



Chaotic Analysis with Lyapunov Exponent of Five-Dimensional Influenza Model with Vaccination

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ABSTRACT: In this article we analyse a model of influenza infectious disease in the presence of vaccination strategies using different parameter coefficients. To identify the chaos Lyapunov exponent is computed using various parameter inputs, and the model's chaos is determined using the exponent's values. This article uses concepts from chaos theory to examine the dynamics of the influenza virus from a chaotic viewpoint. The model has been analyzed for different parameters, and it has been tried to control the chaos in the dynamical system. Here the chaos control is measured with the help of Lyapunov exponent.

Keywords: Influenza, Lyapunov exponent, chaos, control, mathematical model.

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1. Introduction

Influenza is a disease spread by a virus; it is one of the most widespread epidemic diseases. Its symptoms are fever, muscle pain, dizziness, and weakness. Influenza affects the respiratory system of animal species, including humans. The influenza virus belongs to the family of Orthomyxoviridae. Influenza viruses are categorised based on negative-strand RNA. There are three main categories of influenzas: Type A, Type B, and Type C. Influenza virus of Type A is further subcategorised based on hemagglutinin (HA) and neuraminidase (NA). Type A has almost 16 HA types and 9 NA types. Of all these types, the most common types in humans are H1N1 and H3N2. Influenza outbreak can happen with a novel subtype of the influenza virus that can inflict serious disease in humans, and it needs to continue to travel readily from one individual to another. While there have been larger influenza outbreaks, the 1918–19 Spanish flu outbreak was the most severe. The H1N1 virus strain is the cause of this outbreak. Subsequently, there was the 1957–58 Asian flu outbreak and the 1968–70 Hong Kong flu outbreak. Numerous lives have been lost due to the influenza.

The world faced an outbreak of H1N1 in 2009, where influenza has the genes of swine, humans and avians. Avian influenza is also known as bird flu, as this virus has spread by aquatic birds worldwide. It has spread among the domestic poultry as well. Influenza threatens the global public health system;

2020 *Mathematics Subject Classification:* 34H10, 37B25, 97M60.

Submitted September 10, 2025. Published June 03, 2026.

if we look at the data of the United States alone, the influenza epidemic in the United States alone causes an average of 114,000 hospital admissions and 20,000 fatalities annually throughout the winter season every year [1]. Over time, influenza surveillance in the United States has undergone substantial changes because of technological improvements that facilitate the monitoring of influenza occurrence and vaccination efficacy (VE) [2]. Annually, millions of people are afflicted by influenza, and the virus is responsible for thousands of fatalities, indicating the significant impact it has on public health [3]. Experts agree that egg modifications negatively affect vaccination efficacy, and recent research has demonstrated the influence of egg-based production on influenza VE [4]. In addition, fewer instances were reported during the 2020–2021 influenza season, potentially because of COVID-19 safety precautions such as mask use and physical separation that lowered influenza transmission rates [5]. A unique method of influenza surveillance has been demonstrated using wearable mobile health devices to track population-level reactions to influenza-like disease [6]. Mathematical models are used to monitor the spread of the epidemic and, above all, its control. Some recent works of Kim et al. [7] studied the impact of vaccination time and vaccination in preventing influenza. Mossong et al. [8] showed that transmission through contact is the highest among people aged 5–19 years in the initial epidemic stage, and people in this age group are susceptible to infection. Pedro et al. [9] have done a quantitative study of the influenza model and investigated the impact of biomedical intervention on the population. Ainslie, K. E., Riley, S. [10] have studied the effects of annual vaccination against influenza. Sabir z. et al. [11] have solved the nonlinear influenza model using the integration of Levenberg-Marquardt backpropagation (LMB) and the supervised neural networks (NN). Li, L. et al. [12] Have used the machine learning techniques of autoregressive (AR) and Recurrent Neural Networks 2 (RNN) for the useful prediction of the influenza data so that it can be controlled. Guan, X. et al. [13] have studied the global stability of an influenza model with vaccination and asymptomatic patients. Bhowmick, S. et al. [14] have studied the mathematical model of co-infection dynamics of Covid-19 and influenza illness. Abdoon, M. A. et al. [15] have analysed a fractional-order influenza model. Pongsumpun, P. [16] utilises the SEIR model of influenza and Cellular Automata for analysis, showcasing parameters' effects on disease spread, understanding, and predicting influenza transmission dynamics. Seroussi, I. et al. [17] proposes a multiple-compartment SIR model to analyse influenza dynamics in Europe, estimating infection and recovery rates, revealing stable disease parameters, and suggesting minimal impact of inter-country human mobility. Kadhim, M. S. [18] presents a fractional SIR model with time delay for controlling Influenza A (H1N1) outbreaks through an optimal vaccination strategy based on numerical simulations using Moroccan data.

Chaos is a phenomenon which can be characterised as an order in the disorder. The concept of chaos was first coined in the study of three body problems by Henri Poincare in the 19th century. Edward Lorenz, in 1960, used this term in his popular work on the prediction of weather conditions. He popularly coined the term butterfly effect, which exactly represents the sensitive dependence to the initial conditions of the chaotic dynamical system. Chaos in influenza models is a well-documented phenomenon, with various studies highlighting the complex dynamics associated with the spread of the virus. Roberts, M. G. et al. [19] proposed that influenza dynamics may not follow regular patterns, making predicting future seasons' behaviour challenging. Stollenwerk N. et al. [20] investigated models for respiratory diseases with fast mutating virus pathogens, showing complex dynamics with seasonality leading to period-doubling bifurcations into chaos. O'Regan, S. M. et al. [21] develop a seasonally perturbed SIR model for avian influenza in a seabird colony, demonstrating chaotic recurrent epidemics. Thornley, J. H., France, J. [22] reveals that seasonal forcing leads to various regular and chaotic behaviours, emphasising the importance of seasonality in understanding and controlling influenza outbreaks. Seasonal variations are crucial in driving chaotic behaviour in these models, leading to recurrent epidemics with chaotic patterns. Multiple attractors, period-doubling bifurcations, and deterministic chaos have been observed, especially with small parameter changes and seasonal forcing. Models have shown that influenza dynamics can exhibit various behaviors, including fixed points, periodic solutions, and chaotic dynamics, emphasizing the unpredictability of future epidemics based on past data. The wave of chaos in influenza spread is a dominant mode of disease dispersal, influencing spatial pattern formation and the bidirectional spread of epidemics.

In the current investigation, we examine the influenza model given forth by Jonnalagadda in [23] to analyse the development and, if feasible, manage chaos using Lyapunov exponents. These dynamical sys-

tems are solved using Adams-Bashforth methods under various values of the epidemiological parameters given in the model.

2. Preliminaries: Lyapunov Exponent

A dynamical system is a mathematical concept used to describe a system that evolves over time according to a set of fixed rules. We can consider a dynamical system as

$$\begin{cases} \frac{dx}{dt} = f_1(x, y, z, t), \\ \frac{dy}{dt} = f_2(x, y, z, t), \\ \frac{dz}{dt} = f_3(x, y, z, t). \end{cases} \quad (2.1)$$

To identify chaos in any dynamical system, different techniques are used, and one of the most popular is to evaluate the Lyapunov exponent. The Lyapunov exponent, which is a mathematical indicator of sensitivity to the initial condition, is the rate at which the trajectories of the dynamical system diverge exponentially from altered initial conditions. The Lyapunov exponent can be considered as one of the parameters for deciding the movement characteristics of the system. The positive value ($\lambda_i > 0$) of the Lyapunov exponent is described as a divergent and negative value ($\lambda_i < 0$) represents the convergence in the degree sense of average velocity with the time if we consider the separation of trajectories as function of time. If the dynamical system is multidimensional then we get many Lyapunov exponent in number equal to the dimension of the system. Among all these Lyapunov exponents, if the largest Lyapunov exponent is positive then it will be responsible for the divergence of the trajectories which is termed as the chaotic behavior.

Definition 2.1 *Alimi, M. et al. [24]* $\sigma_i(t, x_0)$ ($i = 1, \dots, n$) represent the singular values of the Jacobian matrix $J^t(x_0)$ (i.e., square roots of the eigenvalues of $J^t(x_0)^T J^t(x_0)$). The logarithms of the eigenvalues of the limit Λ_{x_0} , are the precise values of the Lyapunov's exponents $\lambda_{(\sigma_i(., x_0))}$ of the dynamical system where

$$\Lambda_{x_0} = \lim_{t \rightarrow +\infty} \left(J^t(x_0)^T J^t(x_0) \right)^{\frac{1}{2t}}. \quad (2.2)$$

2.1. Method to calculate Lyapunov exponent

The Lyapunov exponents are calculated with the method proposed by [25]. Consider the Dynamical system:

$$\frac{dX}{dt} = G(X, t). \quad (2.3)$$

X and G are n -dimensional vector with the initial condition $X(0)$. To calculate the Lyapunov exponent, we must determine the separation rate of infinitesimally close trajectories evolving around $X(0)$. The tangent map will be:

$$\frac{d\delta X}{dt} = J\delta X \quad (2.4)$$

Here J is $n \times n$ Jacobian matrix. such that

$$J_{i,j} = \frac{\partial G_i}{\partial X_j} \quad (2.5)$$

The solution of the equation 2.4 can be written as

$$\delta X(t) = M(X(t), t)\delta X(0) \quad (2.6)$$

Where M is the tangent map, from this, the n copies of the tangential map can be achieved, its evolution equation gives us the n values of the Lyapunov exponent.

3. Model

The purpose of all epidemiological models is twofold: first, describe the spread of the epidemic and, subsequently, evaluate the effectiveness of interventions to combat it, typically through population vaccination. We divide the population, at time t , between the following compartments

- $S(t)$ susceptible individuals
- $L(t)$ latent individuals
- $I(t)$ infected individuals
- $V(t)$ vaccinated individuals
- $R(t)$ recovered individuals

The dynamical system proposed in [23] is:

$$\begin{cases} \frac{ds}{dt} = A + bR + cV - dSI - (e + f)S \\ \frac{dL}{dt} = dSI - (e + g)L \\ \frac{dI}{dt} = gL - (h + e + k)I \\ \frac{dR}{dt} = hI - (b + e)R \\ \frac{dV}{dt} = fS - (c + e)V \end{cases} \quad (3.1)$$

The parameters in 3.1 are specified as follows:

1. A Input rate or birth rate
2. $\frac{1}{b}$ Period of immunity of the recovered
3. $\frac{1}{c}$ Period of immunity of the vaccinated susceptible
4. d Coefficient of transmission from the susceptible to the latent stage
5. e Natural death rate in all the stages, i.e., $S(t)$, $L(t)$, $I(t)$, $R(t)$, $V(t)$
6. f Rate at which the susceptible are vaccinated
7. g Progression coefficient from the latent stage to the infected stage
8. h Recovery rate coefficient
9. k Death rate due to the disease

3.1. Evaluation of Lyapunov Exponent:

To find the value of the Lyapunov exponent of the model in 3.1, the following parameter values have been considered: $A = 100$, $b = 0.0000833$, $c = \frac{1}{360}$, $d = 0.00018$, $e = 0.0000548$, $f = 0.0001$, $g = 10$, $h = 0.000014$, $k = 0.00001$. Notice that these values were suggested [23]. The values of the Lyapunov exponent of the five-dimensional system at a time interval of 5 seconds are presented in Table 1.

Table 1: : Lyapunov Exponent with the value of Coefficient of Transmission $d = 0.00018$

Time(t)	S	L	I	R	V
t=5.0	-0.000588	0.041749	-1.774989	-2.131078	-6.164865
t=10.0	-0.000989	0.086167	-0.887565	-1.079426	-8.148063
t=15.0	-0.002621	0.130879	-0.591755	-0.728879	-8.837824
t=20.0	-0.010556	0.179747	-0.443851	-0.553607	-9.203121
t=25.0	0.013818	0.187274	-0.355108	-0.448446	-9.434614
t=30.0	0.051543	0.140265	-0.295947	-0.378341	-9.58162
t=35.0	0.017368	0.099312	-0.253752	-0.328271	-9.659255
t=40.0	0.000914	0.040389	-0.234825	-0.290851	-9.706847
t=45.0	-0.000564	0.032005	-0.263613	-0.279032	-9.742555
t=50.0	-0.000834	0.02881	-0.242765	-0.326988	-9.77079
t=55.0	-0.000981	0.026172	-0.223232	-0.376787	-9.793682
t=60.0	-0.001077	0.023973	-0.206947	-0.425557	-9.812607
t=65.0	-0.001142	0.022111	-0.193167	-0.473419	-9.828507
t=70.0	-0.001187	0.020514	-0.181355	-0.520486	-9.842046
t=75.0	-0.001219	0.019129	-0.171118	-0.566847	-9.85371
t=80.0	-0.001242	0.017916	-0.162161	-0.612577	-9.86386
t=85.0	-0.001258	0.016845	-0.154257	-0.657736	-9.872771
t=90.0	-0.001269	0.015892	-0.147232	-0.702377	-9.880653
t=95.0	-0.001277	0.015039	-0.140945	-0.74654	-9.887675
t=100.0	-0.001282	0.01427	-0.135288	-0.790261	-9.893969

Table 1 shows that the largest Lyapunov exponent is positive as observed in case of 'L'. The Lyapunov exponent is calculated till $t = 500$. Although the values initially increased, they started to reduce while remaining positive. The positive value of the Lyapunov exponent indicates the presence of chaos in the dynamical system. The value of the Lyapunov exponent concerning the variable L (latent individuals) for different values of the coefficient of transmission d is calculated and represented in fig.1. The spectrum of the Lyapunov exponent for five dimensional system is calculated using the method proposed in section 2.

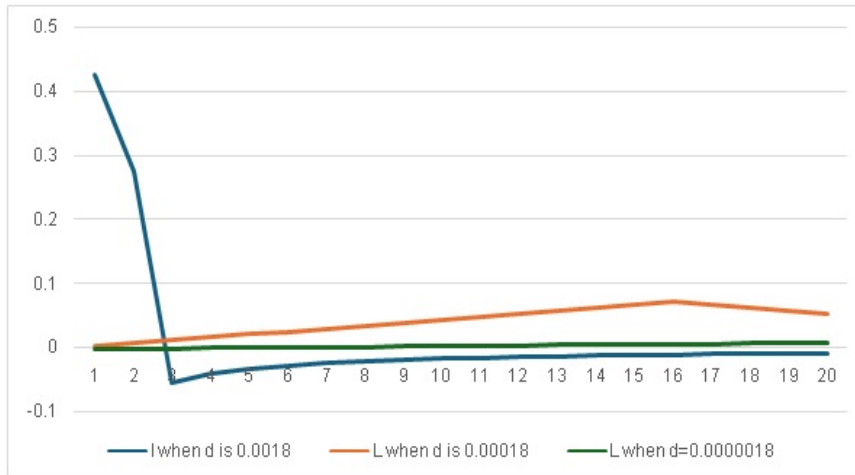


Figure 1: Lyapunov exponent plot concerning different values of the coefficient of transmission for $t = 100$

3.2. Evaluation of the equilibrium Point:

The equilibrium point of dynamical system 3.1 is detected, solving the following stationary system for the unknown S, L, I, R and V :

$$\begin{cases} A + bR + cV - dSI - (e + f)S = 0 \\ dSI - (e + g)L = 0 \\ gL - (h + e + k)I = 0 \\ hI - (b + e)R = 0 \\ fS - (c + e)V = 0 \end{cases} \quad (3.2)$$

Considering the values of the parameters as earlier: $A = 100, b = 0.0000833, c = 1/360, d = 0.00018, e = 0.0000548, f = 0.0001, g = 10, h = 0.000014, k = 0.00001$ system 3.2 has two equilibrium points $E_1(0.438, 11.200, 1421345.489, 144090.057, 0)$ and $E_2(690513.579, 0, 0, 0, 248.791)$. The Jacobian matrix of the equation 3.2 is

$$J = \begin{bmatrix} -dI - e - f & 0 & -dS & b & c \\ dI & -e - g & dS & 0 & 0 \\ 0 & g & -h - e - k & 0 & 0 \\ 0 & 0 & h & -b - e & 0 \\ f & 0 & 0 & 0 & -c - e \end{bmatrix} \quad (3.3)$$

The eigenvalues of the Jacobian matrix 3.3 at the first Equilibrium point E_1 are $\lambda_1 = -25.5844, \lambda_2 = -10.0000, \lambda_3 = -0.0001, \lambda_4 = -0.0015, \lambda_5 = -0.0278$, all negative. Therefore, the equilibrium E_1 is stable, whereas on calculating, the eigenvalues at E_2 we find $\lambda_1 = -0.0001, \lambda_2 = -0.0279, \lambda_3 = -17.2190, \lambda_4 = 7.2175, \lambda_5 = -0.0001$. Among all the eigenvalues one is indeed positive; hence, the system is unstable at this equilibrium point.

3.3. Control of the chaos:

We have tuned and varied the parameters to control the chaos in the dynamical system 3.1. With every varied value of the parameter, the Lyapunov exponent is calculated. The chaos is identified based on the value of the Lyapunov exponent as the positive value of Lyapunov exponent represents the presence of chaos and if all the Lyapunov exponents are negative then it represents absence of chaos in the system. The calculation of the Lyapunov exponent at different values of the parameter at various times is represented in the tables from Table 3 to Table 11. We get, in general, a positive Lyapunov exponent for the variable L , even if the Lyapunov exponent is negative in Tables 6 and 11. In Table 6, the transmission coefficient d is reduced to 0.00000018; in Table 11, the death rate value due to disease coefficient k is increased to 0.1. In both cases, we observe that the Lyapunov exponent concerning L remains negative till time $t = 100$. So, the Lyapunov exponent is calculated till $t = 500$ for $d = 0.00000018$ and $k = 0.1$. We found that the Lyapunov exponent remains negative; hence, the dynamical system 3.1 is non chaotic for such parameter values. To check the stability of the system, the eigenvalues of the Jacobian matrix in equation 3.3 are calculated at the second equilibrium point E_2 with the parameter $A = 100, b = 0.0000833, c = 0.0277, d = 0.0000000018, e = 0.0000548, f = 0.0001, g = 10, h = 0.0014, k = 0.1$ are $(-0.0001, -0.0279, -10.0002, -0.1013, -0.0001)$. Since all the eigenvalues are negative, the dynamical system will be stable. The dynamical system 3.1 with these parameters is solved using the Adams-Basforth-Moulton method, and the attractors are plotted in the fig. 2 to fig.6, which are presented.

Table 2: : Evaluation of Eigenvalues of Jacobian matrix at second equilibrium point with different values of the parameters

Sr. No.	Parameters and eigenvalues	Case 1	Case 2	Case 3	Case 4	Case 5
1	A	100	100	100	100	100
2	b	0.0000833	0.0000833	0.0000833	0.0000833	0.0000833
3	c	0.0277	0.0277	0.0277	0.02777	0.0277
4	d	0.000018	0.000018	0.00000018	0.0000000018	0.000018
5	e	0.0000548	0.0000548	0.0000548	0.0000548	0.0000548
6	f	0.0001	0.0001	0.0001	0.0001	0.0001
7	g	10	10	10	10	10
8	h	0.0014	0.0014	0.0014	0.0014	0.0014
9	k	0.00001	0.1	0.00001	0.00001	14.1
10	λ_1	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001
11	λ_2	-0.0279	-0.0279	-0.0279	-0.0279	-0.0279
12	λ_3	-17.219	-17.2486	-10.1229	-10.0013	-23.3864
13	λ_4	7.2175	7.1471	0.1213	-0.0002	-0.7151
14	λ_5	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001

4. Result and Discussion

It is known that, in the case in which the choice of parameters in the system 3.1 generates positive Lyapunov exponents, this system has chaotic behaviour. Therefore, the chaos is avoided by choosing the parameters in system 3.1 to generate negative Lyapunov exponents. We get a positive Lyapunov exponent for two state variables: the susceptible population $S(t)$ and the latent population $L(t)$. The dynamical system at its first equilibrium point is stable as the eigenvalues of the Jacobian matrix of the dynamical system 3.1 are negative. The system is unstable at the second equilibrium point as the eigenvalues of the Jacobian matrix are positive with the same parameters. The eigenvalues are reflected in Table 2. From Table 2, it can be confirmed that when the value of transmission coefficient d is reduced and the value of parameter k (death due to the disease) is increased, the system shows stable behaviour. Even if the value of k is not increased, the coefficient of transmission d from the susceptible stage to the latent stage is reduced. Then, the system shows stable behaviour, as can be verified with the eigenvalues. The system 3.1 shows the absence of chaos by changing the value of the Lyapunov exponent from positive to negative when we reduce the value of the coefficient of transmission d . The considered dynamical system is sensitive to the coefficient of transmission, and Table 5 shows that as we reduce the value of d from 0.000000018 to 0.0000000018, the Lyapunov exponent changes its sign from positive to negative. It indicates that Chaos can be completely removed from the dynamical system 3.1 by reducing the values of the coefficient of transmission, which can be done by maintaining distance, wearing masks, isolating the infected individuals, and keeping track of them. The Lyapunov exponent is calculated using the following parameters: $A = 100, b = 0.0000833, c = 1/360, d = 0.0018, e = 0.0000548, f = 0.0001, g = 10, h = 0.000014, k = 0.00001$. and it is observed that the values of majority of parameters are not affecting the Lyapunov exponent except the coefficient of transmission which can be observed from Table 3 to Table 11.

5. Tables of Lyapunov Exponents and Figures

In this section, we present the numerical determinations, in tabular form, obtained for the Lyapunov exponents for different sets of parameters of the dynamical system 3.1 and in the fig. 2-fig6 the trajectories of the dynamical system 3.1 obtained using the Adams-Basforth-Moulton method, and the attractors are plotted.

Table 3: : Lyapunov exponents for S, L, I, R, V when input parameters are reduced to $\mathbf{A} = \mathbf{10}$, $b = 0.0000833, c = 0.0277, d = 0.000018, e = 0.0000548, f = 0.0001, g = 10, h = 0.0014, k = 0.00001$

Time(t)	S	L	I	R	V
20	-0.00017	0.000364	-0.44385	-0.7659	-8.42224
40	-0.00016	0.002174	-0.22202	-0.39685	-9.01146
60	-0.00015	0.003976	-0.14808	-0.27385	-9.19313
80	-0.00015	0.005771	-0.1111	-0.21235	-9.30818
100	-0.00017	0.007562	-0.08891	-0.17545	-9.36804

Table 4: : Lyapunov exponents for S, L, I, R, V when input parameters are reduced to $A = 100$, $\mathbf{b} = \mathbf{0.00833}, c = 0.0277, d = 0.000018, e = 0.0000548, f = 0.0001, g = 10, h = 0.0014, k = 0.00001$

Time(t)	S	L	I	R	V
20	-0.00018	0.016346	-0.45205	-0.66856	-8.50183
40	-0.00026	0.034222	-0.23022	-0.34819	-9.03117
60	-0.00091	0.05241	-0.15627	-0.2414	-9.19768
80	-0.00403	0.07166	-0.1193	-0.18801	-9.27811
100	0.016991	0.053148	-0.09711	-0.15598	-9.33233

Table 5: : Lyapunov exponents for S, L, I, R, V when input parameters are reduced to $A = 100$, $b = 0.0000833, c = \mathbf{0.000277}, d = 0.000018, e = 0.0000548, f = 0.0001, g = 10, h = 0.0014, k = 0.00001$

Time(t)	S	L	I	R	V
20	-0.00019	0.00036	-0.44385	-0.73873	-8.42197
40	-0.00019	0.002166	-0.22202	-0.36953	-9.01133
60	-0.0002	0.003964	-0.14808	-0.24646	-9.19677
80	-0.00021	0.005753	-0.1111	-0.18493	-9.31094
100	-0.00023	0.007538	-0.08891	-0.14801	-9.37034

Table 6: : Lyapunov exponents for S, L, I, R, V when input parameters are reduced to $A = 100$, $b = 0.0000833, c = 0.0277, \mathbf{d} = \mathbf{0.00000018}, e = 0.0000548, f = 0.0001, g = 10, h = 0.0014, k = 0.00001$

Time(t)	S	L	I	R	V
20	-0.00016	-0.00142	-0.44385	-0.96793	-8.19829
40	-0.00015	-0.00138	-0.22203	-0.48413	-8.90387
60	-0.00015	-0.00135	-0.1481	-0.32286	-9.13902
80	-0.00015	-0.00131	-0.11115	-0.24223	-9.25656
100	-0.00015	-0.00127	-0.08899	-0.19385	-9.32706
440	-0.000147	-0.000579	-0.0208	-0.04432	-9.54384
460	-0.000147	-0.000546	-0.01994	-0.04241	-9.54652
480	-0.000146	-0.000513	-0.01916	-0.04066	-9.54897
500	-0.000146	-0.000482	-0.01843	-0.03905	-9.55121

Table 7: : Lyapunov exponents for S, L, I, R, V when input parameters are reduced to $A = 100$, $b = 0.0000833$, $c = 0.0277$, $d = 0.000018$, $e = \mathbf{0.000000548}$, $f = 0.0001$, $g = 10$, $h = 0.0014$, $k = 0.00001$

Time(t)	S	L	I	R	V
20	-0.00014	0.000414	-0.4438	-0.73867	-8.42193
40	-0.00014	0.002223	-0.22196	-0.36948	-9.01129
60	-0.00015	0.004024	-0.14802	-0.24641	-9.19288
80	-0.00016	0.005818	-0.11105	-0.18488	-9.30799
100	-0.00018	0.007608	-0.08886	-0.14796	-9.3679

Table 8: : Lyapunov exponents for S, L, I, R, V when input parameters are reduced to $A = 100$, $b = 0.0000833$, $c = 0.0277$, $d = 0.000018$, $e = 0.0000548$, $f = \mathbf{0.1}$, $g = 10$, $h = 0.0014$, $k = 0.00001$

Time(t)	S	L	I	R	V
20	-0.006812	-0.000414	-0.44385	-0.48678	-8.76775
40	-0.000584	-0.000059	-0.22202	-0.29643	-9.18533
60	-0.000148	0.000104	-0.14809	-0.23133	-9.3245
80	-0.000098	0.0002	-0.11114	-0.19862	-9.39408
100	-0.000087	0.000266	-0.08897	-0.17897	-9.4358

Table 9: : Lyapunov exponents for S, L, I, R, V when input parameters are reduced to $A = 100$, $b = 0.0000833$, $c = 0.0277$, $d = 0.000018$, $e = 0.0000548$, $f = 0.0001$, $g = \mathbf{2}$, $h = 0.0014$, $k = 0.00001$

Time(t)	S	L	I	R	V
20	-0.00019	0.000314	-0.36338	-0.5933	-1.04561
40	-0.00019	0.002103	-0.18178	-0.29682	-1.52548
60	-0.0002	0.00389	-0.12125	-0.19799	-1.68662
80	-0.00021	0.005667	-0.09099	-0.14858	-1.76807
100	-0.00023	0.007436	-0.07282	-0.11893	-1.81764

Table 10: : Lyapunov exponents for S, L, I, R, V when input parameters are reduced to $A = 100$, $b = 0.0000833$, $c = 0.0277$, $d = 0.000018$, $e = 0.0000548$, $f = 0.0001$, $g = 10$, $h = \mathbf{0.14}$, $k = 0.00001$

Time(t)	S	L	I	R	V
20	-0.00017	-0.002916	-0.34819	-0.96793	-8.47709
40	-0.00016	0.000002	-0.24293	-0.48413	-9.06105
60	-0.00016	0.000026	-0.20567	-0.32288	-9.24783
80	-0.00016	-0.000011	-0.18431	-0.24404	-9.33091
100	-0.00016	-0.000036	-0.15431	-0.21319	-9.39397

Table 11: : Lyapunov exponents for S, L, I, R, V when input parameters are reduced to $A = 100$, $b = 0.0000833$, $c = 0.0277$, $d = 0.000018$, $e = 0.0000548$, $f = 0.0001$, $g = 10$, $h = 0.0014$, $k = 0.1$

Time(t)	S	L	I	R	V
20	-0.00017	-0.09894	-0.44402	-0.73873	-8.42278
40	-0.00016	-0.09284	-0.22672	-0.36953	-9.01201
60	-0.00016	-0.07083	-0.17306	-0.24647	-9.19469
80	-0.00016	-0.05343	-0.1517	-0.18496	-9.30976
100	-0.00016	-0.04277	-0.13773	-0.14871	-9.37134
440	-0.000148	-0.009827	-0.03386	-0.07302	-9.52614
460	-0.000147	-0.009406	-0.03241	-0.07085	-9.52769
480	-0.000147	-0.009019	-0.03107	-0.06871	-9.52856
500	-0.000147	-0.008664	-0.02984	-0.06662	-9.52877

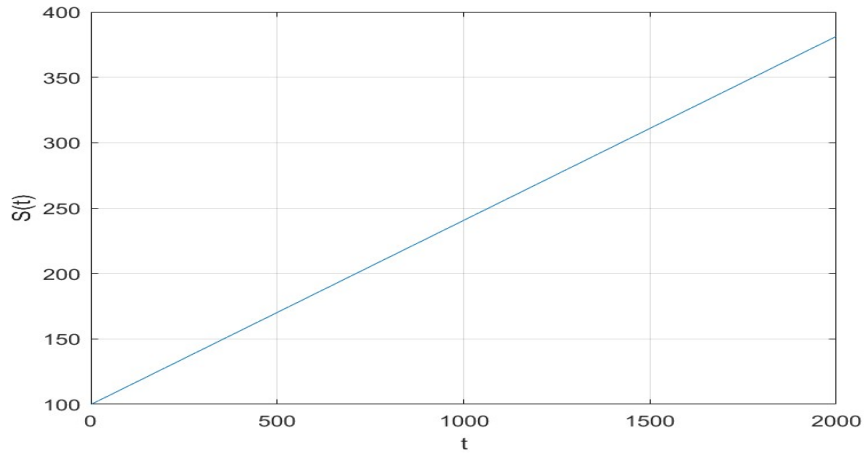


Figure 2: Plot of $s(t)$ with respect to different values of $A = 100$, $b = 0.0000833$, $c = 0.0277$, $d = 0.0000000018$, $e = 0.0000548$, $f = 0.00001$, $g = 10$, $h = 0.0014$, $k = 0.1$, $fort = 100$

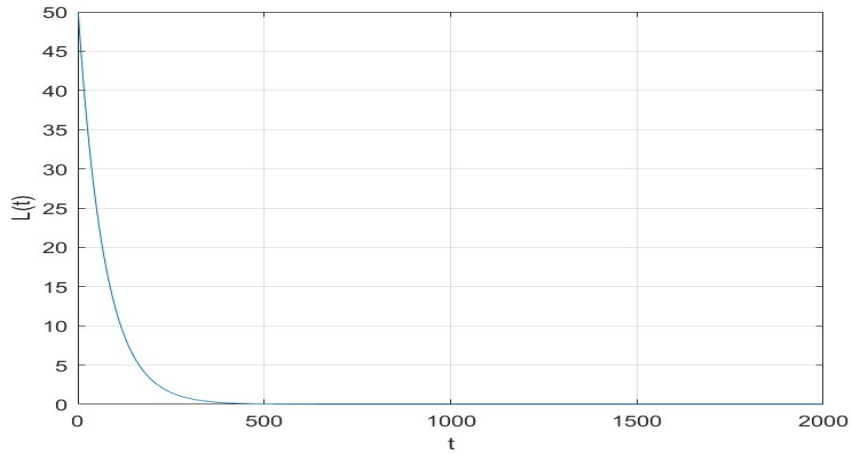


Figure 3: Plot of $L(t)$ with respect to different values of $A = 100, b = 0.0000833, c = 0.0277, d = 0.0000000018, e = 0.0000548, f = 0.00001, g = 10, h = 0.0014, f = 0.1, fort = 100$

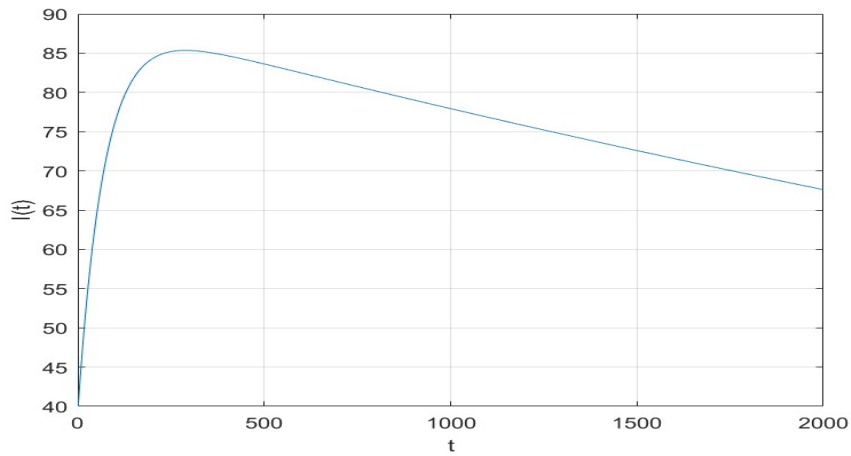


Figure 4: Plot of $I(t)$ with respect to different values of $A = 100, b = 0.0000833, c = 0.0277, d = 0.0000000018, e = 0.0000548, f = 0.00001, g = 10, h = 0.0014, f = 0.1, fort = 100$

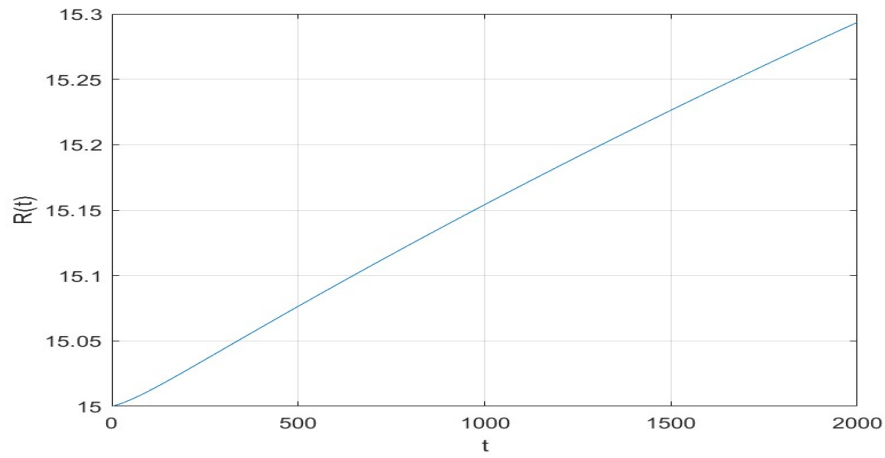


Figure 5: Plot of $R(t)$ with respect to different values of $A = 100, b = 0.0000833, c = 0.0277, d = 0.0000000018, e = 0.0000548, f = 0.00001, g = 10, h = 0.0014, f = 0.1, fort = 100$

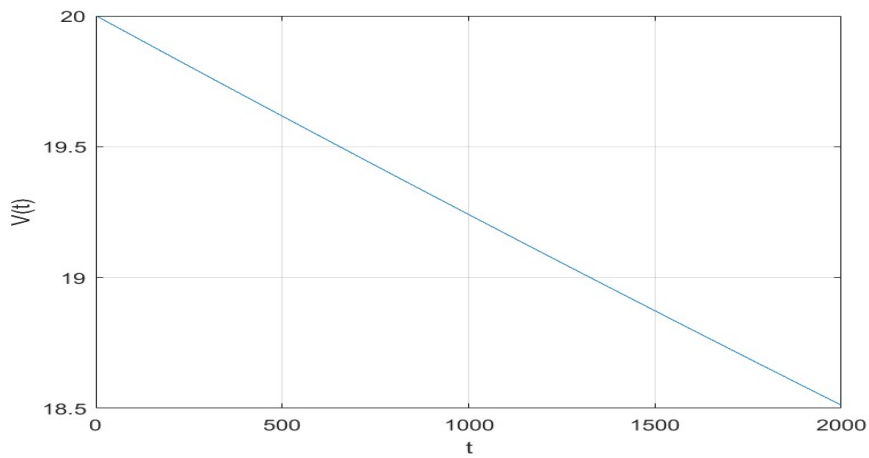


Figure 6: Plot of $V(t)$ with respect to different values of $A = 100, b = 0.0000833, c = 0.0277, d = 0.0000000018, e = 0.0000548, f = 0.00001, g = 10, h = 0.0014, f = 0.1, fort = 100$

6. Conclusion

The current article is an endeavor to control the chaos, not by any traditional controller design method. Rather, the effort is taken to prevent chaos using the parameters of the dynamical system [3.1](#); nine parameters rule this dynamical system, and chaos has been established in the presence of a positive Lyapunov exponent. We identified among the parameters those who generate the chaos. Our numerical approach shows that the transmission coefficient d is the parameter that can be tuned to control the chaos; this is done by reducing its value; thus, for small values of the transmission coefficient, we find values smaller than 0.0000000018, then the dynamical system [3.1](#) is chaos free, and stable.

7. Acknowledgements

This work is supported by the Ajman University Internal Research Grant No. [DRGS Ref. 2025-IRG-DRG-2].

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