



## Cosmological Transition of LRS Universe Model in Saez-Ballester Theory of Gravitation with Time Varying $G$ and $\Lambda$

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**ABSTRACT:** The presence of cosmic strings and dissipative effects with time varying gravitational constant  $G$  and cosmological constant  $\Lambda$ , during the evolution of Universe, have been investigated by taking spatially homogeneous and locally rotationally symmetric (LRS) Bianchi type-II space-time. Moreover, by considering suitable physical assumptions, cosmological solutions to Einstein’s field equations have been found out and their kinematical behaviour have been analysed in a theory proposed by Saez and Ballester (Phys. Lett. A 113: 467, 1986). Further, it has been noticed that  $\Lambda$  is positive (but small) and of decreasing nature in scalar-tensor theory of gravitation but it shows an exact reverse behaviour in general theory of relativity for the entire evolution of Universe.

**Keywords:** LRS Bianchi type-II metric, an accelerating universe, time varying  $G$ , time varying  $\Lambda$ , scalar-tensor theory of gravitation.

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### 1. Introduction

In cosmology, a problem to consider the variability of various fundamental constants is an unsettled one. One of such fundamental constant is  $G$  i.e. the gravitational constant and it acts as a coupling constant between geometry and matter. Another considerable fundamental constant is  $\Lambda$  and it was introduced as universal repulsion by Einstein in 1917 to justify the static nature of Universe in accordance with generally accepted model of that time. These two fundamental constants are an important part of Einstein’s field equations. On historical grounds, Dirac [1,2] and Milne [3,4] started their work on the space-time variation of fundamental constants and Milne [3,4] presented an idea of time dependent  $G$ . After that, a very famous hypothesis known as “Large Number Hypothesis” given by Dirac [1] also points towards a cosmology with gravitational constant as a decreasing function of time  $t$ . Moreover, it has also been shown by Canuto and Narlikar [5] that  $G$ -varying cosmology is consistent with the cosmological observations present at that time.

On the basis of preliminary analysis by Perlmutter *et al.* [6,7], it has been observed that cosmic expansion favours a Universe with decelerating expansion with  $\Omega_M = 1$ . But later on, it was suspected by the apparent magnitude-redshift data of Type Ia Supernovae (SNeIa) [8]–[10] that expansion of the Universe is accelerating at present scenario and some component of universe, which is undetected and large with negative pressure, must be there to justify such type of Universe model with cosmic acceleration. These observations pointed out that some form of “dark energy” or signature of cosmological constant  $\Lambda$  is the required physical entity which is found to be undetected or invisible and it might be the same entity which is responsible for transition of the Universe from deceleration to acceleration, around some 7 billion

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years ago i.e. at redshift  $z = 0.5$ . Moreover, recent observations from Cosmic Microwave Background (CMB) [11]–[13] have also converged the work of researchers towards the Universe dominated by dark energy. These SNeIa and CMB observations have detected a new model of the Universe with  $\Omega_M \approx 0.3$  and  $\Omega_\Lambda \approx 0.7$  and strongly ruled out the traditional  $(\Omega_M, \Omega_\Lambda) = (1, 0)$  Universe model. Such a new model Universe also corresponds to small, nonzero and positive  $\Lambda$ , i.e.,  $\Lambda \approx 10^{-52} m^{-2} \approx 10^{-35} s^{-2}$ . Hence, in modern cosmological theories, this dynamical cosmological term  $\Lambda(t)$  is of great interest as it has been proven very strong term to solve the problem of cosmological constant  $\Lambda$  in some natural way. This “cosmological constant problem” can also be seen as a problem identifying discrepancy between a small but negligible value of  $\Lambda$  which it possesses for present age Universe (as can be observed by the successes of Newton’s theory of gravitation [14]) and the values around  $10^{50}$  times larger as expected in a model by Glashow-Salam-Weinberg [15]. This cosmological term that expresses the measuring of empty space’s energy, also acts as a special force which is of repulsive nature and it opposes natural gravitational pull between galaxies in the Universe. A Universe with such a cosmological term would expand at much faster rate with time because of the push generated from such cosmological term  $\Lambda$  [16] but it is not like this in case of standard inflation. Such a cosmological term remains a constant in the absence of interactions with matter or radiation. But if interactions with matter or radiation are there, then cosmological solutions to Einstein’s field equations exist. Moreover, for time-varying  $\Lambda$ , an assumed equation of covariant conservation of stress-energy can also be found out. Hence, to determine appropriate dependence of  $\Lambda$  upon scale factor  $a$  or cosmic time  $t$  is the main problem in our context.

The theories discussed in the recent literature such as scalar-tensor theories of gravitation given by Brans and Dicke [17]; Barber [18]; Saez and Ballester [19] etc. are alternative theories of gravitation and these theories are of great interest to discuss about. Mathiazhagan and Johri [20], in their inflationary models have proposed an extension to La and Steinhardt [21] inflationary models that are based upon these scalar-tensor theories of gravitation. In a particularly interesting theory proposed by Saez and Ballester [19], the space-time metric is simply coupled with a scalar field which is of dimensionless behaviour. Such type of coupling describes weak fields in which an accelerated expanding regime appears despite of the dimensionless scalar field. A technique to solve the problem concerning missing matter in non-flat FRW cosmology has been also suggested by this theory.

During a phase transition in the history of Universe, there might be an existence of topologically stable objects called cosmic strings [22]. The Grand Unified Theories (GUT) [22]–[26] have speculated that these cosmic strings arise after the big bang explosion as the temperature goes down below some critical temperature. It is also believed that the formation of galaxies [27] is also followed by the density perturbations caused due to the cosmic strings and the string tension is responsible candidate for an effective pressure with anisotropic nature. As the cosmic strings possess stress-energy and have coupling with the gravitational field, so it may be considered as an important problem to observe the effects which are gravitational and came into existence due to these strings. In literature, Letelier [28] has considered the formation of massive strings by geometric strings with particles attached along its extension and thus he formulated an energy-momentum tensor for classical massive strings. In particular context of Bianchi type-II metric, some researchers [29]–[32] have investigated various cosmological models with massive cosmic strings in different physical aspects.

Bulk viscosity as a part of dissipative effects also acts significantly during evolution of the Universe in early phases. When neutrino decoupling occurred during early stage evolution of the Universe, matter in the Universe behaved like viscous fluid and this fact supports the presence of bulk viscosity. In literature, many cosmologists [33]–[35] have described the relativistic theory of viscosity.

Yadav *et al.* [31], Amirhashchi and Mohamadian [32], have analysed the effect of time decaying cosmological term  $\Lambda(t)$  for Bianchi type-II space-time. Rao *et al.* [36] have studied Bianchi type-II, VIII and IX cosmological models with the presence of cosmic strings in Saez-Ballester theory of gravitation. Amirhashchi [37] has investigated about bulk viscous effects and presence of cosmic strings during the evolution of Universe by considering Bianchi type-II space-time and variation of deceleration parameter ( $q$ ) with cosmic time ( $t$ ). Further, an evolution of cosmological models with Bianchi type-II space-time by considering time varying  $G(t)$  and  $\Lambda(t)$  has been studied by Chakraborty and Roy [38], and Belinchon [39]. Naidu *et al.* [40] also constructed LRS Bianchi type-II cosmological models in Saez-Ballester theory of gravitation by considering the presence of cosmic strings and effects that came into due to bulk viscosity

during evolution of the Universe in early phases. Rao *et al.* [41] have studied evolution of cosmological models representing anisotropic effects with bulk viscosity and wet dark fluid in Saez-Ballester theory of gravitation. Panov *et al.* [42] have discussed about evolution of the Universe by considering Bianchi type-II line element and observed an accelerating phase at present scenario. Recently, Godani [43] has worked for the solutions of Einstein's field equations for LRS Bianchi type-II space-time by considering different forms of scale factor. Also, Banerjee *et al.* [44] have considered bulk viscous effects for presenting a cosmological model with Bianchi type-II space-time. Motivated from these cosmological models, in this paper, we have worked to present a modification to our constructed models [45] on LRS Bianchi type-II space-time with cosmic strings and bulk viscosity, by considering the time evolution of Universe under the effect of variation of  $G$  and  $\Lambda$  w.r.t. cosmological time  $t$ , in Saez-Ballester theory of gravitation. The outline of this paper is as discussed: In section (2), we have described cosmological model and Einstein's field equations in Saez-Ballester theory of gravitation. Section (3) deals with cosmological solutions to Einstein's field equations by using some physically viable conditions or assumptions. Physical validation of constructed models along with some physical and geometrical aspects have been discussed in detail under section (4). Finally, a summary of concluding remarks have been given in the last section (5).

## 2. Cosmological Model and its Field Equations

An expression for spatially homogeneous and LRS Bianchi type-II space-time is given by

$$ds^2 = -dt^2 + A_1^2 dx^2 + A_2^2 dy^2 + 2A_2^2 x dy dz + (A_2^2 x^2 + A_1^2) dz^2. \quad (2.1)$$

Here  $A_1$ ,  $A_2$  are metric potentials and vary with cosmic time  $t$  only.

By considering variable gravitational constant  $G(t)$  and cosmological constant  $\Lambda(t)$ , the Einstein's field equations (with  $c = 1$ ) and scalar field equation in the scalar-tensor theory of gravitation proposed by Saez and Ballester [19] are expressed as

$$R_{ij} - \frac{1}{2} R g_{ij} - \eta \phi^r \left( \phi_{,i} \phi_{,j} - \frac{1}{2} g_{ij} \phi_{,hk} \phi^{,k} \right) = -8\pi G(t) T_{ij} + \Lambda(t) g_{ij}, \quad (2.2)$$

with

$$2\phi^r \phi_{;i}^i + r \phi^{r-1} \phi_{,k} \phi^{,k} = 0, \quad (2.3)$$

where  $r$  being an arbitrary constant and  $\eta$  being a dimensionless coupling constant. Moreover,  $T_{ij}$  represents an energy-momentum tensor for a massive strings cloud and bulk viscous fluid, expressed as

$$T_{ij} = (\rho + P) u_i u_j + P g_{ij} - \lambda x_i x_j. \quad (2.4)$$

where

$$P = p - 3\xi H, \quad (2.5)$$

Where  $\rho$  is the proper energy density for a cloud of strings with particles attached to them;  $P$  is the effective pressure;  $p$  is the isotropic pressure;  $\lambda$  is the string tension density;  $\xi$  is the coefficient of bulk viscosity determining magnitude of viscous stress relative to expansion;  $u^i = (0, 0, 0, 1)$  is the four velocity of particles and  $x^i$  is a unit space-like vector representing direction of strings. The vectors  $u^i$  and  $x^i$  satisfy following conditions:  $g_{ij} u^i u^j = -1$ ,  $u^i x_i = 0$ . Further, we choose  $x^i$  parallel to  $\partial/\partial x$ , i.e.  $x^i = (A^{-1}, 0, 0, 0)$ . Also, the mean Hubble parameter  $H$  for LRS Bianchi type-II line element is defined as

$$H = \frac{\dot{a}}{a} = \frac{1}{3} \left( 2 \frac{\dot{A}_1}{A_1} + \frac{\dot{A}_2}{A_2} \right) = \frac{1}{3} (2H_1 + H_2). \quad (2.6)$$

Here an over dot denotes derivative w.r.t. cosmic time  $t$ . Also,  $H_1 = \frac{\dot{A}_1}{A_1}$  and  $H_2 = \frac{\dot{A}_2}{A_2}$  are directional Hubble factors in the directions of  $x$  and  $y$  axes respectively.  $a$  is the average scale factor for LRS Bianchi type-II space-time given by

$$a = (A_1^2 A_2)^{\frac{1}{3}}. \quad (2.7)$$

If  $\rho_p$  denotes particle density of configuration, then

$$\rho = \rho_p + \lambda. \quad (2.8)$$

The Einstein's field equations (2.2) along with scalar field equation (2.3) in Saez-Ballester theory [19] and equation of energy conservation i.e.  $T_{;j}^{ij} = 0$ , for Bianchi type-II space-time (2.1) and energy distribution (2.4), yield following system of independent differential equations:

$$\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\dot{A}_1 \dot{A}_2}{A_1 A_2} + \frac{1}{4} \frac{A_2^2}{A_1^4} - \frac{1}{2} \eta \phi^r \dot{\phi}^2 = \lambda - 8\pi G P + \Lambda, \quad (2.9)$$

$$2 \frac{\ddot{A}_1}{A_1} + \frac{\dot{A}_1^2}{A_1^2} - \frac{3}{4} \frac{A_2^2}{A_1^4} - \frac{1}{2} \eta \phi^r \dot{\phi}^2 = -8\pi G P + \Lambda, \quad (2.10)$$

$$2 \frac{\dot{A}_1 \dot{A}_2}{A_1 A_2} + \frac{\dot{A}_1^2}{A_1^2} - \frac{1}{4} \frac{A_2^2}{A_1^4} + \frac{1}{2} \eta \phi^r \dot{\phi}^2 = 8\pi G \rho + \Lambda, \quad (2.11)$$

$$\ddot{\phi} + \dot{\phi} \left( \frac{2\dot{A}_1}{A_1} + \frac{\dot{A}_2}{A_2} \right) + \frac{r}{2} \frac{\dot{\phi}^2}{\phi} = 0, \quad (2.12)$$

and

$$\dot{\rho} + 3(P + \rho)H = 0. \quad (2.13)$$

Further, covariant divergence of the Einstein tensor yields

$$8\pi G [\dot{\rho} + 3H(P + \rho)] + 8\pi \dot{G}\rho + \dot{\Lambda} = 0, \quad (2.14)$$

which along with energy conservation equation (2.13) provides a coupling relationship between gravitational constant  $G$  and cosmological constant  $\Lambda$ , given by:

$$8\pi \dot{G}\rho + \dot{\Lambda} = 0. \quad (2.15)$$

The expressions of dynamical scalars such as the expansion scalar  $\theta$ , the shear scalar  $\sigma$  and the anisotropy parameter  $A_m$  are given by

$$\theta = u_{;i}^i = \frac{3\dot{a}}{a} \quad (2.16)$$

$$\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{1}{2} \left[ 2 \left( \frac{\dot{A}_1}{A_1} \right)^2 + \left( \frac{\dot{A}_2}{A_2} \right)^2 \right] - \frac{1}{6} \theta^2, \quad (2.17)$$

and

$$A_m = \frac{2\sigma^2}{3H^2}, \quad (2.18)$$

where

$$\sigma_{ij} = u_{i;j} + \frac{1}{2} (u_{i;k} u^k u_j + u_{j;k} u^k u_i) + \frac{1}{3} \theta (g_{ij} + u_i u_j).$$

### 3. Solutions of the Field Equations

The field equations (2.9)–(2.12) and energy conservation equation (2.13) collectively constitute a system of four independent relations and one dependent relation that involves a total of nine unknown variables viz.  $A_1$ ,  $A_2$ ,  $\phi$ ,  $p$ ,  $\rho$ ,  $\xi$ ,  $\lambda$ ,  $G$  and  $\Lambda$ . To evaluate an exact implicit solution for this system, we require four additional relations/constraints that relates unknown variables.

Firstly, physical assumption related to time variation of deceleration parameter is considered, i.e.

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = c(t) \quad \text{say,} \quad (3.1)$$

which may be rewritten as

$$\frac{\ddot{a}}{a} + c \frac{\dot{a}^2}{a^2} = 0. \quad (3.2)$$

To solve above relation, we further assume  $c = c(a)$  i.e.  $c = c(t) = c(a(t))$ . On integrating twice, we get the following solution:

$$\int \left( \exp \int \frac{c}{a} da \right) da = \alpha t + \beta, \quad (3.3)$$

where  $\alpha$  and  $\beta$  are integrating constant. In order to integrate the above equation, we choose  $\int \frac{c}{a} da$  as

$$\int \frac{c}{a} da = \ln h(a). \quad (3.4)$$

This is not affecting nature of generality of solutions at all. Hence from (3.3) and (3.4), we obtain

$$\int h(a) da = \alpha t + \beta. \quad (3.5)$$

As we know that the choice of  $h(a)$  in above equation is arbitrary, but for obtaining physically viable models of the universe having consistency with recent observations, we may choose  $h(a) = \frac{na^{n-1}}{\sqrt{1+a^{2n}}}$ , where  $n$  being a positive constant. For this particular form of  $f(a)$ , equation (3.5) yields following expression (where  $\beta$  is considered as zero for mathematical convenience) for average scale factor  $a$  as a function of cosmic time  $t$ :

$$a(t) = (\sinh(\alpha t))^{\frac{1}{n}}. \quad (3.6)$$

A relation of above form (3.6) gives a generalisation to the one obtained by Pradhan et al. [46] for describing dark energy models in context of Bianchi type- $VI_0$  space-time and the one suggested by Amirhashchi et al. [47] representing cosmological models with dark energy for FRW space-time, by choosing a time varying deceleration parameter  $q$ .

Secondly, a power law relation between gravitational constant  $G$  and average scale factor  $a$  is considered as

$$8\pi G = G_1 a^\gamma, \quad (3.7)$$

where  $G_1 > 0$  is the constant of proportionality and  $\gamma$  is some positive constant. Many researchers [48]–[50] have suggested similar variations of  $G$  and various researchers [51]–[53] have considered a relation in the form of power law for variable  $G(t)$ , for constructing physically viable cosmological models in context of FRW space-time and Bianchi space-times. With the help of equation (3.6), equation (3.7) may be rewritten as an implicit function of cosmic time  $t$  as

$$8\pi G = G_1 (\sinh(\alpha t))^{\frac{\gamma}{n}}. \quad (3.8)$$

Now, our system of equations is constrained by the direct proportionality relation between component of shear tensor and expansion scalar i.e.  $\sigma_{11} \propto \theta$  for LRS Bianchi type-II space-time and this yields an equation of the form

$$\frac{1}{3} \left( \frac{2\dot{A}_1}{A_1} - \frac{\dot{A}_2}{A_2} \right) = \delta \left( \frac{2\dot{A}_1}{A_1} + \frac{\dot{A}_2}{A_2} \right), \quad (3.9)$$

which further gives

$$\frac{\dot{A}_1}{A_1} = \kappa \left( \frac{\dot{A}_2}{A_2} \right), \quad (3.10)$$

where  $\kappa = \frac{1+3\delta}{2(1-3\delta)}$  with  $\delta$  being a constant of proportionality. After integration and without any loss of generality, choosing integration constant equal to 1 for mathematical simplicity, above eq. (3.10) reduces to

$$A_1 = (A_2)^\kappa. \quad (3.11)$$

Now, using above derived relation (3.11) in equation (2.7), we obtain following relationship:

$$A_1 = (A_2)^\kappa = a^{\frac{3\kappa}{2\kappa+1}}, \quad (3.12)$$

which further with the help of equation (3.6), yield following implicit expressions for directional scale factors and directional Hubble parameters:

$$A_1(t) = (A_2(t))^\kappa = [\sinh(\alpha t)]^{\frac{3\kappa}{n(2\kappa+1)}}, \quad (3.13)$$

$$H_1 = \kappa H_2 = \frac{3\kappa\alpha}{n(2\kappa+1)} \coth(\alpha t). \quad (3.14)$$

For the above derived directional Hubble parameters, equation (2.12) can be expressed as

$$\frac{\ddot{\phi}}{\dot{\phi}} + \frac{3\alpha}{n} \coth(\alpha t) + \frac{r}{2} \frac{\dot{\phi}}{\phi} = 0. \quad (3.15)$$

The above equation, on integrating twice, provides following implicit function for scalar field  $\phi$ :

$$\phi = \left[ \frac{r+2}{2} \left( \phi_1 \int \frac{dt}{(\sinh(\alpha t))^{\frac{3}{n}}} + \phi_2 \right) \right]^{\frac{2}{r+2}}, \quad (3.16)$$

with  $\phi_1 (\neq 0)$  and  $\phi_2$  being the constants of integration.

Further, using equation (3.13) into equations (2.10) and (2.11), we obtain the following results

$$8\pi GP - \Lambda = \frac{3\kappa\alpha^2 [2n(2\kappa+1) - 9\kappa]}{n^2(2\kappa+1)^2} \coth^2(\alpha t) - \frac{6\kappa\alpha^2}{n(2\kappa+1)} + \frac{3}{4} (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}} + \frac{\eta\phi_1^2}{2} (\sinh(\alpha t))^{-\frac{6}{n}}, \quad (3.17)$$

$$8\pi G\rho + \Lambda = \frac{9\kappa(\kappa+2)\alpha^2}{n^2(2\kappa+1)^2} \coth^2(\alpha t) - \frac{1}{4} (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}} + \frac{\eta\phi_1^2}{2} (\sinh(\alpha t))^{-\frac{6}{n}}. \quad (3.18)$$

By using equations (3.6), (3.8) and (2.13) into above equations (3.17) and (3.18), we obtain the following expressions for energy density  $\rho$ , cosmological constant  $\Lambda$  and effective pressure  $P$ :

$$\begin{aligned} \rho = & \frac{3(1+2\kappa)}{2G_1[\gamma(1+2\kappa) - 6(1-2\kappa)]} (\sinh(\alpha t))^{\frac{6(1-2\kappa) - \gamma(1+2\kappa)}{n(1+2\kappa)}} + \frac{54\kappa(1-\kappa)\alpha^2}{G_1\gamma n^2(2\kappa+1)^2} \frac{1}{(\sinh(\alpha t))^{\frac{\gamma}{n}}} \\ & + \frac{9\kappa\alpha^2 [2n(1+2\kappa) + 6(1-\kappa)]}{G_1(\gamma+2n)n^2(2\kappa+1)^2} (\sinh(\alpha t))^{-\frac{\gamma+2n}{n}} + \frac{3\eta\phi_1^2}{G_1(\gamma+6)} (\sinh(\alpha t))^{-\frac{\gamma+6}{n}}, \end{aligned} \quad (3.19)$$

$$\begin{aligned} \Lambda = & \frac{9\kappa(\kappa+2)\alpha^2}{n^2(2\kappa+1)^2} \coth^2(\alpha t) + \frac{54\kappa(\kappa-1)\alpha^2}{\gamma n^2(2\kappa+1)^2} + \frac{9\kappa\alpha^2 [6(\kappa-1) - 2n(2\kappa+1)]}{(\gamma+2n)n^2(2\kappa+1)^2} \frac{1}{(\sinh(\alpha t))^2} \\ & + \frac{[24\kappa + \gamma(1+2\kappa)]}{4[6(1-2\kappa) - \gamma(1+2\kappa)]} (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}} + \frac{\eta\phi_1^2\gamma}{2(\gamma+6)} (\sinh(\alpha t))^{-\frac{6}{n}}, \end{aligned} \quad (3.20)$$

and

$$\begin{aligned} P = & \frac{1}{G_1} (\sinh(\alpha t))^{-\frac{\gamma}{n}} \left[ \frac{3\kappa\alpha^2 [6(1-\kappa) + 2n(1+2\kappa)]}{n^2(2\kappa+1)^2} \coth^2(\alpha t) + \frac{\eta\phi_1^2(2\gamma+6)}{2(\gamma+6)} \frac{1}{(\sinh(\alpha t))^{\frac{6}{n}}} \right. \\ & \left. + \frac{9\kappa\alpha^2 [6(\kappa-1) - 2n(2\kappa+1)]}{(\gamma+2n)n^2(2\kappa+1)^2} (\sinh(\alpha t))^{-2} + \frac{[6(3-2\kappa) - 2\gamma(1+2\kappa)]}{4[6(1-2\kappa) - \gamma(1+2\kappa)]} (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}} \right] \end{aligned}$$

$$\left. + \frac{6\kappa[9(\kappa - 1) - n\gamma(2\kappa + 1)]\alpha^2}{\gamma n^2(2\kappa + 1)^2} \right]. \quad (3.21)$$

Now subtracting equation (2.10) from equation (2.9) and using equation (3.13), we may obtain an expression for  $\lambda(t)$  as

$$\lambda = \frac{3(1 - \kappa)(3 - n)\alpha^2}{n^2(2\kappa + 1)} \coth^2(\alpha t) + \frac{3(1 - \kappa)\alpha^2}{n(2\kappa + 1)} + (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}}, \quad (3.22)$$

Hence, an expression for particle density  $\rho_p$  ( $\rho - \lambda$ ) with the help of equations (3.22) and (3.19) is given by

$$\begin{aligned} \rho_p = & \frac{3(1 + 2\kappa)}{2G_1[\gamma(1 + 2\kappa) - 6(1 - 2\kappa)]} (\sinh(\alpha t))^{\frac{6(1-2\kappa) - \gamma(1+2\kappa)}{n(1+2\kappa)}} + \frac{54\kappa(1 - \kappa)\alpha^2}{G_1\gamma n^2(2\kappa + 1)^2} \frac{1}{(\sinh(\alpha t))^{\frac{\gamma}{n}}} \\ & + \frac{9\kappa\alpha^2 [2n(1 + 2\kappa) + 6(1 - \kappa)]}{G_1(\gamma + 2n)n^2(2\kappa + 1)^2} (\sinh(\alpha t))^{-\frac{\gamma+2n}{n}} + \frac{3\eta\phi_1^2}{G_1(\gamma + 6)} (\sinh(\alpha t))^{-\frac{\gamma+6}{n}} - \\ & \frac{3(1 - \kappa)(3 - n)\alpha^2}{n^2(2\kappa + 1)} \coth^2(\alpha t) - \frac{3(1 - \kappa)\alpha^2}{n(2\kappa + 1)} - (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}}. \end{aligned} \quad (3.23)$$

Further, for the evaluation of other remaining parameters which are isotropic pressure  $p$  and bulk viscosity coefficient  $\xi$ , it is required to have another physically viable condition expressing relationship between  $p$  and  $\rho$  or  $\xi$  and  $\rho$ . So, the following two models representing bulk viscous fluid have been constructed:

### 3.1. Model A: With Barotropic Equation of State $p = \omega\rho$

Here, we assume cosmic fluid satisfying barotropic equation of state expressed as:

$$p = \omega\rho. \quad (3.24)$$

Here  $\omega$  is an Equation of State (EoS) parameter and it is a constant. There exist different physically relevant models for different values of  $\omega$ , as discussed below:

- For  $\omega = 0$ , there exists a model dominated by matter or a dust model i.e. model with pressure less fluid. Pure gravitation exists in this case and non-linear gravitational effects are also excluded from all fluid dynamical effects.
- For  $\omega = \frac{1}{3}$ , we have a model dominated by radiation in which there is an existence of relativistic particles in the Universe.
- For  $\omega = -1$ , there exists false vacuum or degenerate vacuum or  $\rho$  vacuum model [54].
- For  $\omega = 1$ , we have  $\rho = p$  and it is called Zel'dovich fluid or stiff fluid model [55].

Now, we obtain an implicit function for isotropic pressure  $p$  by substituting  $\rho$  from equation (3.19) into above relation (3.24), as

$$p = \frac{9\kappa(\kappa + 2)\omega\alpha^2}{n^2(2\kappa + 1)^2} \coth^2(\alpha t) - \frac{\omega}{4} (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}} + \frac{\eta\phi_1^2\omega}{2} (\sinh(\alpha t))^{\frac{-6}{n}}. \quad (3.25)$$

Further, using equations (3.6), (3.21) and (3.25) in  $P = p - 3\xi H$ , we obtain an expression for bulk viscosity coefficient  $\xi$  as

$$\begin{aligned} \xi = & \frac{[\kappa^2(3\omega + 5) + 2\kappa(3\omega - 1)]\alpha}{n(2\kappa + 1)^2} \coth(\alpha t) - \frac{n(\omega + 3)}{12\alpha} \tanh(\alpha t) (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}} \\ & + \frac{(\omega - 1)\eta\phi_1^2 n}{6\alpha} \frac{\tanh(\alpha t)}{(\sinh(\alpha t))^{\frac{6}{n}}} + \frac{2\kappa\alpha}{2\kappa + 1} \left[ \left( \frac{1 - n}{n} \right) \coth(\alpha t) + \tanh(\alpha t) \right]. \end{aligned} \quad (3.26)$$

### 3.2. Model B: With $\xi(t) = \xi_1 \rho^\zeta$

The bulk viscosity coefficient's ( $\xi$ ) dependence upon time is only for spatially homogeneous models of the Universe and many recent investigations by Pavon *et al.* [56]; Maartens [57]; Zimdahl [58]; Santos *et al.* [59], have assumed it to be a simple power function of energy density  $\rho$ , expressed as

$$\xi(t) = \xi_1 \rho^\zeta, \quad (3.27)$$

Here  $\zeta$  is some positive constant and  $\xi_1 > 0$  is a proportionality constant. Murphy, in his work [60] has analysed that for small density,  $\zeta$  may be 1 which is in correspondence with radiative fluid [14]. Further Belinskii and Khalatnikov [61] have also speculated that an appropriate assumption to obtain realistic results near the big bang singularity is given by  $0 \leq \zeta \leq \frac{1}{2}$ . By using expression for  $\rho$  from (3.19) into relation (3.27), we obtain an expression for  $\xi$  as

$$\begin{aligned} \xi = \xi_1 \left[ \frac{3(1+2\kappa)}{2G_1[\gamma(1+2\kappa) - 6(1-2\kappa)]} (\sinh(\alpha t))^{\frac{6(1-2\kappa) - \gamma(1+2\kappa)}{n(1+2\kappa)}} + \frac{54\kappa(1-\kappa)\alpha^2}{G_1\gamma n^2(2\kappa+1)^2} \frac{1}{(\sinh(\alpha t))^{\frac{2}{n}}} \right. \\ \left. + \frac{9\kappa\alpha^2 [2n(1+2\kappa) + 6(1-\kappa)]}{G_1(\gamma+2n)n^2(2\kappa+1)^2} (\sinh(\alpha t))^{-\frac{\gamma+2n}{n}} + \frac{3\eta\phi_1^2}{G_1(\gamma+6)} (\sinh(\alpha t))^{-\frac{\gamma+6}{n}} \right]^\zeta. \end{aligned} \quad (3.28)$$

Further, by using equations (3.6), (3.21) and (3.28) into  $P = p - 3\xi H$ , an expression for  $p$  is obtained as

$$\begin{aligned} p = \frac{1}{G_1} (\sinh(\alpha t))^{-\frac{\gamma}{n}} \left[ \frac{3\kappa\alpha^2 [6(1-\kappa) + 2n(1+2\kappa)]}{n^2(2\kappa+1)^2} \coth^2(\alpha t) + \frac{\eta\phi_1^2(2\gamma+6)}{2(\gamma+6)} \frac{1}{(\sinh(\alpha t))^{\frac{6}{n}}} \right. \\ \left. + \frac{9\kappa\alpha^2 [6(\kappa-1) - 2n(2\kappa+1)]}{(\gamma+2n)n^2(2\kappa+1)^2} (\sinh(\alpha t))^{-2} + \frac{[6(3-2\kappa) - 2\gamma(1+2\kappa)]}{4[6(1-2\kappa) - \gamma(1+2\kappa)]} (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}} \right. \\ \left. + \frac{6\kappa[9(\kappa-1) - n\gamma(2\kappa+1)]\alpha^2}{\gamma n^2(2\kappa+1)^2} \right] + \frac{3\xi_1\alpha}{n} \coth(\alpha t) \times \\ \xi_1 \left[ \frac{3(1+2\kappa)}{2G_1[\gamma(1+2\kappa) - 6(1-2\kappa)]} (\sinh(\alpha t))^{\frac{6(1-2\kappa) - \gamma(1+2\kappa)}{n(1+2\kappa)}} + \frac{54\kappa(1-\kappa)\alpha^2}{G_1\gamma n^2(2\kappa+1)^2} \frac{1}{(\sinh(\alpha t))^{\frac{2}{n}}} \right. \\ \left. + \frac{9\kappa\alpha^2 [2n(1+2\kappa) + 6(1-\kappa)]}{G_1(\gamma+2n)n^2(2\kappa+1)^2} (\sinh(\alpha t))^{-\frac{\gamma+2n}{n}} + \frac{3\eta\phi_1^2}{G_1(\gamma+6)} (\sinh(\alpha t))^{-\frac{\gamma+6}{n}} \right]^\zeta. \end{aligned} \quad (3.29)$$

Now, for vacuum energy density  $\rho_\Lambda$ , critical density  $\rho_c$ , density parameters  $\Omega_M$  and  $\Omega_\Lambda$  and total density parameter  $\Omega (= \Omega_M + \Omega_\Lambda)$ , following respective expressions have been obtained:

$$\begin{aligned} \rho_\Lambda = \frac{1}{G_1} (\sinh(\alpha t))^{-\frac{\gamma}{n}} \left[ \frac{9\kappa(\kappa+2)\alpha^2}{n^2(2\kappa+1)^2} \coth^2(\alpha t) + \frac{54\kappa(\kappa-1)\alpha^2}{\gamma n^2(2\kappa+1)^2} + \frac{\eta\phi_1^2\gamma}{2(\gamma+6)} \frac{1}{(\sinh(\alpha t))^{\frac{6}{n}}} \right. \\ \left. + \frac{9\kappa\alpha^2 [6(\kappa-1) - 2n(2\kappa+1)]}{(\gamma+2n)n^2(2\kappa+1)^2} \frac{1}{(\sinh(\alpha t))^2} + \frac{[24\kappa + \gamma(1+2\kappa)]}{4[6(1-2\kappa) - \gamma(1+2\kappa)]} \frac{1}{(\sinh(\alpha t))^{\frac{6(2\kappa-1)}{n(2\kappa+1)}}} \right], \end{aligned} \quad (3.30)$$

$$\rho_c = \frac{3\alpha^2}{G_1 n^2} \frac{\coth^2(\alpha t)}{(\sinh(\alpha t))^{\frac{2}{n}}}, \quad (3.31)$$

$$\begin{aligned} \Omega_M = \frac{n^2}{3\alpha^2} \tanh^2(\alpha t) \left[ \frac{3(1+2\kappa)}{2G_1(\gamma(1+2\kappa) - 6(1-2\kappa))} (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}} + \frac{54\kappa(1-\kappa)\alpha^2}{G_1\gamma n^2(2\kappa+1)^2} \right. \\ \left. + \frac{9\kappa\alpha^2 (2n(1+2\kappa) + 6(1-\kappa))}{G_1(\gamma+2n)n^2(2\kappa+1)^2} (\sinh(\alpha t))^{-2} + \frac{3\eta\phi_1^2}{G_1(\gamma+6)} (\sinh(\alpha t))^{-\frac{6}{n}} \right], \end{aligned} \quad (3.32)$$

$$\Omega_{\Lambda} = \frac{n^2}{3\alpha^2} \tanh^2(\alpha t) \left[ \frac{9\kappa(\kappa+2)\alpha^2}{n^2(2\kappa+1)^2} \coth^2(\alpha t) + \frac{54\kappa(\kappa-1)\alpha^2}{\gamma n^2(2m+1)^2} + \frac{9\kappa\alpha^2[6(\kappa-1) - 2n(2\kappa+1)]}{(\gamma+2n)n^2(2\kappa+1)^2} (\sinh(\alpha t))^{-2} + \frac{[24\kappa + \gamma(1+2\kappa)]}{4[6(1-2\kappa) - \gamma(1+2\kappa)]} (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}} + \frac{\eta\phi_1^2\gamma}{2(\gamma+6)} (\sinh(\alpha t))^{-\frac{6}{n}} \right], \quad (3.33)$$

and

$$\Omega = \frac{n^2}{3\alpha^2} \tanh^2(\alpha t) \left[ \frac{9\kappa(\kappa+2)\alpha^2}{n^2(2\kappa+1)^2} \coth^2(\alpha t) - \frac{1}{4} (\sinh(\alpha t))^{\frac{6(1-2\kappa)}{n(1+2\kappa)}} + \frac{\eta\phi_1^2}{2} (\sinh(\alpha t))^{-\frac{6}{n}} \right]. \quad (3.34)$$

#### 4. Physical Validation of Models

The presented models exhibit point type singularity [62] because the directional scale factors given by (3.13) become zero at an initial epoch ( $t = 0$ ). Moreover, contribution of scalar field function  $\phi$  is found to be significant in various physical parameters involved such as  $P$ ,  $\rho$ ,  $p$ ,  $\xi$  and  $\rho_p$ . Further, expressions for observational physical quantities such as shear scalar  $\sigma$  and mean anisotropy parameter  $A_m$  are given by

$$\sigma^2 = 3 \left[ \frac{\alpha(\kappa-1)}{n(2\kappa+1)} \coth(\alpha t) \right]^2, \quad (4.1)$$

$$A_m = 2 \left( \frac{\kappa-1}{2\kappa+1} \right)^2. \quad (4.2)$$

From above equation (4.1), we may notice that shear scalar  $\sigma$  becomes infinite at  $t = 0$  and it is also validating with big bang singularity. Also, with expansion of the Universe i.e. as  $t \rightarrow \infty$ ,  $\sigma$  approaches to zero. It may also be observed from equation (4.2) that  $A_m$  is constant attaining small positive value which indicates that presented models of the Universe maintain an anisotropic behaviour throughout the evolution of Universe.

For other cosmological parameters, their physical and geometrical behaviour is discussed below:

**Energy density:**

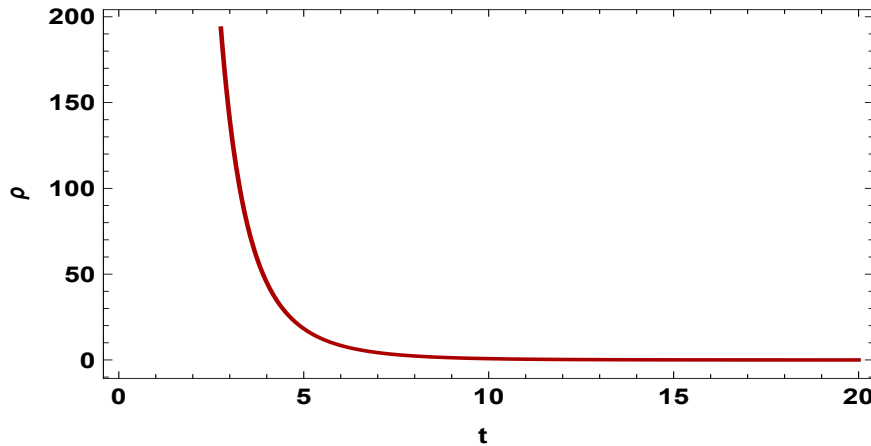


Figure 1: The plot of energy density  $\rho$  versus cosmic time  $t$

From figure 1, it is depicted that  $\rho$  is always positive and decreasing with time. Moreover, it begins from very huge values and move towards zero as  $t \rightarrow \infty$  which is validating with recent observations of SNeIa [8,9,10,63] and CMB anisotropies [11]–[13].

String tension density and particle energy density:

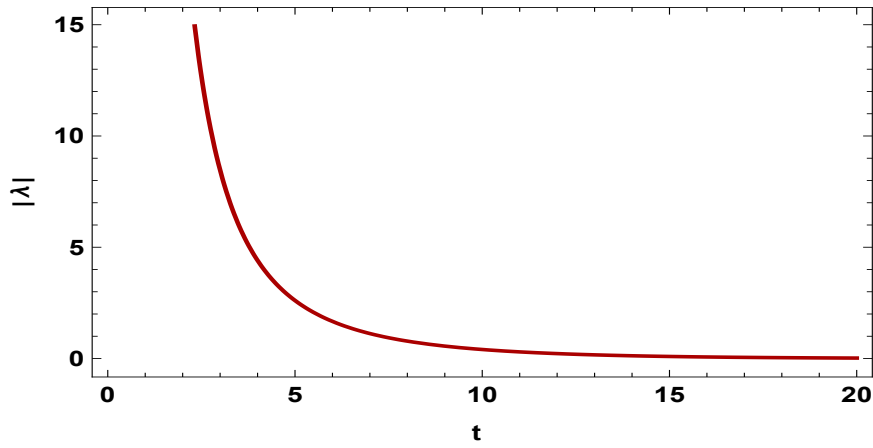


Figure 2: The plot of string energy density  $|\lambda|$  versus cosmic time  $t$

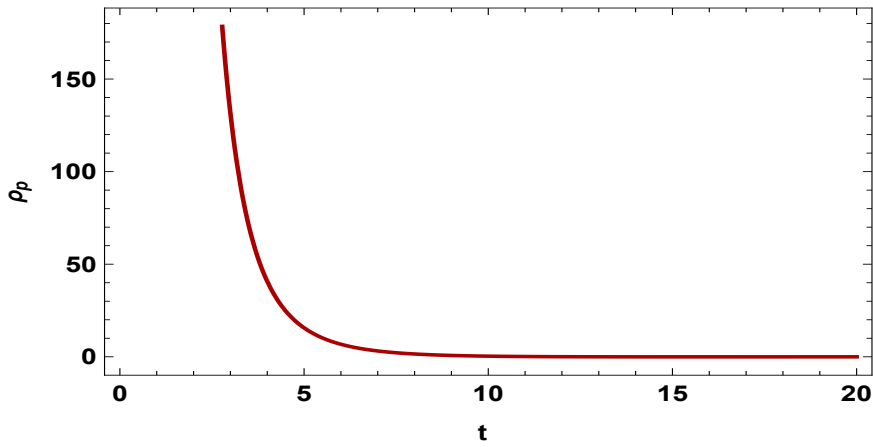


Figure 3: The plot of particle density  $\rho_p$  versus cosmic time  $t$

Figure 2 and figure 3 shows decreasing behaviour of  $|\lambda|$  and  $\rho_p$  respectively with cosmic time  $t$  and on comparing both, it has been noticed that particle energy density ( $\rho_p$ ) is higher than string tension density ( $|\lambda|$ ) during cosmic expansion of the Universe, specially during early phase evolution of the Universe indicating that during that time, Universe was dominated by clouds of massive strings. It is always desirable to construct such type of models of the Universe that shows an evolution from geometric strings or massive strings domination followed by particle domination with or without the remnants of strings as no direct evidence for presence of strings has been found out in the present day Universe. So, above investigated model clearly describes such an evolution of the Universe which is in fine tuning with present day observations. Further, figure 3 also shows a satisfaction of energy condition  $\rho_p \geq 0$  through the entire evolution of universe.

**Pressure parameters:**

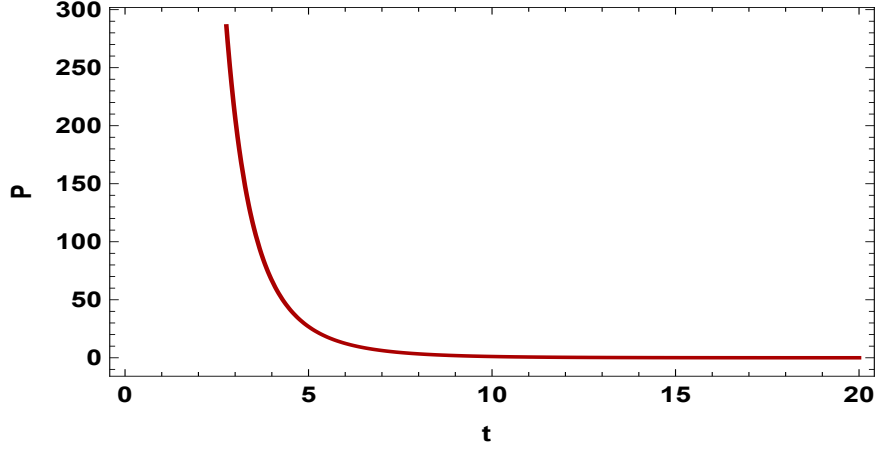


Figure 4: The plot of effective pressure  $P$  versus cosmic time  $t$

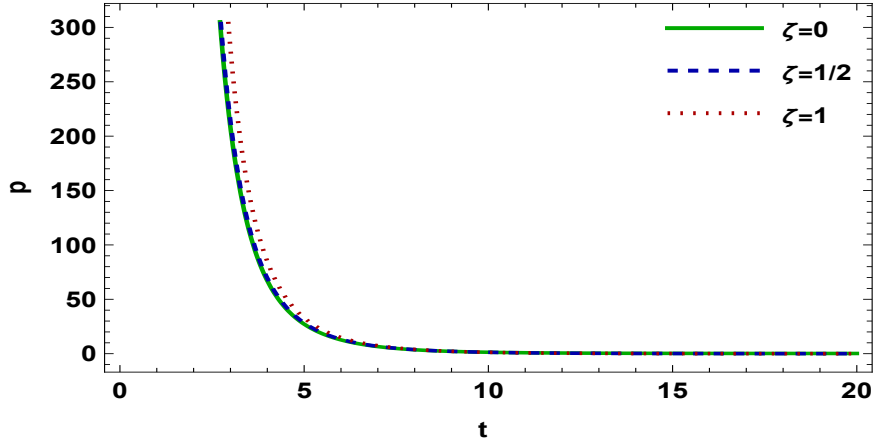


Figure 5: The plot of isotropic pressure  $p$  versus cosmic time  $t$  for Model B

The physically viable positive nature and decreasing behavior of effective pressure  $P$  with time has been depicted through Figure 4. The isotropic pressure for model  $A$  given by (3.25) reveals positive decreasing behaviour for different cosmological eras corresponding to  $\omega = 0$ ,  $\omega = 1/3$  and  $\omega = 1$  during the evolution of Universe, because of the direct proportionality relation between isotropic pressure  $p$  and energy density  $\rho$  via the perfect gas equation of state  $p = \omega\rho$ . Moreover, for model  $B$ , it observed that  $p$  is positive and a decreasing function of time for different values of  $\zeta$  as shown in figure 5 (the plots for  $\zeta = 0$  and  $\zeta = 1/2$  coincides). This figure clearly indicates that  $p$  starts from very large values during the early phases of Universe and approaches to zero as  $t \rightarrow \infty$ . Also, decaying of  $p$  w.r.t. time has been observed to be faster for  $\zeta = 1$  than for  $\zeta = 0$  and  $\zeta = 1/2$ .

**Bulk viscosity coefficient:**

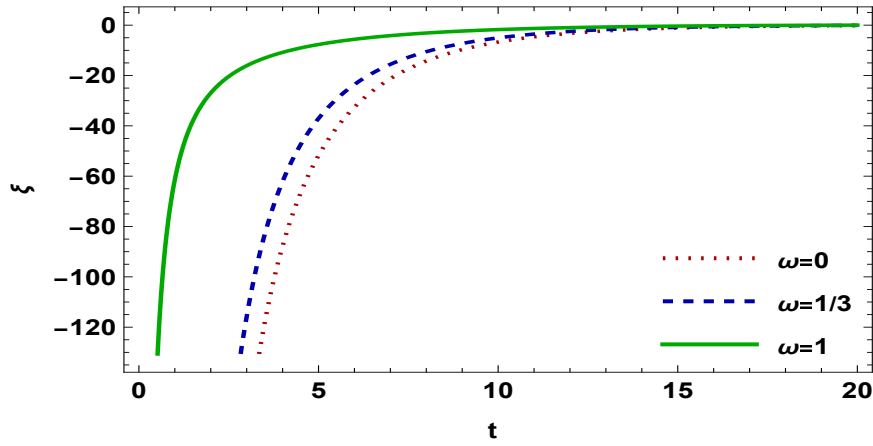


Figure 6: The plot of bulk viscosity coefficient  $\xi$  versus cosmic time  $t$  for Model A

From figure 6 for model A, it has been shown that for  $\omega = 0, 1/3$  and  $1$ ,  $\xi(t)$  is negative and increasing with time. On the other hand, for model B with  $\xi$  expressed as positive power of  $\rho$ , eq. (3.28) clearly reveals positive, decreasing behaviour throughout the evolution of Universe and it is clearly an indication of the presence of viscous effects at an initial epoch near big bang singularity and it also validates decoupling of neutrinos that occurred during radiation era and decoupling of radiation and matter that occurred during recombination era. This is in fine tuning with the GUT which also supports the presence of viscous effects during phase transition and string creation.

#### Cosmological constant:

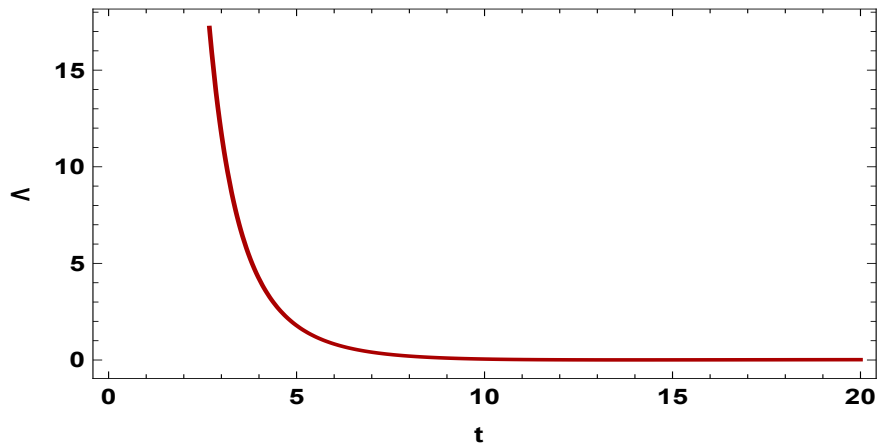


Figure 7: The plot of cosmological constant  $\Lambda$  versus cosmic time  $t$  in Saez-Ballester theory of gravitation

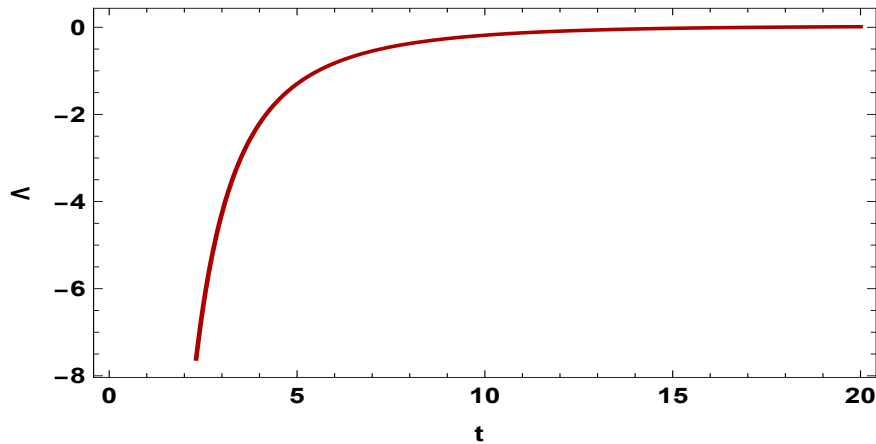


Figure 8: The plot of cosmological constant  $\Lambda$  versus cosmic time  $t$  in GTR

The depiction of positivity and decreasing nature of cosmological constant  $\Lambda$  w.r.t. cosmic time  $t$  can be seen from figure 7. It may be pointed out that  $\Lambda$  attains very small but positive values at late times or for the present day Universe which is in fine tuning with recent observation of SNeIa [8,9,10,63]. Moreover, a graph of  $\Lambda$  versus  $t$  in general theory of relativity (GTR) has been depicted via figure 8 and this figure predicts a negative  $\Lambda$  increasing with time, which is quite unphysical from the view point of recent SNeIa observations.

It is mentioned here that to discuss the graphical behaviour of constructed models, particular values have been assigned to various constants involved such as  $n = 2$ ,  $\alpha = 0.1202$ ,  $\kappa = 3$ ,  $\eta = 2$ ,  $\phi_1 = 2$ ,  $G_1 = 1$ ,  $\xi_1 = 1$  and  $\gamma = 1.5$ . It may be further remarked here that the observed behaviour of the parameters involved is due to this particular chosen set of values for constants and a change may be observed for some other set of values.

## 5. Concluding Remarks

After discussing about the physical and geometrical behaviour of various parameters involved, some concluded results from our work have been summarised below:

- The behaviour of bulk viscosity coefficient  $\xi$  for Model  $A$  is of realistic nature whereas Model  $B$  (for which  $\xi$  is negative) shows an absence of dissipative effects during the evolution of Universe.
- From the behaviour of cosmological constant  $\Lambda$  in Saez-Ballester theory and in GTR, it may be concluded that to reveal the true nature of cosmological constant, a modification in GTR is a necessity to reveal the presence of missing matter i.e., dark energy candidate  $\Lambda$  in the Universe.
- It may be emphasised from the behaviour of anisotropy parameter  $A_m$  that for  $m \neq 1$ , almost FRW Universe models have been obtained i.e., we obtain models with small degree of anisotropy while if we choose  $m = 1$ , the presented model clearly represents an isotropic FRW Universe.

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