[$, *and assume that* $\varphi \in L^2(\mathbb{R})$. *Then the following are equivalent:*

1. $\|\varphi(.+l) - \varphi(l)\|_{L^2(\mathbb{R})} = O\left(l^{\gamma}\right)$ *as* $l \to 0$,
2. $\int_{|\tau|\geq r} |\widehat{\varphi}(\tau)|^2 d\tau = O\left(r^{-2\gamma}\right)$ *as* $r \to +\infty$.

Where $\widehat{\varphi}$ *denotes the Fourier transform of* φ.

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Building on this [8], Younis characterized the set of functions in $L^2(\mathbb{R})$ that meet the Cauchy-Lipschitz condition by analyzing the asymptotic behavior of the norm of their Fourier transform. In particular, the following results are obtained:

Theorem 1.2 *Let* $\gamma \in]0,1[$, *and* $\rho \geq 0$, *assume that* $\varphi \in L^2(\mathbb{R})$. *Then the following are equivalent:*

1. $\|\varphi(.+l)-\varphi(l)\|_{L^2(\mathbb{R})} = O\left(\frac{l^\gamma}{(\log\frac{1}{l})^\rho}\right)$ *as* $l \to 0$,
2. $\displaystyle\int_{|\tau|\geq r} |\widehat{\varphi}(\tau)|^2 d\tau = O\left(\frac{R^{-2\gamma}}{(\log R)^{2\rho}}\right)$ *as* $R \to +\infty$.

Where $\widehat{\varphi}$ *denotes the Fourier transform of* φ.

Our goal is to build on the work conducted in our research laboratory on this theory, as discussed in [18], [19], [20], [21], [22], along with other related studies from the laboratory, as cited in [24], [9], [10].

To achieve this, we will begin by reviewing the key results of this transformation that are essential for further investigation the theory.

M. I. Kontorovich and N. N. Lebedev (see [11], [12]) introduced the Kontorovich-Lebedev transformation:

$$(KL\varphi)(\tau) = G(\tau) = \int_0^\infty \frac{K_{i\tau}(a)}{a}\varphi(a)\,da,$$

and its inversion is given by:

$$\left(KL^{-1}G\right)(a) = \varphi(a) = \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)K_{i\tau}(a)G(\tau)\,d\tau.$$

where $K_{i\tau}(x)$ is the Macdonald function (see [13], [14], [15], [16]).

Now, we define the Kontorovich-Lebedev-Clifford transform (KLC) transform as:

$$\mathbb{K}\varphi(\tau) = \frac{1}{2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{a})\varphi(a)a^{-1}\,da, \quad \tau \in \mathbb{R}^+,$$

and its inversion is given by:

$$\varphi(a) = \frac{4}{\pi^2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{a})\mathbb{K}\varphi(\tau)\sinh(2\pi\sqrt{\tau})\,d\tau, \quad a > 0.$$

We present a few estimates that will be useful in further calculations, cited in [17]:

(i) $\left|K_{2i\sqrt{\tau}}(2\sqrt{a})\right| \leq K_0(2\sqrt{a})$,

(ii) $K_0(2\sqrt{a+b}) \leq K_0(2\sqrt{a})$ or $K_0(2\sqrt{b})$,

(iii) $K_0(2\sqrt{a}) = \frac{1}{2}\int_0^{+\infty} e^{-\left(t+\frac{a}{t}\right)}t^{-1}\,dt \leq \frac{1}{2}\left(e^{-1}+\frac{e^{-a}}{a}\right)$,

(iv) $K_0(2\sqrt{a}) \approx -\frac{1}{2}\ln a, \quad a \to 0^+$.

Let $L^p(\mathbb{R}^+;a^{-1}da)$, $1 \leq p \leq \infty$, denote the space of measurable functions φ on $\mathbb{R}^+$ such that $\|\varphi\|_{L^p(\mathbb{R}^+;a^{-1}da)} < \infty$, with the norm:

$$|\varphi|_{L^p(\mathbb{R}^+;a^{-1}da)} = \begin{cases} \left(\int_0^\infty |\varphi(a)|^p a^{-1}da\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{a>0}|\varphi(a)| & \text{if } p = \infty. \end{cases}$$

$\left(L^p(\mathbb{R}^+;a^{-1}da), \|.\|_{L^p(\mathbb{R}^+;a^{-1}da)}\right)$ is a Banach space.
The Plancherel relation for the KLC-transform can be written as:

$$\int_0^\infty \varphi(a)\overline{\psi(a)}a^{-1}da = \frac{8}{\pi^2}\int_0^\infty (\mathbb{K}\varphi)(\tau)\overline{(K\psi)(\tau)}\sinh(2\pi\sqrt{\tau})d\tau.$$

and the Parseval formula is given by:

$$\int_0^\infty |\varphi(a)|^2a^{-1}da = \frac{8}{\pi^2}\int_0^\infty |(\mathbb{K}\varphi)(\tau)|^2\sinh(2\pi\sqrt{\tau})d\tau.$$

$$\|\varphi\|_{L^2(\mathbb{R}^+;a^{-1}da)} = \|\mathbb{K}\varphi\|_{L^2\left(\mathbb{R}^+;\frac{8}{\pi^2}\sinh(2\pi\sqrt{\tau})d\tau\right)}. \tag{1.1}$$

Thus, the KLC-transform is an isometrically isomorphic operator from $L^2\left(\mathbb{R}^+;a^{-1}da\right)$ to $L^2\left(\mathbb{R}^+;\frac{8}{\pi^2}\sinh(2\pi\sqrt{\tau})d\tau\right)$.

From [17], we have

$$\frac{4}{\pi^2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{a})K_{2i\sqrt{\tau}}(2\sqrt{b})K_{2i\sqrt{\tau}}(2\sqrt{c})\sinh(2\pi\sqrt{\tau})\,d\tau = D(a,b,c),$$

where $D(a,b,c)$ is given by:

$$D(a,b,c) = \frac{1}{2}\exp\left[-\frac{ab+bc+ca}{\sqrt{abc}}\right], \quad a,b,c\in\mathbb{R}^+,$$

which is symmetric in a, b, and c. From [17], we have the product of Macdonald functions as:

$$K_{2i\sqrt{\tau}}(2\sqrt{a})K_{2i\sqrt{\tau}}(2\sqrt{b}) = \frac{1}{2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{c})D(a,b,c)c^{-1}\,dc = \mathbb{K}(D(a,b,.))(\tau),$$

where

$$0 < D(a,b,c) < \frac{e^{-2\sqrt{a}}}{2}.$$

such that $\mathfrak{T}_l$, $l>0$, denotes the translation operator and is defined as:

$$\mathfrak{T}_l(\varphi)(b) = \frac{1}{2}\int_0^\infty D(l,b,c)\varphi(c)c^{-1}\,dc.$$

Proposition 1.1 *[17] If $\varphi\in L^1\left(\mathbb{R}^+;a^{-1}da\right)$, then the KLC-transform of the translation operator satisfies the following property:*

$$\mathbb{K}(\mathfrak{T}_l\varphi)(\tau) = K_{2i\sqrt{\tau}}(2\sqrt{l})\mathbb{K}\varphi(\tau). \tag{1.2}$$

2. Main Result

We now present our first main result, which extends Theorem(1.1) to the context of the Kontorovich-Lebedev-Clifford transform.

Theorem 2.1 *Let $\varphi\in L^1\left(\mathbb{R}^+;a^{-1}da\right)\cap L^2\left(\mathbb{R}^+;a^{-1}da\right)$ and $\gamma\in]0,1[$. Then the following statements are equivalent:*

1. $\left\|\mathfrak{T}_l\varphi + \left(1+K_0(2\sqrt{l})\right)\varphi\right\|_{L^2(\mathbb{R}^+;a^{-1}da)} = O\left(l^\gamma\right)$ *as* $l\to 0$.

2. $\int_R^\infty \sinh(2\pi\sqrt{\tau})\left|(\mathbb{K}\varphi)(\tau)\right|^2d\tau = O(R^{-2\gamma})$ *as* $R\to+\infty$.

Proof: 1) $\Longrightarrow$ 2) Assume that

$$\left\|\mathfrak{T}_l\varphi+\left(1+K_0(2\sqrt{l})\right)\varphi\right\|_{L^2(\mathbb{R}^+;a^{-1}da)}=O\left(l^\gamma\right)\quad \text{as } l\to 0.$$

We have:

$$\begin{aligned}\mathbb{K}\left(\mathfrak{T}_l\varphi+\left(1+K_0(2\sqrt{l})\right)\varphi\right)&=\int_0^\infty\left[\mathfrak{T}_l\varphi(a)+\left(1+K_0(2\sqrt{l})\right)\varphi(a)\right]K_{2i\sqrt{\tau}}(2\sqrt{a})a^{-1}da\\&=\int_0^\infty\mathfrak{T}_l\varphi(a)K_{2i\sqrt{\tau}}(2\sqrt{a})a^{-1}da\\&\quad+\left(1+K_0(2\sqrt{l})\right)\int_0^\infty\varphi(a)K_{2i\sqrt{\tau}}(2\sqrt{a})a^{-1}da\\&=\mathbb{K}\left(\mathfrak{T}_l\varphi\right)(\tau)+\left(1+K_0(2\sqrt{l})\right)\mathbb{K}\varphi(\tau).\end{aligned}$$

By using equality (1.2), we get:

$$\begin{aligned}\mathbb{K}\left(\mathfrak{T}_l\varphi+\left(1+K_0(2\sqrt{l})\right)\varphi\right)&=K_{2i\sqrt{\tau}}(2\sqrt{l})(\mathbb{K}\varphi)(\tau)+\left(1+K_0(2\sqrt{l})\right)\mathbb{K}\varphi(\tau).\\&=\left(1+K_0(2\sqrt{l})+K_{2i\sqrt{\tau}}(2\sqrt{l})\right)\mathbb{K}\varphi(\tau).\end{aligned}$$

By Plancherel's and Parseval's relations (1.1), we have:

$$\begin{aligned}&\left\|\mathbb{K}\left(\mathfrak{T}_l\varphi+\left(1+K_0(2\sqrt{l})\right)\varphi\right)\right\|_{L^2\left(\mathbb{R}^+;\frac{8}{\pi^2}\sinh(2\pi\sqrt{\tau})d\tau\right)}\\&=\left\|\left(\mathfrak{T}_l\varphi+\left(1+K_0(2\sqrt{l})\right)\varphi\right)\right\|_{L^2(\mathbb{R}^+;a^{-1}da)}.\end{aligned}$$

Then:

$$\begin{aligned}&\left\|\left(\mathfrak{T}_l\varphi+\left(1+K_0(2\sqrt{l})\right)\varphi\right)\right\|^2_{L^2(\mathbb{R}^+,a^{-1}da)}\\&=\frac{8}{\pi^2}\int_0^\infty\sinh(2\pi\sqrt{\tau})\left(1+K_0(2\sqrt{l})+K_{2i\sqrt{\tau}}(2\sqrt{l})\right)^2|\mathbb{K}\varphi(\tau)|^2d\tau.\end{aligned}$$

We know that:

$$\left|K_{2i\sqrt{\tau}}(2\sqrt{l})\right|\leq K_0(2\sqrt{l}).$$

So,

$$-K_0(2\sqrt{l})\leq K_{2i\sqrt{\tau}}(2\sqrt{l})\leq K_0(2\sqrt{l}).$$

Then:

$$1\leq 1+K_{2i\sqrt{\tau}}(2\sqrt{l})+K_0(2\sqrt{l})\leq 1+2K_0(2\sqrt{l}).\tag{1.3}$$

It follows that:

$$\left\|\left(\mathfrak{T}_l\varphi+\left(1+K_0(2\sqrt{l})\right)\varphi\right)\right\|^2_{L^2(\mathbb{R}^+;a^{-1}da)}\geq\frac{8}{\pi^2}\int_0^\infty\sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2d\tau.$$

We have:

$$\int_{\frac{1}{l}}^\infty\sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2d\tau\leq\int_0^\infty\sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2d\tau.$$

So:

$$\int_{\frac{1}{l}}^\infty\sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2d\tau\leq\frac{\pi^2}{8}\left\|\left(\mathfrak{T}_l\varphi+\left(1+K_0(2\sqrt{l})\right)\varphi\right)\right\|^2_{L^2(\mathbb{R}^+;a^{-1}da)}.$$

If $l \to 0$, then:

$$\int_{\frac{1}{h}}^{\infty} \sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2 d\tau = O\left(h^{2\gamma}\right).$$

Let $R = \frac{1}{l}$, then $R \to +\infty$,
and:

$$\int_{R}^{\infty} \sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2 d\tau = O\left(R^{-2\gamma}\right).$$

as $R \to +\infty$.
2) $\Longrightarrow$ 1) Suppose now that:

$$\int_{R}^{\infty} \sinh(2\pi\sqrt{\tau})|(\mathbb{K}\varphi)(\tau)|^2 d\tau = O\left(R^{-2\gamma}\right) \quad \text{as } R \to +\infty.$$

We write:

$$\int_{0}^{\infty} \sinh(2\pi\sqrt{\tau})\left(1 + K_0(2\sqrt{l}) + K_{2i\sqrt{\tau}}(2\sqrt{l})\right)^2 |\mathbb{K}\varphi(\tau)|^2 d\tau = J_1 + J_2.$$

where:

$$\begin{aligned} J_1 &= \int_{0}^{\frac{1}{l}} \sinh(2\pi\sqrt{\tau})\left(1 + K_0(2\sqrt{l}) + K_{2i\sqrt{\tau}}(2\sqrt{l})\right)^2 |\mathbb{K}\varphi(\tau)|^2 d\tau, \\ J_2 &= \int_{\frac{1}{l}}^{\infty} \sinh(2\pi\sqrt{\tau})\left(1 + K_0(2\sqrt{l}) + K_{2i\sqrt{\tau}}(2\sqrt{l})\right)^2 |\mathbb{K}\varphi(\tau)|^2 d\tau. \end{aligned}$$

Evaluate the terms J_1 and J_2. From the properties of $K_{2i\sqrt{\tau}}(2\sqrt{l})$, there exists a constant $C > 0$ such that:

$$K_0(2\sqrt{l}) \leq C.$$

By using inequality (1.3), we get:

$$1 + K_0(2\sqrt{l}) + K_{2i\sqrt{\tau}}(2\sqrt{l}) \leq 1 + 2C. \tag{1.4}$$

Then:

$$\begin{aligned} \int_{\frac{1}{l}}^{\infty} \sinh(2\pi\sqrt{\tau})\Big(1 &+ K_0(2\sqrt{l})K_{2i\sqrt{\tau}}(2\sqrt{l})\Big)^2 |\mathbb{K}\varphi(\tau)|^2 d\tau \\ &\leq (1+2C)^2 \int_{\frac{1}{l}}^{\infty} \sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2 d\tau. \end{aligned}$$

Then:

$$J_2 = O\left(l^{2\gamma}\right).$$

To estimate J_1, we define:

$$\phi(a) = \int_{a}^{\infty} \sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2 d\tau.$$

We have:

$$\begin{aligned} \int_{0}^{a} \tau\phi'(\tau)d\tau &\leq a\int_{0}^{a} \phi'(\tau)d\tau \\ &\leq a(\phi(a) - \phi(0)). \end{aligned}$$

It follows that:

$$\int_{0}^{a} \tau\phi'(\tau)d\tau \leq -a\int_{0}^{a} \sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2 d\tau.$$

Then:

$$\int_0^a \sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2 d\tau \leq \frac{-1}{a}\int_0^a \tau\phi'(\tau)d\tau.$$

Upon an integration by parts, we obtain:

$$\int_0^a \tau\phi'(\tau)d\tau = a\phi(a) - \int_0^a \phi(\tau)d\tau.$$

So:

$$\frac{-1}{a}\int_0^a \tau\phi'(\tau)d\tau = \frac{1}{a}\int_0^a \phi(\tau)d\tau - \phi(a).$$

Then:

$$\frac{-1}{a}\int_0^a \tau\phi'(\tau)d\tau \leq \frac{1}{a}\int_0^a \phi(\tau)d\tau.$$

So, if $\tau \to +\infty$, then $a \to +\infty$:

$$\frac{-1}{a}\int_0^a \tau\phi'(\tau)d\tau \leq \frac{1}{a}\int_0^a O\left(\tau^{-2\gamma}\right)d\tau = O\left(a^{-2\gamma}\right).$$

It follows that:

$$\int_0^a \sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2 d\tau = O\left(a^{-2\gamma}\right).$$

We have:

$$\left\|\mathfrak{T}_l\varphi + \left(1 + K_0(2\sqrt{l})\right)\varphi\right\|^2_{L^2(\mathbb{R}^+;a^{-1}da)} = \frac{8}{\pi^2}\int_0^\infty \sinh(2\pi\sqrt{\tau})$$
$$\left(1 + K_0(2\sqrt{l}) + K_{2i\sqrt{\tau}}(2\sqrt{l})\right)^2 |\mathbb{K}\varphi(\tau)|^2 d\tau.$$

Then:

$$\left\|\mathfrak{T}_l\varphi + \left(1 + K_0(2\sqrt{l})\right)\varphi\right\|^2_{L^2(\mathbb{R}^+;a^{-1}da)} = \frac{8}{\pi^2}J_1 + \frac{8}{\pi^2}J_2.$$

This implies that:

$$\left\|\mathfrak{T}_l\varphi + \left(1 + K_0(2\sqrt{l})\right)\varphi\right\|^2_{L^2(\mathbb{R}^+;a^{-1}da)} = O\left(l^{2\gamma}\right) + O\left(l^{2\gamma}\right), \quad \text{as } l \to 0.$$

It follows that:

$$\left\|\mathfrak{T}_l\varphi + \left(1 + K_0(2\sqrt{l})\right)\varphi\right\|^2_{L^2(\mathbb{R}^+;a^{-1}da)} = O\left(l^{2\gamma}\right), \quad \text{as } l \to 0.$$

Which shows that:

$$\left\|\mathfrak{T}_l\varphi + \left(1 + K_0(2\sqrt{l})\right)\varphi\right\|_{L^2(\mathbb{R}^+;a^{-1}da)} = O\left(l^{\gamma}\right), \quad \text{as } l \to 0.$$

□

In the remainder of this paper, we present our second main result, which generalizes Theorem (1.2) to the framework of the Kontorovich-Lebedev-Clifford transform.

Theorem 2.2 *Let $\varphi \in L^1\left(\mathbb{R}^+; a^{-1}da\right) \cap L^2\left(\mathbb{R}^+; a^{-1}da\right)$, $\gamma \in]0,1[$ and $\rho > 0$. Then the following statements are equivalent:*

1. $\left\|\mathfrak{T}_l\varphi + \left(1 + K_0(2\sqrt{l})\right)\varphi\right\|_{L^2(\mathbb{R}^+;a^{-1}da)} = O\left(\frac{l^\gamma}{(\log\frac{1}{l})^\rho}\right)$, *as $l \to 0$,*
2. $\int_R^\infty \sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2 d\tau = O\left(\frac{R^{-2\gamma}}{(\log R)^{2\rho}}\right)$, *as $R \to +\infty$.*

Proof: 1) $\Longrightarrow$ 2) Assume that

$$\left\|\mathfrak{T}_l\varphi+\left(1+K_0(2\sqrt{l})\right)\varphi\right\|_{L^2(\mathbb{R}^+;a^{-1}da)}=O\left(\frac{l^\gamma}{(\log\frac{1}{l})^\rho}\right),\quad \text{as } l\to 0.$$

Similarly to the proof of Theorem(2.1), we can establish the following result:

$$\int_{\frac{1}{l}}^{\infty}\sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2d\tau\leq\frac{\pi^2}{8}\left\|\left(\mathfrak{T}_l\varphi+\left(1+K_0(2\sqrt{l})\right)\varphi\right)\right\|^2_{L^2(\mathbb{R}^+,a^{-1}da)}.$$

If $l\to 0$, then:

$$\int_{\frac{1}{l}}^{\infty}\sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2d\tau=O\left(\frac{l^{2\gamma}}{(\log\frac{1}{l})^{2\rho}}\right).$$

Let $R=\frac{1}{l}$, then $R\to+\infty$ and:

$$\int_{R}^{\infty}\sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2d\tau=O\left(\frac{R^{-2\gamma}}{(\log R)^{2\rho}}\right),\quad \text{as } R\to+\infty.$$

2) $\Longrightarrow$ 1) Suppose now that:

$$\int_{R}^{\infty}\sinh(2\pi\sqrt{\tau})|(\mathbb{K}\varphi)(\tau)|^2d\tau=O\left(\frac{R^{-2\gamma}}{(\log R)^{2\rho}}\right),\quad \text{as } R\to+\infty.$$

We write:

$$\int_{0}^{\infty}\sinh(2\pi\sqrt{\tau})\left(1+K_0(2\sqrt{l})+K_{2i\sqrt{\tau}}(2\sqrt{l})\right)^2|\mathbb{K}\varphi(\tau)|^2d\tau=J_1+J_2.$$

where:

$$\begin{aligned}J_1&=\int_{0}^{\frac{1}{l}}\sinh(2\pi\sqrt{\tau})\left(1+K_0(2\sqrt{l})+K_{2i\sqrt{\tau}}(2\sqrt{l})\right)^2|\mathbb{K}\varphi(\tau)|^2d\tau.\\ J_2&=\int_{\frac{1}{l}}^{\infty}\sinh(2\pi\sqrt{\tau})\left(1+K_0(2\sqrt{l})+K_{2i\sqrt{\tau}}(2\sqrt{l})\right)^2|\mathbb{K}\varphi(\tau)|^2d\tau.\end{aligned}$$

Estimate the summands J_1 and J_2.
By using inequality (1.4), we get:

$$\begin{aligned}\int_{\frac{1}{l}}^{\infty}\sinh(2\pi\sqrt{\tau})\Big(1&+K_0(2\sqrt{l})+K_{2i\sqrt{\tau}}(2\sqrt{l})\Big)^2|\mathbb{K}\varphi(\tau)|^2d\tau\\ &\leq(1+2K)^2\int_{\frac{1}{l}}^{\infty}\sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2d\tau.\end{aligned}$$

This implies that

$$J_2=O\left(\frac{l^{2\gamma}}{(\log\frac{1}{l})^{2\rho}}\right)$$

To estimate J_1, set

$$\phi(a)=\int_{a}^{\infty}\sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2d\tau$$

Similarly to the proof of the Theorem (2.1) we get:

$$\begin{aligned}\int_0^a \sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2 d\tau &\leq \frac{-1}{a}\int_0^a \tau\phi'(\tau)d\tau \\ &\leq \frac{1}{a}\int_0^a \phi(\tau)d\tau.\end{aligned}$$

So, if $\tau \to +\infty$ then $a \to +\infty$, we get

$$\frac{1}{a}\int_0^a \phi(\tau)d\tau \leq \frac{1}{a}\int_0^a O\left(\frac{\tau^{-2\gamma}}{(\log\tau)^{2\rho}}\right)d\tau = O\left(\frac{a^{-2\gamma}}{(\log a)^{2\rho}}\right).$$

It follows that:

$$\int_0^a \sinh(2\pi\sqrt{\tau})|\mathbb{K}\varphi(\tau)|^2 d\tau = O\left(\frac{a^{-2\gamma}}{(\log a)^{2\rho}}\right).$$

We have:

$$\begin{aligned}\left\|\mathfrak{T}_l\varphi + \left(1+K_0(2\sqrt{l})\right)\varphi\right\|^2_{L^2(\mathbb{R}^+,a^{-1}da)} &= \frac{8}{\pi^2}\int_0^\infty \sinh(2\pi\sqrt{\tau})(1 \\ &+K_0(2\sqrt{l}) + K_{2i\sqrt{\tau}}(2\sqrt{l})\Big)^2 |\mathbb{K}\varphi(\tau)|^2 d\tau.\end{aligned}$$

This implies that:

$$\left\|\mathfrak{T}_l\varphi + \left(1+K_0(2\sqrt{l})\right)\varphi\right\|^2_{L^2(\mathbb{R}^+,a^{-1}da)} = \frac{8}{\pi^2}J_1 + \frac{8}{\pi^2}J_2.$$

Then:

$$\left\|\mathfrak{T}_l\varphi + \left(1+K_0(2\sqrt{l})\right)\varphi\right\|^2_{L^2(\mathbb{R}^+;a^{-1}da)} = O\left(\frac{l^{2\gamma}}{\left(\log\frac{1}{l}\right)^{2\rho}}\right) + O\left(\frac{l^{2\gamma}}{\left(\log\frac{1}{l}\right)^{2\rho}}\right),$$

as $l \to 0$.
It follows that:

$$\left\|\mathfrak{T}_l\varphi + \left(1+K_0(2\sqrt{l})\right)\varphi\right\|^2_{L^2(\mathbb{R}^+;a^{-1}da)} = O\left(\frac{l^{2\gamma}}{\left(\log\frac{1}{l}\right)^{2\rho}}\right), \quad \text{as } l \to 0.$$

Which shows that:

$$\left\|\mathfrak{T}_l\varphi + \left(1+K_0(2\sqrt{l})\right)\varphi\right\|_{L^2(\mathbb{R}^+;a^{-1}da)} = O\left(\frac{l^{\gamma}}{\left(\log\frac{1}{l}\right)^{\rho}}\right), \quad \text{as } l \to 0.$$

□

3. Conclusion

This paper successfully extends two fundamental theorems in harmonic analysis—Titchmarsh's theorem and the Dini–Lipschitz theorem—to the context of the Kontorovich–Lebedev–Clifford (KLC) transform. By leveraging a generalized dual translation operator, the authors establish necessary and sufficient conditions for functions in the weighted space $L^1(\mathbb{R}^+;x^{-1}dx) \cap L^2(\mathbb{R}^+;x^{-1}dx)$ to satisfy specific asymptotic behaviors.

The main results demonstrate that:

- The asymptotic decay of the KLC transform in the high-frequency regime is directly linked to the smoothness of the function, as measured by a generalized translation operator.

- Both the classical Titchmarsh and Dini–Lipschitz criteria can be naturally generalized within the KLC framework, preserving the deep connection between the regularity of a function and the decay of its transform.

These findings not only enrich the theoretical foundation of the Kontorovich–Lebedev–Clifford transform but also open avenues for its application in solving integral equations, signal processing, and other areas where such transforms are instrumental. The results also align with and extend previous work done in the field, reinforcing the utility and versatility of the KLC transform in mathematical analysis.

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