Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 6** : 1–9.
ISSN-0037-8712
doi:10.5269/bspm.78910

$\mathcal{N}$-r-Ideals and Semi $\mathcal{N}$-r-Ideals of Commutative $\mathbb{Z}_2$-Graded Rings

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ABSTRACT: Let R be a commutative ring with nonzero unity 1. This article introduces and investigates new classes of ideals in $\mathbb{Z}_2$-graded rings, building on the previously established notion of r-ideals. Using the function $\mathcal{N}: R \to R_0$, defined by $\mathcal{N}(x) = x_0^2 - x_1^2$ for $x = x_0 + x_1 \in R$, we define and study $\mathcal{N}$-r ideals and semi $\mathcal{N}$-r-ideals. A proper ideal I is $\mathcal{N}$-r-ideal if $xy \in I$ implies $\mathcal{N}(x) \in I$ or $y \in zd(R)$, while it is semi $\mathcal{N}$-r-ideal if $x^2 \in I$ implies $\mathcal{N}(x) \in I$ or $x \in zd(R)$, where $zd(R)$ is the set of zero divisors of R. Fundamental properties of these ideals are explored, including their relationships to existing structures in graded ring theory. These results extend the understanding of ideal theory in the context of $\mathbb{Z}_2$-graded rings and offer new perspectives for future research.

Keywords: r-ideals, semi-r-ideals, $\mathcal{N}$-prime ideals.

Contents

1. Introduction

Recall that the set of all zero divisors of R is denoted by $zd(R)$, and is defined by $zd(R) = \{x \in R :$ there exists a nonzero $y \in R$ with $xy = 0\}$. In [7], the concept of r-ideals has been established and investigated. A proper ideal I of a ring R is said to be an r-ideal if whenever $x, y \in R$ with $xy \in I$, then either $x \in I$ or $y \in zd(R)$. On the other hand, the concept of semi r-ideals has been introduced and studied in [5]. A proper ideal I of a ring R is said to be a semi r-ideal if whenever $x \in R$ with $x^2 \in I$, then either $x \in I$ or $x \in zd(R)$.

The study of graded rings and their ideal structures has been a central theme in modern algebra, offering a rich framework for exploring connections between ring theory, homological algebra, and algebraic geometry. A graded ring is a ring R that is decomposed into a direct sum $R = \bigoplus_{g\in G} R_g$ with $R_g R_h \subseteq R_{gh}$, for all $g, h \in G$, where G is a group and R_g is an additive subgroup of R, for all $g \in G$. The elements of each component R_g are called homogeneous elements, and the set of all homogeneous elements is $\cup_{g\in G} R_g$ which is denoted by $h(R)$. For more terminology, one can see [4,8]. In this context, a $\mathbb{Z}_2$-graded ring is a graded ring where G is $\mathbb{Z}_2 = \{0, 1\}$, with each element of R being uniquely decomposed as $x = x_0 + x_1$, where $x_0 \in R_0$ and $x_1 \in R_1$.

Among the most prominent classes of graded rings are $\mathbb{Z}_2$-graded rings, which have significant applications in areas such as representation theory and quantum mechanics. In this context, the interplay between grading and ideal theory has inspired various extensions of classical notions, including the concept of r-ideals and semi r-ideals.

In [1], the notion of $\mathcal{N}$-prime ideals was introduced for $\mathbb{Z}_2$-graded rings using the norm function $\mathcal{N}_R : R \to R_0$, defined by

$$\mathcal{N}_R(x) = x_0^2 - x_1^2,$$

where $x = x_0 + x_1$ is the homogeneous decomposition of $x \in R$. When no confusion occurs, we replace $\mathcal{N}_R$ by $\mathcal{N}$. It is important to note that the function $\mathcal{N}$ was introduced for the first time in ([3], Theorem 5.8). A proper ideal I is $\mathcal{N}$-prime if $xy \in I$ implies $\mathcal{N}(x) \in I$ or $\mathcal{N}(y) \in I$. This novel approach connects the grading structure with the functional mapping $\mathcal{N}$, leading to a deeper understanding of graded prime ideals. Not only in algebra, even in other branches of Mathematics, the norm function has a significance

Submitted September 11, 2025. Published February 17, 2026
2020 *Mathematics Subject Classification:* 13A02, 13A15.

role in analyzing the algebraic structure or other mathematical structures of a $\mathbb{Z}_2$-graded ring like the graphical structure as presented in [6]. The reader should pay attention to that the norm function can be defined for graded rings which are not necessarily $\mathbb{Z}_2$-graded. For example, if R is a first strongly $\mathbb{Z}_4$-graded ring (for first strongly graded rings and modules, we refer the reader to [9]) whose support is $\langle 2\rangle$, then the norm function is defined on R by $\mathcal{N}(r) = r_0^2 - r_2^2$, where $r = r_0 + r_1 + r_2 + r_3 = r_0 + r_2$ is the unique decomposition of r in the $\mathbb{Z}_4$-graded ring R. The existence of the norm function for a wide range of graded rings will definitely leads to new releases in this area of research.

In this article, we extend this framework by introducing and studying two new classes of ideals: $\mathcal{N}$-r-ideals and semi $\mathcal{N}$-r-ideals. These concepts generalize the classical notion of r-ideals in a way that respects the grading structure. Specifically, A proper ideal I is $\mathcal{N}$-r-ideal if $xy \in I$ implies $\mathcal{N}(x) \in I$ or $y \in zd(R)$. A proper ideal I is semi $\mathcal{N}$-r-ideal if $x^2 \in I$ implies $\mathcal{N}(x) \in I$ or $x \in zd(R)$.

These definitions naturally arise from the graded structure of the ring and offer a new perspective on ideal theory in $\mathbb{Z}_2$-graded rings. This work explores the fundamental properties of these new ideals, their interrelations, and their connections to r-ideals.

This study not only enriches the theory of $\mathbb{Z}_2$-graded rings but also lays the groundwork for further exploration of functional mappings in algebraic structures.

2. Preliminaries on the norm function $\mathcal{N}$ and $\mathcal{N}$-prime ideals

In this article, we focus on $\mathbb{Z}_2$-graded ring $R = R_0 \bigoplus R_1$ with $R_iR_j \subseteq R_{i+j}$, for all $i, j \in \mathbb{Z}_2$, where R_0 and R_1 are additive subgroups of R. Actually, R_0 is a subring of R and $1 \in R_0$. Now, for every $x \in R$, x is written uniquely as $x = x_0 + x_1$, for some $x_0 \in R_0$ and $x_1 \in R_1$. Then for every $x \in R$, define $\mathcal{N}(x) = x_0^2 - x_1^2$.

Theorem 2.1 *[1] Let R be a $\mathbb{Z}_2$-graded ring. Then*

1. *$\mathcal{N}(x) \in R_0$, for every $x \in R$. Hence, $\mathcal{N}$ is a function from R to R_0.*
2. *$\mathcal{N}(0) = 0$.*
3. *$\mathcal{N}(1) = 1$.*
4. *$\mathcal{N}(xy) = \mathcal{N}(x)\mathcal{N}(y)$, for every $x, y \in R$.*

Theorem 2.2 *[1] Let R be a $\mathbb{Z}_2$-graded ring and $x \in h(R)$. Then*

$$\mathcal{N}(x) = \begin{cases} x^2, & x \in R_0 \\ -x^2, & x \in R_1 \end{cases}$$

Lemma 2.1 *[1] Let R be a $\mathbb{Z}_2$-graded ring and I be an ideal of R. Then $\mathcal{N}(I) \subseteq I \bigcap R_0 \subseteq I$.*

An ideal I of a graded ring R is said to be a graded ideal if $I = \bigoplus_{g\in G}(I \bigcap R_g)$, i.e., whenever $x \in I$, we have $x_g \in I$, for all $g \in G$. Indeed, not every ideal of a graded ring is a graded ideal, see [8]. Moreover, if I is a graded ideal of a graded ring R, then R/I is graded by $(R/I)_g = (R_g + I)/I$, for all $g \in G$.

Lemma 2.2 *[1] Let R be a $\mathbb{Z}_2$-graded ring and J be a graded ideal of R. Then $\mathcal{N}_{R/J}(x+J) = \mathcal{N}_R(x)+J$, for every $x \in R$.*

Definition 2.1 [1] Let R be a $\mathbb{Z}_2$-graded ring and I be a proper ideal of R. Then I is said to be an $\mathcal{N}$-prime ideal of R if whenever $x, y \in R$ such that $xy \in I$, then either $\mathcal{N}(x) \in I$ or $\mathcal{N}(y) \in I$.

Theorem 2.3 *[1] Let R be a $\mathbb{Z}_2$-graded ring and I be an ideal of R. If I is a prime ideal of R, then I is an $\mathcal{N}$-prime ideal of R.*

Indeed, ([1], Example 5) introduces an $\mathcal{N}$-prime ideal that is not a prime ideal. So, the converse of Theorem 2.3 is not necessarily true.

By Theorem 2.1 (4), $\mathcal{N}$ is a multiplicative function. On the other hand, $\mathcal{N}$ is not additive function by ([1], Remark 4). However, we have the following in general:

Theorem 2.4 *[1] Let R be a $\mathbb{Z}_2$-graded ring and $x, y \in R$. Then*

$$\mathcal{N}(x+y) = \mathcal{N}(x) + \mathcal{N}(y) + 2(x_0y_0 - x_1y_1).$$

Note that, if R is of characteristic 2, $\mathcal{N}$ will be an additive function, and hence $\mathcal{N}$ is a ring homomorphism from R to R_0. As a new result, we prove that if $\mathcal{N}$ is one to one, then R is of characteristic 2, and the converse is not necessarily true.

Theorem 2.5 *Let R be a $\mathbb{Z}_2$-graded ring. If $\mathcal{N}$ is one to one, then R is of characteristic 2.*

Proof: Let $x \in R$. Then $x = x_0 + x_1$, for some $x_0 \in R_0$ and $x_1 \in R_1$, and then $-x = -x_0 - x_1$, which gives that $\mathcal{N}(x) = x_0^2 - x_1^2 = \mathcal{N}(-x)$, and hence $x = -x$ that is $2x = 0$. Thus, $char(R) = 2$. □

As a counterexample to the converse of Theorem 2.5, we present the next example.

Example 2.1 Consider $R = \mathbb{Z}_2[x]/\langle x^2\rangle$. Let $f(x) + \langle x^2\rangle \in R$. Then by division algorithm, $f(x) = x^2q(x) + r(x)$, for some $q(x), r(x) \in \mathbb{Z}_2[x]$ with $r(x) = 0$ or $deg(r(x)) = 1$, that is $r(x) = a + bx$, for some $a, b \in \mathbb{Z}_2$. So, $f(x) + \langle x^2\rangle = a + bx + \langle x^2\rangle$. Now, R is $\mathbb{Z}_2$-graded by $R_0 = \mathbb{Z}_2 + \langle x^2\rangle/\langle x^2\rangle$ and $R_1 = \mathbb{Z}_2x + \langle x^2\rangle/\langle x^2\rangle$. So, for any $f(x)+\langle x^2\rangle \in R$, $(f(x)+\langle x^2\rangle)_0 = a+\langle x^2\rangle$ and $(f(x)+\langle x^2\rangle)_1 = bx+\langle x^2\rangle$, for some $a, b \in \mathbb{Z}_2$, and then $\mathcal{N}(f(x) + \langle x^2\rangle) = a^2 + \langle x^2\rangle = a + \langle x^2\rangle$. Clearly, R is of characteristic 2 but $\mathcal{N}$ is not one to one as $\mathcal{N}(x + \langle x^2\rangle) = 0 + \langle x^2\rangle = \mathcal{N}(0 + \langle x^2\rangle)$ with $x + \langle x^2\rangle \neq 0 + \langle x^2\rangle$.

Corollary 2.1 *Let R be a $\mathbb{Z}_2$-graded ring. If $\mathcal{N}$ is one to one, then $\mathcal{N}$ is a ring isomorphism between R and R_0. Moreover, if R is a finite ring, then R has the trivial gradation by $\mathbb{Z}_2$.*

One can see that ([1], Example 3) introduces a case where $\mathcal{N}$ is one to one.

3. $\mathcal{N}$-r-Ideals

Definition 3.1 Let R be a $\mathbb{Z}_2$-graded ring and I be a proper ideal of R. Then I is said to be an $\mathcal{N}$-r- ideal of R if whenever $x, y \in R$ such that $xy \in I$, then either $\mathcal{N}(x) \in I$ or $y \in zd(R)$.

Theorem 3.1 *Let R be a $\mathbb{Z}_2$-graded ring. If I is an r-ideal of R, then I is an $\mathcal{N}$-r-ideal of R.*

Proof: Let $x, y \in R$ such that $xy \in I$. Then either $x \in I$ or $y \in zd(R)$. If $x \in I$, then by Lemma 2.1, $\mathcal{N}(x) \in \mathcal{N}(I) \subseteq I$. Hence, I is an $\mathcal{N}$-r-ideal of R. □

The next example shows that an $\mathcal{N}$-r-ideal is not necessarily an r-ideal.

Example 3.1 Consider $R = K[x, y]$, where K is a field, and $G = \mathbb{Z}_2$. Then R is trivially G-graded by $R_0 = R$ and $R_1 = \{0\}$. Consider the graded ideal $I = \langle xy\rangle$ of R. Then R/I is a G-graded ring by $(R/I)_n = (R_n + I)/I$, for all $n \in \mathbb{Z}_2$. Consider the prime ideals $P = \langle x + I\rangle$ and $Q = \langle y + I\rangle$ of R/I. We show that $zd(R/I) = P \bigcup Q$. Let $f + I \in zd(R/I)$. Then, there exists $g + I \in R/I$ such that $g + I \neq 0 + I$ and $(f + I)(g + I) = 0 + I$. Thus, $fg \in I$ with $g \notin I$. So, $fg = xyh$, for some $h \in R$, and hence x divides fg and y divides fg, which implies x divides f or x divides g, and y divides f or y divides g. If x divides g and y divides g, then xy divides g, and then $g \in I$, which is a contradiction. So, x divides f or y divides f, or only one of x or y divides g. If x only divides g, then $g = \mu x$ for some $\mu \in R$. Thus, $fg = xyh$ implies $x\mu f = xyh$. Canceling x from both sides of the last equality yields $\mu f = yh$. Now, y divides μf and y does not divide μ (otherwise y will divide g, which contradicts that x only divides g). Therefore, y divides f. In the same manner, we can show that if y only divides g, then x divides f. So in either case, x or y divides f. This implies $f + I \in P \bigcup Q$. Thus, $zd(R/I) \subseteq P \bigcup Q$. Let $f + I \in P \bigcup Q$. Then $f + I \in P$ or $f + I \in Q$. If $f + I \in P$, then $f + I = (x + I)(h + I) = xh + I$, for some $h \in R$, and then $f - xh \in I$ which implies $f - xh = xyt$, for some $t \in R$, and hence $f = xh + xyt$. Now, $yf = xy(h + yt) \in I$, and thus $(y + I)(f + I) = yf + I = 0 + I$ with $y + I \neq 0 + I$ as $y \notin I$, which means that $f + I \in zd(R/I)$. Similarly, if $f + I \in Q$, then $f + I \in zd(R/I)$. We conclude that, $zd(R/I) = P \bigcup Q$. Now, we show that P^2 is an $\mathcal{N}_{R/I}$-r-ideal of R/I. Let $f + I, g + I \in R/I$ such that

$(f+I)(g+I) \in P^2$. Assume that $g+I \notin zd(R/I)$. Then $g+I \notin P$. Since $(f+I)(g+I) \in P^2 \subseteq P$ and P is prime, we have $f+I \in P$, which implies $\mathcal{N}_{R/I}(f+I) = \mathcal{N}_R(f)+I = f^2+I = (f+I)^2 \in P^2$. Hence, P^2 is an $\mathcal{N}_{R/I}$-r-ideal of R/I. On the other hand, P^2 is not an r-ideal of R/I since $x+I, x+y+I \in R/I$ such that $(x+I)(x+y+I) = x^2+xy+I = x^2+I = (x+I)^2 \in P^2$, $x+I \notin P^2$ and $x+y+I \notin zd(R/I)$.

The ideals in the next two examples are r-ideals, so they are $\mathcal{N}$-r-ideals by Theorem 3.1. However, we introduce a classical proof using the definition.

Example 3.2 Let R be a $\mathbb{Z}_2$-graded ring. Then $I = \{0\}$ is an $\mathcal{N}$-r-ideal of R. Let $x, y \in R$ such that $xy \in I$ and $y \notin zd(R)$. Then $xy = 0$. So, $x = 0$ because $y \notin zd(R)$, this implies $\mathcal{N}(x) = \mathcal{N}(0) = 0 \in I$.

Example 3.3 Let R be a $\mathbb{Z}_2$-graded ring and $0 \neq x \in R$. Then $Ann(x) = \{r \in R : rx = 0\}$ is an $\mathcal{N}$-r-ideal of R. Let $a, b \in R$ such that $ab \in Ann(x)$ and $b \notin zd(R)$. Then $abx = 0$, and then $ax = 0$, which implies $a \in Ann(x)$, and hence $\mathcal{N}(a) \in \mathcal{N}(Ann(x)) \subseteq Ann(x)$ by Lemma 2.1.

Even though the next result is an immediate consequence of the definition of $\mathcal{N}$-r-ideals, it is an important fact since it emphasizes that the components of the $\mathcal{N}$-r-ideals are entirely consisting of zero divisors.

Theorem 3.2 *Let R a $\mathbb{Z}_2$-graded ring and I an $\mathcal{N}$-r-ideal of R. Then, $I \subseteq zd(R)$.*

Proof: Let $x \in I$. Then $1.x = x \in I$, and then since I is an $\mathcal{N}$-r-ideal and $\mathcal{N}(1) = 1 \notin I$, we have that $x \in zd(R)$. Hence, $I \subseteq zd(R)$. □

The next two examples show that $\mathcal{N}$-r-ideals and $\mathcal{N}$-prime ideals are completely different concepts:

Example 3.4 Consider $R = \mathbb{Z}_6[i]$. Then R is $\mathbb{Z}_2$-graded by $R_0 = \mathbb{Z}_6$ and $R_1 = i\mathbb{Z}_6$. We have $I = \{0\}$ is an $\mathcal{N}$-r-ideal by Example 3.2. However, I is not $\mathcal{N}$-prime since $2, 3 \in R$ with $2 \cdot 3 \in I$ but $\mathcal{N}(2) = 4 \notin I$ and $\mathcal{N}(3) = 3 \notin I$.

Example 3.5 Consider the graded ring given in ([1], Example 5). $I = pR$, where p is a prime integer, is an $\mathcal{N}$-prime ideal. However, if I is an $\mathcal{N}$-r-ideal, then by Theorem 3.2, $I \subseteq zd(R) = \{0\}$, a contradiction. Hence, I is not an $\mathcal{N}$-r-ideal.

Let R be a ring, $I \subseteq R$ an ideal, and $a \in R$. The *colon ideal* or *ideal quotient* $(I : a)$ is defined by $(I : a) = \{r \in R \mid ra \in I\}$. For more details, one can look at ([2], Chapter 1, Section 7).

Theorem 3.3 *Let R a $\mathbb{Z}_2$-graded ring and I an $\mathcal{N}$-r-ideal of R. Then for every $a \in R$, either $\mathcal{N}(a) \in I$ or $(I : a) \subseteq zd(R)$.*

Proof: Let $a \in R$ such that $\mathcal{N}(a) \notin I$. Assume that $b \in (I : a)$. Then $ab \in I$, and then since I is an $\mathcal{N}$-r-ideal, we have that $b \in zd(R)$. Hence, $(I : a) \subseteq zd(R)$. □

The element $a \in R$ is said to be regular if it is not a zero divisor. The set of all regular elements of R is denoted by $r(R)$. So, $R = zd(R) \bigcup r(R)$. In terms of the regular elements, we can reshape Definition 3.1 as follows: A proper ideal I of a $\mathbb{Z}_2$-graded ring R is an $\mathcal{N}$-r-ideal if whenever $ab \in I$, and a is regular, then $\mathcal{N}(b) \in I$.

Theorem 3.4 *Let R be a $\mathbb{Z}_2$-graded ring. If P is an $\mathcal{N}$-r-ideal of R, then $\mathcal{N}(aR \bigcap P) \subseteq \mathcal{N}(a)P$, for every regular $a \in R$.*

Proof: Let $a \in R$ be regular. Assume that $y \in \mathcal{N}(aR \bigcap P)$. Then there exists $x \in aR \bigcap P$ such that $\mathcal{N}(x) = y$. Now, $x = az \in P$, for some $z \in R$. Since $za \in P$ with a is regular, we get $\mathcal{N}(z) \in P$. Thus, $y = \mathcal{N}(x) = \mathcal{N}(az) = \mathcal{N}(a)\mathcal{N}(z) \in \mathcal{N}(a)P$. □

The next theorem gives a partial converse of Theorem 3.4.

Theorem 3.5 *Let R a $\mathbb{Z}_2$-graded ring and P a proper ideal of R. Suppose that $\mathcal{N}(aR\bigcap P) \subseteq \mathcal{N}(a)P$, for every regular $a \in R$. If $\mathcal{N}(r(R)) \subseteq r(R)$, then P is an $\mathcal{N}$-r-ideal of R.*

Proof: Let $a, b \in R$ such that $ab \in P$ and a is regular. Then $ab \in aR\bigcap P$, this gives $\mathcal{N}(a)\mathcal{N}(b) \in \mathcal{N}(aR\bigcap P) \subseteq \mathcal{N}(a)P$. Thus, $\mathcal{N}(a)\mathcal{N}(b) = \mathcal{N}(a)z$, for some $z \in P$. So, $\mathcal{N}(a)(\mathcal{N}(b) - z) = 0$, which implies $\mathcal{N}(b) = z \in P$ since $\mathcal{N}(a)$ is regular. Consequently, P is an $\mathcal{N}$-r-ideal of R. □

The next example shows that the assumption $\mathcal{N}(r(R)) \subseteq r(R)$ in Theorem 3.5 does not hold in general. For a $\mathbb{Z}_2$-graded ring R, $\aleph = \{x \in R : \mathcal{N}(x) = 0\}$ [1].

Example 3.6 Consider $R = M_2(\mathbb{R})$ (the ring of all 2×2 matrices with real entries). Then R is $\mathbb{Z}_2$-graded by:

$R_0 = \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{bmatrix}$ and $R_1 = \begin{bmatrix} 0 & \mathbb{R} \\ \mathbb{R} & 0 \end{bmatrix}$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R$. Then $A_0 = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and $A_1 = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$, and then $\mathcal{N}(A) = A_0^2 - A_1^2 = \begin{bmatrix} a^2 - bc & 0 \\ 0 & d^2 - bc \end{bmatrix}$. Now, choose $A = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$. Then, $A \in r(R)$ as A is invertible. On the other hand, $\mathcal{N}(A) = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \notin r(R)$ as $\begin{bmatrix} \alpha & 0 \\ \beta & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, for all $\alpha, \beta \in \mathbb{R}$. Hence, $\mathcal{N}(r(R)) \nsubseteq r(R)$. Note that, $\aleph = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a^2 = bc = d^2\right\} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as $\begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix} \in \aleph$.

Lemma 3.1 *Let R be a $\mathbb{Z}_2$-graded ring such that $\aleph = \{0\}$. Then $\mathcal{N}(r(R)) \subseteq r(R)$.*

Proof: Let $y \in \mathcal{N}(r(R))$. Then there exists $x \in r(R)$ such that $\mathcal{N}(x) = y$. Let $z \in R$ such that $yz = 0$. Then $\mathcal{N}(x)z = 0$. So, $0 = \mathcal{N}(0) = \mathcal{N}(\mathcal{N}(x)z) = \mathcal{N}(\mathcal{N}(x))\mathcal{N}(z) = \mathcal{N}(x^2)\mathcal{N}(z) = \mathcal{N}(x^2z)$. Therefore, $x^2z \in \aleph = \{0\}$. Since $x \in r(R)$, we obtain $z = 0$. Hence, $y \in r(R)$. □

Now, we can state the following result whose proof comes from Theorems 3.4 and 3.5, and Lemma 3.1.

Theorem 3.6 *Let R be a $\mathbb{Z}_2$-graded ring such that $\aleph = \{0\}$ and P be a proper ideal of R. Then P is an $\mathcal{N}$-r-ideal of R if and only if $\mathcal{N}(aR\bigcap P) \subseteq \mathcal{N}(a)P$, for every regular $a \in R$.*

Proposition 3.1 *Let R be a $\mathbb{Z}_2$-graded ring. Then,*

$$\mathcal{N}(r(R)\bigcap h(R)) \subseteq r(R)\bigcap h(R).$$

Proof: Let $y \in \mathcal{N}(r(R)\bigcap h(R))$. Then there exists $x \in r(R)\bigcap h(R)$ such that $\mathcal{N}(x) = y$. Since $x \in h(R)$, $y = \mathcal{N}(x) = x^2$ or $-x^2$. Assume that $y = x^2$. Let $z \in Ann(y)$. Then $0 = zy = zx^2 = zx.x$, which gives $zx \in Ann(x) = \{0\}$. So, $zx = 0$. Again, $z \in Ann(x) = \{0\}$, i.e., $z = 0$. Hence, $Ann(y) = \{0\}$, i.e., $y \in r(R)$. Similarly, if $y = -x^2$, then $y \in r(R)$. On the other hand, $\mathcal{N}(x) \in h(R)$. So, $y \in r(R)\bigcap h(R)$. □

Theorem 3.7 *Let R a $\mathbb{Z}_2$-graded ring and P a proper ideal of R. Then P is an $\mathcal{N}$-r-ideal of R if and only if $\mathcal{N}((P : a)) \subseteq P$, for every regular $a \in R$.*

Proof: Suppose that P is an $\mathcal{N}$-r-ideal of R. Let $a \in r(R)$ and $y \in \mathcal{N}((P:a))$. Then there exists $x \in (P:a)$ such that $\mathcal{N}(x) = y$. Now, $xa \in P$, so $y = \mathcal{N}(x) \in P$. Conversely, let $a, b \in R$ such that $ab \in P$ and a is regular. Then $b \in (P:a)$. Since $\mathcal{N}(b) \in \mathcal{N}((P:a)) \subseteq P$, we have P is an $\mathcal{N}$-r-ideal of R. □

By Lemma 2.1, $\mathcal{N}(P) \subseteq P$, for every ideal P of R. Moreover, if P is an $\mathcal{N}$-r-ideal of R, then we can put something between $\mathcal{N}(P)$ and P:

Corollary 3.1 *Let R be a $\mathbb{Z}_2$-graded ring. If P is an $\mathcal{N}$-r-ideal of R, then $\mathcal{N}(P) \subseteq \mathcal{N}((P:a)) \subseteq P$, for every regular $a \in R$.*

Proof: Let $a \in r(R)$. Then $P \subseteq (P:a)$, and then $\mathcal{N}(P) \subseteq \mathcal{N}((P:a)) \subseteq P$ by Theorem 3.7. □

Theorem 3.8 *Let R a Z_2-graded ring and P a proper ideal of R. Then P is an $\mathcal{N}$-r-ideal of R if and only if whenever I and J are ideals of R such that $IJ \subseteq P$ and $J \bigcap r(R) \neq \emptyset$, then $\mathcal{N}(I) \subseteq P$.*

Proof: Suppose that P is an $\mathcal{N}$-r-ideal of R. Let I and J be two ideals of R such that $IJ \subseteq P$ and $J \bigcap r(R) \neq \emptyset$. Without loss of generality, suppose $I \neq \{0\}$. Then there exists $b \in J \bigcap r(R)$. Let $y \in \mathcal{N}(I)$. Then there exists $a \in I$ such that $y = \mathcal{N}(a)$. Now, $ab \in P$, so $y = \mathcal{N}(a) \in P$. Conversely, let $a, b \in R$ such that $ab \in P$ and $b \in r(R)$. Set $I = Ra$ and $J = Rb$. Then I and J are ideals of R such that $IJ \subseteq P$ and $J \bigcap r(R) \neq \emptyset$, which implies $\mathcal{N}(I) \subseteq P$. In particular, $\mathcal{N}(a) \in P$. We conclude that, P is an $\mathcal{N}$-r-ideal of R. □

Theorem 3.9 *Let R be a Z_2-graded ring. If R is an integral domain, then $\{0\}$ is the only $\mathcal{N}$-r-ideal of R.*

Proof: As stated in Example 3.2, $\{0\}$ is an $\mathcal{N}$-r-ideal of R. Let P be a nonzero $\mathcal{N}$-r-ideal of R. Then there exists $0 \neq b \in P$. Since R is an integral domain, $b \in r(R)$. Now, $1.b \in P$, so we have $\mathcal{N}(1) = 1 \in P$, a contradiction. □

Theorem 3.10 *Let R be a $\mathbb{Z}_2$-graded ring. If P_1 and P_2 are $\mathcal{N}$-r-ideals of R, then $P_1 \bigcap P_2$ is an $\mathcal{N}$-r-ideal of R.*

Proof: Let $a, b \in R$ such that $ab \in P_1 \bigcap P_2$ and $a \in r(R)$. Then $ab \in P_1$. Since P_1 is an $\mathcal{N}$-r-ideal, $\mathcal{N}(b) \in P_1$. Similarly, $\mathcal{N}(b) \in P_2$ and hence $\mathcal{N}(b) \in P_1 \bigcap P_2$. Therefore, $P_1 \bigcap P_2$ is an $\mathcal{N}$-r-ideal of R. □

Definition 3.2 Let R be a $\mathbb{Z}_2$-graded ring and P be a proper ideal of R. Then P is said to be a semi $\mathcal{N}$-r-ideal of R if whenever $x \in R$ such that $x^2 \in P$, then either $\mathcal{N}(x) \in P$ or $x \in zd(R)$.

Clearly, every $\mathcal{N}$-r-ideal is a semi $\mathcal{N}$-r-ideal. On the other hand, the next example shows that a semi $\mathcal{N}$-r-ideal is not necessarily an $\mathcal{N}$-r-ideal.

Example 3.7 Consider $R = \mathbb{Z}[i]$. Then R is $\mathbb{Z}_2$-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Consider the ideal $P = 6R$ of R. Let $x \in R$ such that $x^2 \in P$. Then $x^2 = 6(a + ib)$, for some $a, b \in \mathbb{Z}$. Thus, $(\mathcal{N}(x))^2 = \mathcal{N}(x^2) = 36(a^2 + b^2)$. Now, $2, 3$ divide $(\mathcal{N}(x))^2$, so $2, 3$ divide $\mathcal{N}(x)$ which implies 6 divides $\mathcal{N}(x)$. That is, $\mathcal{N}(x) \in P$. Hence, P is a semi $\mathcal{N}$-r-ideal of R. On the other hand, P is not an $\mathcal{N}$-r-ideal of R since $2 \cdot 3 \in P$ but $\mathcal{N}(2) = 4 \notin P$, $\mathcal{N}(3) = 9 \notin P$, $2 \notin zd(R)$ and $3 \notin zd(R)$.

Theorem 3.11 *Let R be a $\mathbb{Z}_2$-graded ring. If P is a semi r-ideal of R, then P is a semi $\mathcal{N}$-r-ideal of R.*

Proof: Let $x \in R$ such that $x^2 \in P$. Then either $x \in P$ or $x \in zd(R)$. If $x \in P$, then $\mathcal{N}(x) \in \mathcal{N}(P) \subseteq P$ by Lemma 2.1. Hence, P is a semi $\mathcal{N}$-r-ideal of R. □

On the other hand, the next example shows that a semi $\mathcal{N}$-r-ideal is not necessarily a semi r-ideal.

Example 3.8 Consider the $\mathbb{Z}_2$-graded ring given in Example 3.7. Let $P = 2R$ and $x \in R$ such that $x^2 \in P$. Then $x^2 = 2(a + ib)$, for some $a, b \in \mathbb{Z}$, and then $(\mathcal{N}(x))^2 = \mathcal{N}(x^2) = 4(a^2 + b^2)$. Now, 2 divides $(\mathcal{N}(x))^2$ which implies 2 divides $\mathcal{N}(x)$ that is $\mathcal{N}(x) \in 2R = P$. Hence, P is a semi $\mathcal{N}$-r-ideal. On the other hand, P is not a semi r-ideal since $1+i \in R$ with $(1+i)^2 \in P$, $1+i \notin zd(R)$ and $(1+i) \notin P$.

Theorem 3.12 *Let P be a proper ideal of a $\mathbb{Z}_2$-graded ring R. Then P is a semi $\mathcal{N}$-r-ideal of R if and only if whenever $x \in R$ with $0 \neq x^2 \in P$ and $x \in r(R)$, then $\mathcal{N}(x) \in P$.*

Proof: Suppose that whenever $x \in R$ with $0 \neq x^2 \in P$ and $x \in r(R)$, then $\mathcal{N}(x) \in P$. Let $x \in R$ with $x^2 \in P$ and $x \in r(R)$. If $x^2 = 0$, then $x = 0$ as $x \in r(R)$, and then $\mathcal{N}(x) = 0 \in P$. If $x^2 \neq 0$, then $\mathcal{N}(x) \in P$ by assumption. Hence, P is a semi $\mathcal{N}$-r-ideal of R. The converse is clear. □

Theorem 3.13 *Let P and Q be two semi $\mathcal{N}$-r-ideals of a $\mathbb{Z}_2$-graded ring R. Then $P \bigcap Q$ is a semi $\mathcal{N}$-r-ideal of R.*

Proof: Let $x \in R$ such that $x^2 \in P \bigcap Q$ and $x \in r(R)$. Then $x^2 \in P$ and $x^2 \in Q$. Since P is a semi $\mathcal{N}$-r-ideal, $\mathcal{N}(x) \in P$. Similarly, $\mathcal{N}(x) \in Q$. Hence, $\mathcal{N}(x) \in P \bigcap Q$. Thus, $P \bigcap Q$ is a semi $\mathcal{N}$-r-ideal of R.

□

Lemma 3.2 *Let R and S be two $\mathbb{Z}_2$-graded rings. Then*
$\mathcal{N}_{R\times S}(x, y) = (\mathcal{N}_R(x), \mathcal{N}_S(y))$, for every $(x, y) \in R \times S$.

Proof: Let $(x, y) \in R \times S$. Then $(x, y) = (x, y)_0 + (x, y)_1 = (x_0, y_0) + (x_1, y_1)$. Now, $\mathcal{N}(x, y) = (x_0, y_0)^2 - (x_1, y_1)^2 = (x_0^2, y_0^2) - (x_1^2, y_1^2) = (x_0^2 - x_1^2, y_0^2 - y_1^2) = (\mathcal{N}_R(x), \mathcal{N}_S(y))$.

□

Theorem 3.14 *Let R and S be two $\mathbb{Z}_2$-graded rings and $T = R \times S$. If P is a semi $\mathcal{N}_R$-r-ideal of R, then $P \times S$ is a semi $\mathcal{N}_T$-r-ideal of T.*

Proof: Let $(x, y) \in T$ such that $(x, y)^2 \in P \times S$ and $(x, y) \in r(R \times S)$. Then $x^2 \in P$. If $Ann_R(x) \neq \{0_R\}$, then there exists a nonzero $t \in R$ such that $tx = 0_R$. Since $(t, 0_R)(x, y) = (0_R, 0_S)$, we get $t = 0_R$, which is a contradiction. So, $Ann_R(x) = \{0_R\}$. Since P is a semi $\mathcal{N}$-r-ideal, $\mathcal{N}_R(x) \in P$, and hence $\mathcal{N}_T((x, y)) \in P \times S$. Thus, $P \times S$ is a semi $\mathcal{N}$-r-ideal of T.

□

Similarly, we can prove the following theorem.

Theorem 3.15 *Let R and S be two $\mathbb{Z}_2$-graded rings and $T = R \times S$. If Q is a semi $\mathcal{N}_S$-r-ideal of S, then $R \times Q$ is a semi $\mathcal{N}_T$-r-ideal of T.*

The converses of Theorems 3.14 and 3.15 hold true.

Theorem 3.16 *Let R and S be two $\mathbb{Z}_2$-graded rings and $T = R \times S$. If $P \times S$ is a semi $\mathcal{N}_T$-r-ideal of T, then P is a semi $\mathcal{N}_R$-r-ideal of R.*

Proof: Let $x \in R$ such that $x^2 \in P$ and $x \in r(R)$. Then $(x, 1) \in T$ with $(x, 1)^2 \in P \times S$. If $Ann_T((x, 1)) \neq \{(0_R, 0_S)\}$, then there exists a nonzero $(t, s) \in T$ such that $(x, 1)(t, s) = (0_R, 0_S)$, which implies $xt = 0_R$ and $s = 0_S$, so $t \in Ann_R(x) = \{0_R\}$ and $s = 0_S$. Hence, $(t, s) = (0_R, 0_S)$, which is a contradiction. So, $Ann_T((x, 1)) = \{(0_R, 0_S)\}$. Since $P \times S$ is a semi $\mathcal{N}$-r-ideal, $\mathcal{N}_T((x, 1)) \in P \times S$, and hence $\mathcal{N}_R(x) \in P$. Thus, P is a semi $\mathcal{N}_R$-r-ideal of R.

□

Similarly, the following Theorem is demonstrated.

Theorem 3.17 *Let R and S be two $\mathbb{Z}_2$-graded rings and $T = R \times S$. If $R \times Q$ is a semi $\mathcal{N}_T$-r-ideal of T, then Q is a semi $\mathcal{N}_S$-r-ideal of S.*

Theorem 3.18 *Let R and S be two $\mathbb{Z}_2$-graded rings and $T = R \times S$. If P and Q are semi $\mathcal{N}_R$-r-ideal of R and $\mathcal{N}_S$-r-ideal of S, respectively, then $P \times Q$ is a semi $\mathcal{N}_T$-r-ideal of T.*

Proof: Let $(x, y) \in T$ such that $(x, y)^2 \in P \times Q$ and $(x, y) \in r(R \times S)$. A same argument as in the proof of Theorem 3.14 leads to that $x \in r(R)$, and then since P is a semi $\mathcal{N}_R$-r-ideal, $\mathcal{N}_R(x) \in P$. Similarly, $y \in r(S)$. Therefore, $\mathcal{N}_S(y) \in Q$. Hence, $\mathcal{N}_T((x, y)) \in P \times Q$. Thus, $P \times Q$ is a semi $\mathcal{N}_T$-r-ideal of T. □

Theorem 3.19 *Let R and S be two $\mathbb{Z}_2$-graded rings and $T = R \times S$. Suppose that P and Q are proper ideals of R and S, respectively. If $P \times Q$ is a semi $\mathcal{N}_T$-r-ideal of T, then either P is a semi $\mathcal{N}_R$-r-ideal of R or Q is a semi $\mathcal{N}_S$-r-ideal of S.*

Proof: Suppose that P is not a semi $\mathcal{N}_R$-r-ideal of R and Q is not a semi $\mathcal{N}_S$-r-ideal of S. Then, there exist $x \in R$ and $y \in S$ such that $x^2 \in P$, $y^2 \in Q$, $Ann_R(x) = \{0_R\}$, $Ann_S(y) = \{0_S\}$, $\mathcal{N}_R(x) \notin P$, $\mathcal{N}_S(y) \notin Q$. So, $(x, y)^2 \in P \times Q$. If $Ann_T((x, y)) \neq \{(0_R, 0_S)\}$, then there exists a nonzero $(t, s) \in T$ such that $(x, y)(t, s) = (0_R, 0_S)$, which implies $xt = 0_R$ and $ys = 0_S$. So $t \in Ann_R(x) = \{0_R\}$ and $s \in Ann_S(y) = \{0_S\}$. Hence, $(t, s) = (0_R, 0_S)$, which is a contradiction. Thus, $Ann_T((x, y)) = \{(0_R, 0_S)\}$. Since $P \times Q$ is a semi $\mathcal{N}_T$-r-ideal, we have $\mathcal{N}_T((x, y)) \in P \times Q$, which is a contradiction. Hence, either P is a semi $\mathcal{N}_R$-r-ideal of R or Q is a semi $\mathcal{N}_S$-r-ideal of S. □

Acknowledgments

We sincerely thank the reviewers for their excellent suggestions and valuable improvements. Their insightful feedback has greatly enhanced the quality of our article, and we truly appreciate the time and effort they invested in reviewing our work.

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