Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 2** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.78902

Solvability of Infinite system of q-Difference Equations in c_0 space

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ABSTRACT: We investigate the existence of solutions of an infinite system of fractional q-difference equations with integral conditions in c_0 space which involves a q-derivative of the Caputo type. The result is obtained by using a generalization of Darbo's fixed point theorem and measure of noncompactness (MNC in short). This method has demonstrated significant potential in the analysis of these types of problem. It also introduces essential elements of fractional q-calculus. Finally, an example is presented to validate the proposed findings.

Key Words: Measure of noncompactness, fixed point, BVP, fractional q-difference equation.

Contents

1. Introduction

The fractional differential equations have proven to be highly effective tools for modeling complex phenomena across various scientific and engineering disciplines including physics, chemistry, biology, economics, engineering, and control theory. A wealth of foundational work on the subject can be found in [13,14,15]. In recent years, a growing body of literature has focused on the existence of solutions to fractional boundary value problems, as evidenced by studies [3].

The existence theory for BVPs involving fractional differential equations in Banach spaces has also seen significant progress. Relevant discussions and recent contributions in this area can be found in works such as [7].

The quantum calculus, often referred to as q-calculus, was originally introduced by Jackson in 1910 [11,12]. The key definitions and fundamental properties of this calculus are well documented in [8, 10]. The fractional version of q-calculus later emerged through the efforts of Al-Salam [5] and Agarwal [2]. Since then, both fractional q-difference calculus and associated nonlinear boundary value problems have attracted the attention of numerous researchers. Recent advancements in this field, particularly in connection with boundary value problems, were discussed in [1,3,18,20]. In this article, we investigate the following system of BVPs:

$$({}^C D_q^\alpha p_i)(s) = \eta_i(s, p(s)), s \in I = [0, \tau],\ \alpha \in (1, 2], \tag{1.1}$$

$$p_i(0) - p_i'(0) = \int_0^\tau \xi_i(t, p(t))dt, \tag{1.2}$$

$$p_i(\tau) + p_i'(\tau) = \int_0^\tau \zeta_i(t, p(t))dt, \tag{1.3}$$

$\tau > 0, 0 < q < 1, \eta_i,\ \xi_i\zeta_i : [0, \tau] \times c_0 \to \mathbb{R},\ {}^C D_q^\alpha$ is the Caputo fractional α^{th}-order q-derivative.

In our analysis of the existence of solutions for the aforementioned problem, we apply a technique that combines the concept of MNC with extended Darbo's fixed point result. This methodology proves to be highly effective in establishing the presence of solutions across various classes of fractional differential equations. The MNC-based framework was initially introduced by Banaś and Goebel [6].

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2010 *Mathematics Subject Classification:* 26A33, 39A13, 47H10.
Submitted September 10, 2025. Published September 30, 2025

2. Preliminaries

We begin by outlining the essential definitions, notational conventions, and preliminary results that will serve as the foundation for the subsequent analysis in this paper.

Suppose $I = [0, \tau]$, where$\tau > 0$, and let $C(I, c_0)$ be the space of continuous functions from I to c_0 endowed with the norm

$$\|p\|_{c_0} = \sup\{|p(s)| : s \in I\}.$$

We take $C^2(I, c_0)$ as the space of all differentiable functions $u : I \to c_0$, whose both 1^{st} and 2^{nd} derivatives are continuous, and take $L^1(I, c_0)$ as the Banach space consisting of the measurable functions $u : I \to c_0$ that are Bochner integrable with the norm

$$\|u\|_{L^1} = \int_I |u(s)| ds.$$

Here, we briefly mention some key definitions and results from fractional q-calculus [8,10] that are useful in our later discussion. For $l \in \mathbb{R}$ and $q \in (0, 1)$, we have

$$[l]_q = \frac{1 - q^l}{1 - q}.$$

The q-analogue of $(l - m)^{(n)}$ is given as

$$(l - m)^{(0)} = 1, (l - m)^{(n)} = \prod_{k=0}^{n-1} \left(l - mq^k\right), l, m \in \mathbb{R}, n \in \mathbb{N}.$$

In general,

$$(l - m)^{(\beta)} = l^\beta \prod_{k=0}^{\infty} \left(\frac{l - mq^k}{l - mq^{k+\beta}}\right), l, m, \beta \in \mathbb{R}.$$

It is easy to see if $m = 0$, then $l^{(\beta)} = l^\beta$.

Definition 2.1 *[8] The q-gamma function is defined as*

$$\Gamma_q(\gamma) = \frac{(1 - q)^{(\gamma-1)}}{(1 - q)^{\gamma-1}}, \gamma \in \mathbb{R} - \{0, -1, -2, -3 \ldots\}.$$

We observe that the q-gamma function obeys the identity $\Gamma_q(\beta + 1) = [\beta]_q \Gamma_q(\beta)$.

Definition 2.2 *[8] The $k^{th} (k \in \mathbb{N})$ order q-derivative of a function $\eta : I \to \mathbb{R}$ is given by $\left(D_q^0 \eta\right)(s) = \eta(s)$,*

$$\left(D_q \eta\right)(s) = \left(D_q^1 \eta\right)(s) = \frac{\eta(s) - \eta(qs)}{(1 - q)s}, s \neq 0, \left(D_q \eta\right)(0) = \lim_{s \to 0} \left(D_q \eta\right)(s),$$

and

$$\left(D_q^n \eta\right)(s) = \left(D_q^1 D_q^{n-1} \eta\right)(s), s \in I, n \in \{1, 2, 3 \ldots\}.$$

Now let $I_s = \{sq^n : n = 1, 2, 3, \ldots\} \cup \{0\}$.

Definition 2.3 *[8] The q-integration of any function $\eta : I_s \to \mathbb{R}$ is given by*

$$\left(I_q \eta\right)(s) = \int_0^s \eta(t) d_q t = \sum_{n=0}^{\infty} s(1 - q) q^n \eta\left(sq^n\right)$$

assuming the series is convergent.

We observe that $(D_q I_q \eta)(s) = \eta(s)$, when η is continuous at the point 0, then

$$(I_q D_q \eta)(s) = \eta(s) - \eta(0).$$

Definition 2.4 *[2] The β^{th} order ($\beta \in \mathbb{R}_+$) Riemann-Liouville fractional q-integration of the function $\eta : I \to \mathbb{R}$ is given by $\left(I_q^0 \eta\right)(s) = \eta(s)$, and*

$$\left(I_q^\beta \eta\right)(s) = \int_0^s \frac{(s-qt)^{(\beta-1)}}{\Gamma_q(\beta)} \eta(t) d_q t, s \in I.$$

For $\beta = 1$, we have $\left(I_q^1 \eta\right)(s) = (I_q \eta)(s)$.

Lemma 2.1 *[16] If $\beta \in \mathbb{R}_+$ and $\rho \in (-1, +\infty)$, then*

$$\left(I_q^\beta (s-b)^{(\rho)}\right)(s) = \frac{\Gamma_q(\rho+1)}{\Gamma_q(\beta+\rho+1)} (s-b)^{(\beta+\rho)}, 0 < b < s < \tau.$$

It is easy to see that

$$\left(I_q^\beta 1\right)(s) = \frac{1}{\Gamma_q(\beta+1)} s^{(\beta)}.$$

We denote $[\beta]$ as the integer part of β.

Definition 2.5 *[17] The β^{th} order ($\beta \in \mathbb{R}_+$)) Riemann-Liouville fractional q-derivative of the function $\eta : I \to \mathbb{R}$ is given by $\left(D_q^0 \eta\right)(s) = \eta(s)$, and*

$$\left(D_q^\beta \eta\right)(s) = \left(D_q^{[\beta]} I_q^{[\beta]-\beta} \eta\right)(s), s \in I.$$

Definition 2.6 *[17] The β^{th} order ($\beta \in \mathbb{R}_+$) Caputo fractional q-derivative of the function $\eta : I \to \mathbb{R}$ is defined by $\left(D_q^0 \eta\right)(s) = \eta(s)$, and*

$$\left({}^C D_q^\beta \eta\right)(s) = \left(I_q^{[\beta]-\beta} D_q^{[\beta]} \eta\right)(s), s \in I.$$

Lemma 2.2 *[17] Let $\beta, \rho \in \mathbb{R}_+$ and η be a function defined on I. Then*

1. $\left(I_q^\beta I_q^\rho \eta\right)(s) = \left(I_q^{\beta+\rho} \eta\right)(s)$
2. $\left(D_q^\beta I_q^\beta \eta\right)(s) = \eta(s)$.

Lemma 2.3 *[17] For $\beta \in \mathbb{R}_+$ and η is a function defined on I. Then*

$$\left(I_q^{\beta C} D_q^\beta \eta\right)(s) = \eta(s) - \sum_{n=0}^{[\beta]-1} \frac{s^n}{\Gamma_q(n+1)} \left(D_q^n \eta\right)(0).$$

If $0 < \beta < 1$, then

$$\left(I_q^{\beta C} D_q^\beta \eta\right)(s) = \eta(s) - \eta(0).$$

To proceed, we restate the definition of the Kuratowski MNC and highlight several of its essential properties.

Definition 2.7 *[6] Let us consider the Banach space $\mathfrak{M}$ and take $M_{\mathfrak{M}}$ as the collection of subsets of $\mathfrak{M}$ which are bounded. The Kuratowski MNC is given as $\mu : M_{\mathfrak{M}} \to [0, \infty)$ defined as*

$$\mu(\mathcal{Q}) = \inf \left\{\delta > 0 : \mathcal{Q} \subset \cup_{n=1}^m \mathcal{Q}_i \text{ and diam } (\mathcal{Q}_i) \leq \delta\right\}; \mathcal{Q} \in M_{\mathfrak{M}}.$$

The Hausdorff MNC is defined as $\chi(\mathcal{Q}) = \inf\{\delta > 0 : \mathcal{Q}$ admits a δ net in $\mathfrak{M}\}$.

Definition 2.8 *[6] A map $\mu : M_{\mathfrak{M}} \to [0,\infty)$ is said to be MNC if the following criteria are satisfied*

1. $ker\mu = \{\mathcal{P} \in M_{\mathfrak{M}} : \mu(\mathcal{P}) = 0\} \neq \phi$ *and* $Ker\mu \subseteq N_{\mathfrak{M}}$.
2. $\mathcal{P} \subseteq \mathfrak{Q} \implies \mu(\mathcal{P} \leq \mu(\mathfrak{Q})$.
3. $\mu(Conv(\mathcal{P}) = \mu(\mathcal{P})$.
4. $\mu(\bar{\mathcal{P}}) = \mu(\mathcal{P})$.
5. $\mu(\rho\mathcal{P} + (1-\rho)\mathfrak{Q}) \leq \rho\mu(\mathcal{P}) + (1-\rho)\mu(\mathfrak{Q}); \rho \in [0,1]$.
6. *If* $\{\mathcal{P}_i$ *forms a sequence consisting of closed sets in* $M_{\mathfrak{M}}$ *with* $\mathcal{P}_{i+1} \subseteq \mathcal{P}, i = 1,2,3,\dots$ *with* $\lim_{i\to\infty}\mu(\mathcal{P}_i) = 0$, *then* $\cap_{i=1}^{\infty}\mathcal{P}_i \neq \phi$.

Here $\bar{\mathcal{P}}$ and $conv(\mathcal{P})$ represent the closure and the convex hull of $\mathcal{P}$, respectively. Also, $N_{\mathfrak{M}}$ is the family of all relatively compact sets of $\mathfrak{M}$.

Theorem 2.1 *[9] Consider $\mathcal{P} \neq \phi$ closed, convex and bounded subset of the Banach space $\mathfrak{M}$ and $C : \mathcal{P} \to \mathcal{P}$ is a continuous map. Suppose that there is a constant $0 \leq l < 1$ with $\mu(C\mathfrak{Q}) \leq \mu(l\mathfrak{Q})$ for $\mathfrak{Q}(\neq \phi) \subseteq \mathcal{P}$, where μ represents a MNC in $\mathcal{P}$. Then C possesses a fixed point in $\mathcal{P}$.*

Theorem 2.2 *[19] If $\mathcal{P}$ is a closed, convex and bounded subset of the Banach space $\mathfrak{M}$. Suppose that $V : \mathcal{P} \to \mathcal{P}$ is continuous with*

$$\psi(\mu(\mathfrak{Q})) + \mathfrak{F}(\mu(V(\mathfrak{Q}))) \leq \mathfrak{F}(\mu(\mathfrak{Q}))$$

for each nonempty subset $\mathfrak{Q}$ of $\mathcal{P}$, μ is a MNC defined on $\mathfrak{M}$, $\psi : (0,\infty) \to (0,\infty)$, $(\psi,\mathfrak{F}) \in \Delta$. Then V has a fixed point in $\mathcal{P}$.

In the above result, Δ denotes the set of the pairs $(\psi,\mathfrak{F})$ and the following hold:

1. $\psi(x_n) \nrightarrow 0$ for any non-decreasing sequence $\{x_n\}$.
2. $\mathfrak{F}(x_n) < \mathfrak{F}(x_{n+1})$.
3. $\lim_n x_n = 0$ iff $\lim_n \mathfrak{F}(x_n) = -\infty$, where $x_n > 0$.
4. $\{y_n\}$ is a sequence $(y_n \geq y_{n+1})$ with $y_n \to 0$ and $\psi(y_n) < \mathfrak{F}(y_n) - \mathfrak{F}(y_{n+1}) \implies \sum_n y_n < \infty$.

Lemma 2.4 *[4] Suppose $g, \mathfrak{L}_1, \mathfrak{L}_2 : I \to \mathfrak{M}$ are two continuous functions. The solution to the BVP*

$$\left({}^C D_q^{\alpha} p\right)(s) = g(s), s \in I = [0,\tau], \quad 1 < \alpha \leq 2$$
$$p(0) - p'(0) = \int_0^{\tau} \mathfrak{L}_1(t)dt$$
$$p(\tau) + p'(\tau) = \int_0^{\tau} \mathfrak{L}_2(s)ds$$

is given by

$$p(t) = K(s) + \int_0^{\tau} H(s,t)g(t)d_qt \tag{7}$$

so that

$$K(s) = \frac{(1+\tau-s)}{(2+\tau)}\int_0^{\tau}\mathfrak{L}_1(t)dt + \frac{(1+s)}{(2+\tau)}\int_0^{\tau}\mathfrak{L}_2(t)dt,$$

and

$$H(s,t) = \begin{cases} \frac{(s-qt)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(1+s)(\tau-qt)^{(\alpha-1)}}{(2+\tau)\Gamma_q(\alpha)} - \frac{(1+s)(\tau-qt)^{(\alpha-2)}}{(2+\tau)\Gamma_q(\alpha-1)}, & 0 \leq t < s \\ -\frac{(1+s)(\tau-qt)^{(\alpha-1)}}{(2+\tau)\Gamma_q(\alpha)} - \frac{(1+s)(\tau-qt)^{(\alpha-2)}}{(2+\tau)\Gamma_q(\alpha-1)}, & s \leq t \leq \tau \end{cases}$$

3. Main Result

We now examine the existence of solutions to the infinite system (1.1)-(1.3) in the space c_0, which consists null sequences, having the norm $\|u\|_{c_0} = \sup\{|u_n| : n = 1, 2, 3...\}$. Let M_{c_0} represent the family of all nonempty and bounded subsets of the space c_0. For the space $(c_0, \|.\|_{c_0})$, Hausdorff MNC is given by

$$\chi(\mathcal{W}) = \lim_{n\to\infty} \{\sup_{x\in\mathcal{W}} \{\max_{i\geq n} |x_i|\}\},$$

where $\mathcal{W} \in M_{c_0}$ and $u(t) = (u_n(t))_{n=\infty}^{\infty} \in c_0$. We consider the following assumptions

(A1) The function $\eta_n : [0, \tau] \times c_0 \to \mathbb{R}$ are continuous with

$$|\eta_n(s, x(s)) - \eta_n(s, y(s))| \leq e^{\kappa} \sup_{n\geq 1}\{|x_i(t) - y_i(t)|; i \geq n\},$$

for $s \in [0, \tau], x(s) = (x_i(s)), y = (y_i(s)) \in c_0, i \in \mathbb{N}$.

(A2) The function $s \to \eta_i(s, 0)$ is bounded on $[0, \tau]$. Let

$$H_1 = \sup |\eta_i(s, 0)|; s \in [0, \tau], i \geq 1,$$

$$\lim_{i\to\infty} \eta_i(s, 0) = 0.$$

(A3) The functions $\xi_n, \zeta_n : [0, \tau] \times c_0 \to \mathbb{R}$ are continuous and there exists continuous functions $\nu_n : [0, \tau] \to \mathbb{R}$ such that

$$|\xi_n(s, p(s))| + |\zeta_n(s, p(s))| \leq |\nu_n(s)|, q = \sup\{|\nu_n(s)|; s \in [0, \tau]\},$$

for each $s \in [0, \tau]$,$p \in C(I, c_0)$ $p(s) = (p_i(s)) \in c_0; i = 1, 2, \ldots$.
Moreover, $\lim_{n\to\infty} \int_0^{\tau} |\nu_n(t)|dt = 0$.

(A4) There is $r_0 > 0$ satisfying the following inequality

$$\tau q + H^* e^{-\kappa} r\tau + H^* H_1 \tau < r.$$

Moreover, let $\tau H^* < 1$.

Put, $H^* = \sup_{(s,t)\in I\times I} |H(s, t)|$.

Theorem 3.1 *Under the above conditions, the infinite system (1.1)-(1.3) possesses at least one solution $p(t) = (p_i(t))_{i=1}^{\infty}$ such that $p(t) \in c_0 \forall t \in [0, \tau]$.*

Proof: Consider the following operator N on the space $C(I, c_0)$ defined as follows

$$(Np)(s) = K_i(s) + \int_0^{\tau} H_i(s, t)\eta_i(t, p(t))d_q t; \forall s \in [0, \tau],$$

where $C(I, c_0)$ denotes the space of all continuous functions on $[0, \tau]$ with values in c_0 space with the norm $||p|| = \sup\{||p(s)||_{c_0} : s \in [0, \tau]\}$.

$$K_i(s) = \frac{(1 + \tau - s)}{(2 + \tau)} \int_0^{\tau} \xi_i(t, p(t))dt + \frac{(1 + s)}{(2 + \tau)} \int_0^{\tau} \zeta_i(t, p(t))dt,$$

$$H_i(s, t) = H(s, t)\forall i.$$

We now prove that $(Np)(s) \in c_0$. For any fixed $s \in [0,\tau]$, we get

$$\begin{aligned}\|(Np)(s)\|_{c_0} &= \sup_{n\geq 1}|K_i(s) + \int_0^\tau H(s,t)\eta_i(t,p(t))d_qt| \\ &\leq \sup|K_i(s)| + \sup_{n\geq 1}\int_0^\tau |H(s,t)||\eta_i(t,p(t)) - \eta_i(t,0)|d_qt + \sup_{n\geq 1}\int_0^\tau |H(s,t)|\eta_i(t,0)|d_qt \\ &\leq \sup_{n\geq 1}\int_0^\tau (|\xi_i(t,p(t))| + |\zeta_i(t,p(t)|)dt + H^*e^{-\kappa}\int_0^\tau \sup\{|p_i(t)|\}d_qt + H^*H_1\tau.\end{aligned}$$

Hence

$$\|Np\| \leq \tau q + H^*e^{-\kappa}\|p\|\tau + H^*H_1\tau.$$

Next, we prove that Np is a continuous $[0,\tau]$ in c_0 space. Take $s, s_0 \in c_0$ and $\epsilon > 0$. From the continuity of $H(s,t)$, $\exists\delta > 0$ such that $|H(s,t) - H(s_0,t)| < \frac{\epsilon}{\|p\|\tau e^{-\kappa}+H_1\tau}$,

$$\begin{aligned}\|(Np)(s) - (Np)(s_0)\|_{c_0} &\leq |\frac{s_0 - s}{2+\tau}|(\|\int_0^\tau \xi_i(t,p(t))dt\| + \|\int_0^\tau \zeta_i(t,p(t)dt\| \\ &+ \|\int_0^\tau (H(s,t) - H(s_0,t))(\eta_i(t,p(t) - \eta_i(t,0) + \eta_i(t,0))d_qt\| \\ &\leq \sup_{n\geq 1}\int_0^\tau (|\xi_i(t,p(t))| + |\zeta_i(t,p(t))|)dt + \epsilon \\ &\leq \sup_{n\geq 1}\int_0^\tau (|\nu_i(t)|dt + \epsilon.\end{aligned}$$

Thus Np is a continuous map from $[0,\tau]$ into c_0.
Moreover, N maps the ball B_{r_0} into itself, where r_0 is the constant obtained from $(A4)$ and $B_{r_0} = \{p \in C(I,c_0); \|p\| \leq r, p(0) - p'(0) = \int_0^\tau \xi(t,p(t))dt, p(\tau) + p'(\tau) = \int_0^\tau \zeta(t,p(t))dt\}$.

Again, for $\epsilon > 0, s \in [0,\tau]$ and $p, x \in B_{r_0}$ with $\|p - x\|_{C(I,c_0)} < \epsilon$

$$\begin{aligned}&\|(Np)(s) - (Nx)(s)\|_{c_0} \\ &= \|\frac{1+\tau+s}{2+\tau}\int_0^\tau \xi_i(t,p(t))dt + \int_0^\tau \zeta_i(t,p(t))dt - \frac{1+\tau+s}{2+\tau}\int_0^\tau \xi_i(t,x(t))dt - \frac{1+s}{2+\tau}\zeta_i(t,x(t))dt \\ &+ \int_0^\tau H(s,t)\eta_i(t,p(t))d_qt - \int_0^\tau H(s,t)\eta_i(t,x(t))d_qt\| \\ &\leq \sup\int_0^\tau |\xi_i(t,p(t) - \xi_i(t,x(t))|dt + \sup\int_0^\tau |\zeta_i(t,p(t)) - \zeta_it,x(t))|dt \\ &+ (\sup\int_0^\tau |H(s,t)||\eta_i(t,p(t)) - \eta_i(t,x(t))|d_qt \\ &\leq \sup\int_0^\tau l_1(\xi,\epsilon)dt + \sup\int_0^\tau l_2(\zeta,\epsilon)dt + H^*\int_0^\tau e^{-\kappa}\sup|p_i(t) - x_i(t)|d_qt \\ &\leq \sup\int_0^\tau l_1(\xi,\epsilon)dt + \sup\int_0^\tau l_2(\zeta,\epsilon)dt + H^*(s,t)e^{-\kappa}\|p - x\|\tau,\end{aligned}$$

where

$$l_1(\xi,\epsilon) = \sup\{|\xi_i(s,p) - \xi_i(s,x)| : s \in [0,\tau], p.x \in c_0, \|p - x\|_{c_0} < \epsilon\}$$

$$l_2(\zeta_i,\epsilon) = \sup\{|\zeta_i(s,p) - \zeta_i(s,x)| : s \in [0,\tau], p.x \in c_0, \|p - x\|_{c_0} < \epsilon\}.$$

Also, $l_1(\xi_i.\epsilon), l_2(\zeta_i,\epsilon) \to 0$ as $\xi_i.\zeta_i$ are coninuous. Thus, continuity of N is allowed. Take K as a non empty set in B_{r_0}. Then

$$\begin{aligned}\chi_{c_0}(NK)(s) &= \lim_{n\to\infty}\{\sup_{y\in K}(\max_{i\geq n}|K_i(s)+\int_0^\tau H(s,t)\eta_i(t,p(t))d_qt|)\}\\ &\leq \lim_{n\to\infty}\sup_{y\in K}(\max_{i\geq n}[|K_i(s)|+\int_0^\tau |H(s,t)(\eta_i(t,p(t))-\eta_i(t,0)+\eta_i(t,0))|d_qt])\\ &\leq \lim_{n\to\infty}\{\sup_{p\in K}\max_{i\geq n}[\int_0^\tau |\nu_i(t)|dt+H^*e^{-\kappa}\int_0^\tau \sup_{i\geq n}|p_i(t)|d_qt+H^*\int_0^\tau |\eta_i(t,0)|d_qt]\}.\end{aligned}$$

Therefore,

$$\chi_{C(I,C_0)(NK)} \leq \tau H^*e^{-\kappa}\sup_{s\in I}\lim_{n\to\infty}\sup\{\sup_{y\in K}(\max_{i\geq n}|p_i(s)|)\}.$$

Since $H^*\tau<1$, we have

$$\kappa+\ln(\chi_{C(I,c_0)})NK \leq \chi_{C(I,c_0)}(X).$$

□

Thus, Theorem (2.2) is satisfied with $\mathfrak{F}(s)=\ln(s)$ and $\psi(s)=\kappa$. Therefore, N has a fixed point in $C(I,c_0)$, which is a solution of the system. This completes the proof.

4. Example

Example 4.1 Let us consider the following equation

$$({}^CD^{\frac{3}{2}}_{\frac{1}{4}}p_n)(s) = e^{-\kappa}\frac{\sin ns}{n+1}p_n(s)+\frac{\cos s}{n}; s\in[0,\tau], 1<\alpha\leq 2. \tag{4.1}$$

$$p_n(0)-p_n'(0)=\int_0^\tau \xi_n(s,p(s))ds, \tag{4.2}$$

$$p_n(\tau)+p_n(\tau)=\int_0^\tau \zeta_n(s,p(s))ds, \tag{4.3}$$

where

$$\xi_n(s,p(s)) = \begin{cases}\frac{1}{3}[1-n|s-s_0|]\tanh x_1, & \text{if } |s-s_0|\leq\frac{1}{n},\\ & 0, \text{ otherwise.}\end{cases}$$

$$\zeta_n(s,p(s)) = \begin{cases}\frac{2}{3}[1-n|s-s_0|]\tanh x_2, & \text{if } |s-s_0|\leq\frac{1}{n},\\ & 0, \text{ otherwise.}\end{cases}$$

for fixed $s_0\in[0,\tau]$.
Here,

$$\eta_n(s,p)=e^{-\kappa}\frac{\sin ns}{n+1}p_n(s)+\frac{\cos s}{n} \text{ for } s\in[0,\tau].$$

Again,

$$|\eta(s,p(s))-\eta(s,x(s))| \leq e^{-\kappa}\{\sup_{n\geq 1}|p_i(t)-x_i(t)| : i\geq n\}$$

$$|\xi_n(s,p(s))|+|\zeta_n(s,p(s))| \leq |\nu(s)|,$$

where

$$\nu_n(s)=\begin{cases}1-n|s-s_0|, & \text{if } |s-s_0|\leq\dfrac{1}{n},\\ 0, & \text{otherwise.}\end{cases}$$

$$q=1, \lim_n \int_0^\tau |\nu_n(t)|dt=0.$$

The condition in $(A4)$ has the structure $\tau+H^*e^{-\kappa}r\tau+H^*\tau<r$, which has a solution in the positive real numbers. Consequently, using the Theorem 3.1, the system (4.1)-(4.3) has a solution in $C(I,c_0)$.

Acknowledgments

The first author (Dipankar Patgiri) acknowledged the Department of Science and Technology (DST), Govt. of India, for awarding INSPIRE Fellowship award (Code IF220237) in 2023.

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