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Analysis of a novel Class of Nonlinear Boundary Value Langevin Hybrid Fractional Systems

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ABSTRACT: This paper explores the existence and uniqueness of solutions for a novel class of nonlinear boundary value Langevin hybrid fractional integro-differential systems involving the (Υ, Λ)-order Caputo generalized proportional derivative. Our approach is based on a detailed analysis of the properties of the generalized proportional operator. Using Schauder's and Banach's fixed point theorems, we establish the existence and uniqueness of solutions, respectively. To illustrate and support our main findings, we present a concrete example.

Key Words: Generalized Caputo proportional fractional derivative, Langevin equation, hybrid equation, fixed point theorem.

Contents

1. Introduction

Fractional calculus is a topic almost as old as classical calculus. It has important applications not only in pure mathematics but also in a wide range of engineering sciences. In particular, the theory of fractional differential equations has emerged in recent years as a particularly fascinating field of research. It has numerous applications in modeling various real-world phenomena. For example, it is frequently used in engineering, physics, chemistry, biology, and many other fields [19,3,6,18,10,4,5]. The ability of these equations to accurately model a wide range of real-world phenomena has inspired many researchers to explore their quantitative and qualitative properties. Furthermore, hybrid and Langevin fractional differential equations constitute a particularly important and captivating field of study. Interest in these equations continues to grow, as reflected by the increasing number of publications focused on issues of existence and uniqueness of their solutions.

Introduced by Paul Langevin in 1908, the Langevin equation plays a fundamental role in statistical mechanics, providing a framework for describing Brownian motion and systems subject to fluctuating environments. It effectively captures the interplay between deterministic forces and random disturbances, offering valuable insights into particle dynamics under thermal noise [12,11,20,9,2]. A more adaptable approach to modeling fractal processes is the fractional version of the Langevin equation, which extends the classical form and produces a fractional Gaussian process characterized by two parameters [21]. Additional discussions on these equations can be found in references [13,7,8,23,24].

Hybrid equation theory plays a significant role in the study of nonlinear dynamical systems that are difficult to solve or analyze directly. The inherent nonlinearity of these systems often lacks the smoothness required to examine the existence or other properties of their solutions. However, by introducing certain perturbations, the problem can be approached using existing methods to explore various aspects of the

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solutions. Dynamical systems modified in this manner are referred to as hybrid differential equations. Numerous studies have been conducted on the theory of hybrid differential equations, and we direct readers to the relevant literature, see [1,22,15,25] for more details.

In [22], we explored the existence of solutions for a p-Laplacian hybrid fractional differential equation that involves the generalized Caputo proportional fractional derivative:

$$\begin{cases} {}^{\mathfrak{C}}_{\xi}D^{\alpha,\mathfrak{G}}_{0^+}\Phi_p\left({}^{\mathfrak{C}}_{\xi}D^{\beta,\mathfrak{G}}_{0^+}(\frac{y(t)}{\mathcal{F}(t,y(t))})\right) = \mathcal{G}(t,y(t)), & t\in\Sigma=[0,b],\\ \left(\frac{w(t)}{\mathcal{F}(t,y(t))}\right)_{t=0} = y_0, & y_0\in\mathbb{R},\\ \left(\frac{y(t)}{\mathcal{F}(t,y(t))}\right)'_{t=0} = 0, & \end{cases} \tag{1.1}$$

where $0<\alpha<1$, $1<\beta<2$, ${}^{\mathfrak{C}}_{\xi}D^{\alpha,\mathfrak{G}}_{0^+}(\cdot)$ is the generalized Caputo proportional fractional derivative of order α, $\Phi_p(x)=|x|^{p-2}x$, $p>1$ is the p-Laplacian operator, $\mathfrak{G}:\Sigma\to\mathbb{R}$, $\mathcal{F}\in C(\Sigma\times\mathbb{R},\mathbb{R}^*)$, and $\mathcal{G}\in C(\Sigma\times\mathbb{R},\mathbb{R})$.
In [24], we further investigated the existence and uniqueness of solutions for a nonlinear Langevin fractional boundary value integro-differential equation involving the ψ-Caputo derivative with two different variable orders:

$$\begin{cases} {}^{C}D^{\theta(\rho),\psi}_{0^+}\left({}^{C}D^{\lambda(\rho),\psi}_{0^+}+\omega(\rho)\right)y(\rho) = \mathcal{H}(\rho,y(\rho),I^{\lambda(\rho),\psi}_{0^+}y(\rho)), \quad \rho\in\mathfrak{J}=[0,b],\\ {}^{C}D^{\lambda(\rho),\psi}_{0^+}y(0)=y(0)=0, \quad y(b)=\nu\in\mathbb{R}, \end{cases} \tag{1.2}$$

where $1<\theta(\rho)<2$, $0<\lambda(\rho)<1$, ${}^{C}D^{\theta(\rho),\psi}_{0^+}(.)$ is the ψ-Caputo fractional derivative of variable order $\theta(\rho)$, $\omega:\mathfrak{J}\to\mathbb{R}$, and $\mathcal{H}\in C(\mathfrak{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.
Motivated by the aforementioned studies, this paper integrates their concepts to examine the existence and uniqueness of solutions for a nonlinear boundary value Langevin hybrid fractional integro-differential system involving the (Υ,Λ)-order Caputo generalized proportional derivative

$$\begin{cases} {}^{C}_{\xi}D^{\Upsilon,\mathfrak{G}}_{c^+}\left({}^{C}_{\xi}D^{\Lambda,\mathfrak{G}}_{c^+}+\Theta(t)\mathcal{H}(t,y(t))\right)\left(\frac{y(t)-\mathcal{F}(t,y(t),\mathfrak{B}y(t))}{\mathcal{H}(t,y(t))}\right) = \mathcal{M}(t,y(t),\mathfrak{B}y(t)), \quad t\in\mathfrak{T}:=[c,d],\\ \left(\frac{y(t)-\mathcal{F}(t,y(t),\mathfrak{B}y(t))}{\mathcal{H}(t,y(t))}\right)_{t=c} = {}^{C}_{\xi}D^{\Lambda,\mathfrak{G}}_{c^+}\left(\frac{y(t)-\mathcal{F}(t,y(t),\mathfrak{B}y(t)))}{\mathcal{H}(t,y(t))}\right)_{t=c}=0,\\ (y(t)-\mathcal{F}(t,y(t),\mathfrak{B}y(t)))_{t=d}=0, \quad {}^{C}_{\xi}D^{\Lambda,\mathfrak{G}}_{c^+}\left(\frac{y(t)-\mathcal{F}(t,y(t),\mathfrak{B}y(t))}{\mathcal{H}(t,y(t))}\right)_{t=d}=\upsilon\in\mathbb{R}, \end{cases} \tag{1.3}$$

where $\mathfrak{T}:=[c,d]$ a finite interval of $\mathbb{R}$, ${}^{C}_{\xi}D^{\Upsilon,\mathfrak{G}}_{c^+}(.)$, ${}^{C}_{\xi}D^{\Lambda,\mathfrak{G}}_{c^+}(.)$ are the generalized Caputo proportional fractional derivative of order $1<\Upsilon<2$, and $0<\Lambda<1$ respectively, $\mathfrak{G}:[0,b]\to\mathbb{R}$, $\Theta\in C(\mathfrak{T},\mathbb{R}^+)$, $\mathcal{F},\mathcal{M}\in C(\mathfrak{T}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, and $\mathcal{H}\in C(\mathfrak{T}\times\mathbb{R},\mathbb{R}\setminus\{0\})$. The operator $\mathfrak{B}y(t)$ is given by $\mathfrak{B}y(t):=\int_c^t\mathfrak{N}(\tau,s)y(s)ds$ such that $\mathfrak{N}\in C(\mathfrak{D},\mathbb{R})$, where $\mathfrak{D}:=\{(\tau,s):c<s<\tau<d\}$. We consider

$$\mathfrak{B}^* = \max_{\tau\in[c,d]}\int_c^\tau|\mathfrak{N}(\tau,s))|\,ds \quad \text{and} \quad \Theta^*=\sup_{t\in[c,d]}|\Theta(t)|.$$

The novelty of this work lies in the integration of the Langevin equation within a hybrid framework, studied under a new and challenging type of fractional derivative known as the generalized Caputo proportional fractional derivative. This approach offers greater flexibility in modeling systems with fractional orders, thereby expanding the range of potential applications. To the best of our knowledge, this is the first investigation of a nonlinear Langevin hybrid fractional integro-differential system involving the (Υ,Λ)-order Caputo generalized proportional derivative under the boundary conditions (1.3). These conditions provide deeper insight into the impact of initial and terminal constraints on the behavior of the system.

The structure of the paper is as follows: Section 2 introduces essential notations and preliminary results concerning the generalized Caputo proportional fractional derivative and establishes the solution formula for the nonlinear fractional integro-differential system (1.3). In Section 3, we investigate the existence and uniqueness of solutions to the given problem (1.3) by applying Schauder's and Banach's fixed point theorems, respectively. Lastly, Section 4 provides an illustrative example to demonstrate the main findings.

2. Preliminaries

In this section, we present definitions and lemmas associated with the generalized Caputo proportional fractional derivative and derive the solution formula for the nonlinear fractional integro-differential system (1.3). These definitions and lemmas will be consistently applied in the subsequent sections of this work.

- Let $C(\Delta, \mathbb{R})$ be the Banach space of all continuous functions with the norm $\|y\| = \sup\{|y(t)|, t \in \mathfrak{T}\}$.

- Throughout this manuscript, we consider that $\mathfrak{G} : \mathfrak{T} \to \mathbb{R}$ is a strictly positive, increasing, and differentiable function.

Definition 2.1 [16,17] Let $t \in \mathfrak{T}$, $0 < \xi < 1$, $\Upsilon > 0$, and $\Phi \in L^1(\mathfrak{T}, \mathbb{R})$. The left-sided generalized proportional fractional integral of order Υ with respect to $\mathfrak{G}$ of the function Φ is given by

$$ {}_\xi I_{c^+}^{\Upsilon,\mathfrak{G}} \Phi(t) = \frac{1}{\xi^\Upsilon \Gamma(\Upsilon)} \int_c^t e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(s))} (\mathfrak{G}(t) - \mathfrak{G}(s))^{\Upsilon-1} \Phi(s) \mathfrak{G}'(s) ds, \tag{2.1} $$

where $\Gamma(\Upsilon) = \int_0^{+\infty} e^{-\tau} \tau^{\Upsilon-1} d\tau$ is the Euler gamma function.

Definition 2.2 [16,17] Let $0 < \xi < 1$, $\Psi, \varphi : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous functions such that $\lim_{\xi \to 0^+} \Psi(\xi, t) = 0$, $\lim_{\xi \to 1^-} \Psi(\xi, t) = 1$, $\lim_{\xi \to 0^+} \varphi(\xi, t) = 1$, $\lim_{\xi \to 1^-} \varphi(\xi, t) = 0$, and $\Psi(\xi, t) + \varphi(\xi, t) \neq 0$ for each $\xi \in [0, 1]$, and $t \in \mathbb{R}$.

Then, the proportional derivative of order ξ with respect to $\mathfrak{G}$ of the function Φ is given by

$$ {}_\xi D^{\mathfrak{G}} \Phi(t) = \varphi(\xi, t)\Phi(t) + \Psi(\gamma, t) \frac{\Phi'(t)}{\mathfrak{G}'(t)}. $$

In particular, if $\Psi(\xi, t) = \xi$ and $\varphi(\varphi, t) = 1 - \xi$, then we have

$$ {}_\xi D^{\mathfrak{G}} \Phi(t) = (1 - \xi)\Phi(t) + \xi \frac{\Phi'(t)}{\mathfrak{G}'(t)}. $$

Definition 2.3 [16,17] Let $\xi \in (0, 1]$ and $\Phi \in C^n(\mathfrak{T}, \mathbb{R})$. The left-sided generalized Caputo proportional fractional derivative of order $n - 1 < \Upsilon < n$ of the function Φ is defined as follows:

$$ \begin{aligned} {}^C_\xi D_{c^+}^{\Upsilon,\mathfrak{G}}(\Phi)(t) &= {}_\xi I_{c^+}^{n-\Upsilon,\mathfrak{G}}({}_\xi D^{n,\mathfrak{G}} \Phi(t)) \\ &= \frac{1}{\xi^{n-\Upsilon}\Gamma(n - \Upsilon)} \int_c^t e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(s))} (\mathfrak{G}(t) - \mathfrak{G}(s))^{n-\Upsilon-1} ({}_\xi D^{n,\mathfrak{G}} \Phi)(s) \mathfrak{G}'(s) ds, \end{aligned} \tag{2.2} $$

where $n = [\Upsilon] + 1$ and ${}_\xi D^{n,\mathfrak{G}} = \underbrace{{}_\xi D_\gamma^{\mathfrak{G}} D^{\mathfrak{G}} \ldots {}_\xi D^{\mathfrak{G}}}_{n\text{-times}}$.

As a simplification, throughout this paper, we consider

$$ \Xi_{\mathfrak{G}}^k(t, c) = e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))} (\mathfrak{G}(t) - \mathfrak{G}(c))^k. \tag{2.3} $$

Lemma 2.1 *[16,17] Let $t \in \mathfrak{T}$, $\xi \in (0, 1]$, $(\rho, \theta > 0)$, and $\Phi \in L^1(\mathfrak{T}, \mathbb{R})$. Then, we have*

$$ {}_\xi I_{c^+}^{\theta,\mathfrak{G}}({}_\xi I_{c^+}^{\rho,\mathfrak{G}} \Phi(t)) = {}_\xi I_{c^+}^{\rho,\mathfrak{G}}({}_\xi I_{c^+}^{\theta,\mathfrak{G}} \Phi(t)) = {}_\xi I_{c^+}^{\theta+\rho,\mathfrak{G}} \Phi(t). $$

Lemma 2.2 *[16,17] Let $\xi \in (0,1]$, $n-1<\rho<n$, $(n=[\rho]+1)$. Then, we have*

$$ {}_{\xi}I_{c^+}^{\rho,\mathfrak{G}}\left({}_{\xi}^{c}D_{c^+}^{\rho,\mathfrak{G}}\Phi(t)\right) = \mathfrak{G}(t) - \sum_{k=0}^{n-1} \frac{({}_{\xi}D^{k,\mathfrak{G}}\Phi)(c)}{\gamma^k \Gamma(k+1)} \Xi_{\mathfrak{G}}^{k}(t,c), $$

where $\Xi_{\mathfrak{G}}^{k}(t,c)$ is given by (2.3).

Lemma 2.3 *[16,17] Let $t \in [c,d]$, $\xi \in (0,1]$ and $\rho, \Upsilon > 0$. Then, we have*

(i) $\left({}_{\xi}I_{c^+}^{\rho,\mathfrak{G}} e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Upsilon-1}\right)(\tau) = \frac{\Gamma(\Upsilon)}{\xi^{\rho}\Gamma(\rho+\Upsilon)}\Xi_{\mathfrak{G}}^{\rho+\Upsilon-1}(\tau,c).$

(ii) $\left({}_{\xi}^{C}D_{c^+}^{\rho,\mathfrak{G}} e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(0))}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Upsilon-1}\right)(\tau) = \frac{\xi^{\rho}\Gamma(\Upsilon)}{\Gamma(\Upsilon-\rho)}\Xi_{\mathfrak{G}}^{\Upsilon-\rho-1}(\tau,c).$

Lemma 2.4 *[16,17] Let $\xi \in (0,1]$, $\Upsilon > 0$, and $\Phi \in L^1(\mathfrak{T},\mathbb{R})$. Then, we have*

$$ \lim_{\tau \to c}\left({}_{\xi}I_{c^+}^{\Upsilon,\mathfrak{G}}\Phi(\tau)\right) = 0. $$

Theorem 2.1 *(Schauder's Fixed-Point Theorem [14]) Let $\mathfrak{S}$ be a closed, nonempty, bounded, and convex subset of a Banach space $\mathfrak{X}$, and let $\mathfrak{W} : \mathfrak{S} \to \mathfrak{S}$ be a continuous and compact map. Then, $\mathfrak{W}$ has at least a fixed point in $\mathfrak{S}$.*

We are now prepared to present the definition of a solution to problem (1.3), which plays a central role in our work. To that end, we begin with the following lemma, which serves as the foundation for deriving this definition.

Lemma 2.5 *Let $\mathfrak{G} : \mathfrak{T} \longrightarrow \mathbb{R}$, $h \in C(\mathfrak{T},\mathbb{R}^*)$, $f, m \in C(\mathfrak{T},\mathbb{R})$, and $y \in C(\mathfrak{T},\mathbb{R})$. Then the solution of the following problem:*

$$ \begin{cases} {}_{\xi}^{C}D_{c^+}^{\Upsilon,\mathfrak{G}}\left({}_{\xi}^{C}D_{c^+}^{\Lambda,\mathfrak{G}} + \Theta(t)h(t)\right)\left(\frac{y(t)-f(t)}{h(t)}\right) = m(t), \qquad t \in \mathfrak{T} := [c,d], \\ \left(\frac{y(t)-f(t)}{h(t)}\right)_{t=c} = {}_{\xi}^{C}D_{c^+}^{\Lambda,\mathfrak{G}}\left(\frac{y(t)-f(t)}{h(t)}\right)_{t=c} = 0, \\ (y(t)-f(t))_{t=d} = 0, \quad {}_{\xi}^{C}D_{c^+}^{\Lambda,\mathfrak{G}}\left(\frac{y(t)-f(t)}{h(t)}\right)_{t=d} = \upsilon \in \mathbb{R}, \end{cases} \tag{2.4} $$

is given by

$$ y(t) = f(t) + h(t)\left[\widehat{\Delta} e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1} + {}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}m(t) - {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}\Theta(t)\left(y(t)-f(t)\right)\right], $$

where

$$ \widehat{\Delta} = \frac{\upsilon - {}_{\xi}I_{c^+}^{\Upsilon,\mathfrak{G}}m(d)}{\xi^{\Lambda}\Gamma(\Lambda+2)e^{\frac{\xi-1}{\xi}(\mathfrak{G}(d)-\mathfrak{G}(c))}(\mathfrak{G}(d)-\mathfrak{G}(c))}. \tag{2.5} $$

Proof: suppose that $y(t)$ is a solution of the nonlinear boundary value Langevin hybrid fractional integro-differential system (2.4). Then, applying the operator ${}_{\xi}I_{c^+}^{\Upsilon,\mathfrak{G}}(.)$ on both sides of the equation (2.4) and using Lemma 2.2, we get

$$ \begin{aligned} {}_{\xi}^{C}D_{c^+}^{\Lambda,\mathfrak{G}}\left(\frac{y(t)-f(t)}{h(t)}\right) &= {}_{\xi}I_{c^+}^{\Upsilon,\mathfrak{G}}m(t) - \Theta(t)\left(y(t)-f(t)\right) + \lambda_0 e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))} \\ &\quad + \lambda_1 \frac{e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c)). \end{aligned} \tag{2.6} $$

Applying the operator ${}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}(.)$ on both sides of the equation (2.6), thanks to Lemma 2.2, we get

$$\left(\frac{y(t)-f(t)}{h(t)}\right) ={}_{\xi} I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}m(t) -{}_{\xi} I_{c^+}^{\Lambda,\mathfrak{G}}\Theta(t)\left(y(t)-f(t)\right)+\lambda_0({}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}})e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}$$
$$+\lambda_1({}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}})\left(\frac{e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))\right)+\lambda_2 e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}, \tag{2.7}$$

such that λ_0, λ_1, and λ_2 are constants of $\mathbb{R}$.
Thanks to Lemma 2.3, (i), the equation (2.7) becomes

$$\left(\frac{y(t)-f(t)}{h(t)}\right) ={}_{\xi} I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}m(t) -{}_{\xi} I_{c^+}^{\Lambda,\mathfrak{G}}\Theta(t)\left(y(t)-f(t)\right)+\lambda_0\frac{e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}}{\xi^{\Lambda}\Gamma(\Lambda+1)}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda}$$
$$+\lambda_1\frac{e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}}{\xi^{\Lambda+1}\Gamma(\Lambda+2)}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1}+\lambda_2 e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}. \tag{2.8}$$

Next, we determine the constants λ_0, λ_1, and λ_2. By taking $t=c$ in the integral equations (2.6) and (2.8), and applying Lemma 2.4 along with the initial condition $\left(\frac{y(t)-f(t)}{h(t)}\right)_{t=c}={}_{\xi}^{C}D_{c^+}^{\Lambda,\mathfrak{G}}\left(\frac{y(t)-f(t)}{h(t)}\right)_{t=c}=0$, we obtain

$$\lambda_0=\lambda_2=0.$$

We proceed to determine λ_1, taking $t=d$ in the integral equation (2.6) with $(\lambda_0=0)$ and using the initial conditions ${}_{\xi}^{C}D_{c^+}^{\Lambda,\mathfrak{G}}\left(\frac{y(t)-f(t)}{h(t)}\right)_{t=d}=\upsilon$ and $(y(t)-f(t))_{t=d}=0$, we get

$${}_{\xi}^{C}D_{c^+}^{\Lambda,\mathfrak{G}}\left(\frac{y(t)-f(t)}{h(t)}\right)_{t=d}={}_{\xi}I_{c^+}^{\Upsilon,\mathfrak{G}}m(d)+\lambda_1\frac{e^{\frac{\xi-1}{\xi}(\mathfrak{G}(d)-\mathfrak{G}(c))}}{\xi}(\mathfrak{G}(d)-\mathfrak{G}(c)),$$

this implies that

$$\lambda_1=\xi\left(\frac{\upsilon-{}_{\xi}I_{c^+}^{\Upsilon,\mathfrak{G}}m(d)}{e^{\frac{\xi-1}{\xi}(\mathfrak{G}(d)-\mathfrak{G}(c))}(\mathfrak{G}(d)-\mathfrak{G}(c))}\right).$$

Substituting the values of λ_0, λ_1, and λ_2 in (2.8) we get

$$\left(\frac{y(t)-f(t)}{h(t)}\right)=\left(\frac{\upsilon-{}_{\xi}I_{c^+}^{\Upsilon,\mathfrak{G}}m(d)}{\xi^{\Lambda}\Gamma(\Lambda+2)e^{\frac{\xi-1}{\xi}(\mathfrak{G}(d)-\mathfrak{G}(c))}(\mathfrak{G}(d)-\mathfrak{G}(c))}\right)e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1}$$
$$+{}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}m(t)-{}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}\Theta(t)\left(y(t)-f(t)\right).$$

Therefore

$$y(t)=f(t)+h(t)\left[\widehat{\Delta}e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1}+{}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}m(t)-{}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}\Theta(t)\left(y(t)-f(t)\right)\right],$$

where $\widehat{\Delta}$ is given by (2.5).
The opposite follows by direct computation. □

Using the information from the previous lemma, we can now define the solution to nonlinear boundary value Langevin hybrid fractional integro-differential system (1.3).

Definition 2.4 Let $t\in\mathfrak{T}$, $y\in C(\mathfrak{T},\mathbb{R})$, $\mathcal{F},\mathcal{M}\in C(\mathfrak{T}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, and $\mathcal{H}\in C(\mathfrak{T}\times\mathbb{R},\mathbb{R}\setminus\{0\})$. If y is a solution to the nonlinear boundary value Langevin hybrid fractional integro-differential system (1.3), then y is also a solution of the following equation:

$$y(t)=\mathcal{F}(t,y(t),\mathfrak{B}y(t))+\mathcal{H}(t,y(t))\left[\overline{\Delta}e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1}\right.$$
$$\left.+{}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}\mathcal{M}(t,y(t),\mathfrak{B}y(t))-{}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}\Theta(t)\left(y(t)-\mathcal{F}(t,y(t),\mathfrak{B}y(t))\right)\right],$$

where

$$\overline{\Delta}=\frac{\upsilon-\ {}_{\xi}I_{c^+}^{\Upsilon,\mathfrak{G}}\mathcal{M}(d,y(d),\mathfrak{B}y(d))}{\xi^{\Lambda}\Gamma(\Lambda+2)e^{\frac{\xi-1}{\xi}(\mathfrak{G}(d)-\mathfrak{G}(c))}(\mathfrak{G}(d)-\mathfrak{G}(c))}. \tag{2.9}$$

3. Existence Results

In this section, we investigate the existence and uniqueness of the solution to the nonlinear boundary value Langevin hybrid fractional integro-differential system (1.3) by employing the Schauder's and Banachs fixed point theorems, respectively.

The derivation of our results relies on the following assumptions:

$(\mathcal{A}_1)$ $\mathcal{F}\in C(\mathfrak{T}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there are constants $J_{\mathcal{F}}$, and $\widehat{J}_{\mathcal{F}}$ such that for all $x,y,x',y'\in\mathbb{R}$ and $t\in\mathfrak{T}$, we have

(i) $\|\mathcal{F}(t,x,y)-\mathcal{F}(t,x',y')\|\le J_{\mathcal{F}}\left[\|x-x'\|+\|y-y'\|\right].$

(ii) $\|\mathcal{F}(t,x,y)\|\le \widehat{J}_{\mathcal{F}}\left[\|x\|+\|y\|\right].$

$(\mathcal{A}_2)$ $\mathcal{M}\in C(\mathfrak{T}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there are constants $K_{\mathcal{M}}$, and $\widehat{K}_{\mathcal{M}}$ such that for all $x,y,x',y'\in\mathbb{R}$ and $t\in\mathfrak{T}$, we have

(i) $\|\mathcal{M}(t,x,y)-\mathcal{M}(t,x',y')\|\le K_{\mathcal{M}}\left[\|x-x'\|+\|y-y'\|\right].$

(ii) $\|\mathcal{M}(t,x,y)\|\le \widehat{K}_{\mathcal{M}}\left[\|x\|+\|y\|\right].$

$(\mathcal{A}_3)$ $\mathcal{H}\in C(\mathfrak{T}\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and there is a constant $\overline{L}_{\mathcal{H}}$ such that for all $y\in\mathbb{R}$ and $t\in\mathfrak{T}$, we have

$$\|\mathcal{H}(t,y)\|\le\overline{L}_{\mathcal{H}}.$$

Before stating the existence theorem of the solution to our problem (1.3) we prepare the following results:

- We consider the Banach space $\mathfrak{X}:=(C(\mathfrak{T},\mathbb{R}),\|\ .\ \|)$. Define the subset $\mathfrak{S}_{\varpi}$ of $\mathfrak{X}$ as follows:

$$\mathfrak{S}_{\varpi}=\{y\in\mathfrak{X},\ \|\ y\ \|\le\varpi\},$$

with

$$\varpi\ge\frac{\overline{\Pi}}{1-\widehat{\Pi}},\qquad\text{such that}\qquad 1-\widehat{\Pi}>0,$$

where

$$\overline{\Pi}=\overline{L}_{\mathcal{H}}\left|\overline{\Delta}\right|(\mathfrak{G}(d)-\mathfrak{G}(c))^{\Lambda+1}.$$

$$\widehat{\Pi}=\widehat{J}_{\mathcal{F}}(1+\mathfrak{B}^*)+\frac{\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Upsilon+\Lambda}}{\xi^{\Upsilon+\Lambda}\,\Gamma(\Upsilon+\Lambda+1)}\overline{L}_{\mathcal{H}}\widehat{K}_{\mathcal{M}}(1+\mathfrak{B}^*)+\frac{\overline{L}_{\mathcal{H}}\Theta^*\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Lambda}}{\xi^{\Lambda}\,\Gamma(\Lambda+1)}\Big(1+\widehat{J}_{\mathcal{F}}(1+\mathfrak{B}^*)\Big).$$

It is easy to see that $\mathfrak{S}_{\varpi}$ is a closed, convex, bounded, and nonempty subset of the Banach space $\mathfrak{X}$.

- Define the operator $\mathfrak{W}:\mathfrak{X}\to\mathfrak{X}$ as follows:

$$\begin{aligned}(\mathfrak{W})y(t)&=\mathcal{F}(t,y(t),\mathfrak{B}y(t))+\mathcal{H}(t,y(t))\Big[\overline{\Delta}e^{\frac{\xi-1}{\xi}(\mathfrak{G}(t)-\mathfrak{G}(c))}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1}\\&\quad+\ {}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}\mathcal{M}(t,y(t),\mathfrak{B}y(t))-\ {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}\Theta(t)\left(y(t)-\mathcal{F}(t,y(t),\mathfrak{B}y(t))\right)\Big],\end{aligned}$$

Having established all the required arguments, we are now prepared to prove the existence results for the nonlinear boundary value Langevin hybrid fractional integro-differential system (1.3). Accordingly, we present the following existence theorem.

Theorem 3.1 *Suppose that assumptions* $(\mathcal{A}_1)$-$(\mathcal{A}_3)$ *are satisfied. Then, the nonlinear boundary value Langevin hybrid fractional integro-differential system* (1.3) *has at least a solution in* $C(\mathfrak{T},\mathbb{R})$.

Proof: The following steps demonstrate the proof of the above theorem:
Step 1. $\mathfrak{W}(\mathfrak{S}_\varpi)\subset\mathfrak{S}_\varpi$.
Let $t\in\mathfrak{T}$ and $y\in\mathfrak{S}_\varpi$. Thanks to the assumptions $(\mathcal{A}_1, (ii))$, $(\mathcal{A}_2, (ii))$, $(\mathcal{A}_3, (ii))$, and the fact that $e^{\frac{\xi-1}{\xi}(\mathfrak{G}(\cdot)-\mathfrak{G}(\cdot))}<1$, we get

$$
\begin{aligned}
&\|(\mathfrak{W}y)(t)\|\\
&\leq \|\mathcal{F}(t,y(t),\mathfrak{B}y(t))\|+\|\mathcal{H}(t,y(t))\|\left[\left|\overline{\Delta}\right|(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1}\right.\\
&+\frac{1}{\xi^{\Upsilon+\Lambda}\,\Gamma(\Upsilon+\Lambda)}\int_c^t\mathfrak{G}'(s)\big(\mathfrak{G}(t)-\mathfrak{G}(s)\big)^{\Upsilon+\Lambda-1}\|\mathcal{M}(t,y(t),\mathfrak{B}y(t))\|\,ds\\
&\left.+\frac{1}{\xi^{\Lambda}\,\Gamma(\Lambda)}\int_c^t\mathfrak{G}'(s)\big(\mathfrak{G}(t)-\mathfrak{G}(s)\big)^{\Lambda-1}|\Theta(t)|\,\|(y(t)-\mathcal{F}(t,y(t),\mathfrak{B}y(t)))\|\,ds\right]\\
&\leq \widehat{J}_{\mathcal{F}}\left[\|y\|+\|\mathfrak{B}y\|\right]+\overline{L}_{\mathcal{H}}\left[\left|\overline{\Delta}\right|(\mathfrak{G}(d)-\mathfrak{G}(c))^{\Lambda+1}\right.\\
&+\frac{1}{\xi^{\Upsilon+\Lambda}\,\Gamma(\Upsilon+\Lambda)}\int_c^t\mathfrak{G}'(s)\big(\mathfrak{G}(t)-\mathfrak{G}(s)\big)^{\Upsilon+\Lambda-1}\widehat{K}_{\mathcal{M}}\left[\|y\|+\|\mathfrak{B}y\|\right]ds\\
&\left.+\frac{\Theta^*}{\xi^{\Lambda}\,\Gamma(\Lambda)}\int_c^t\mathfrak{G}'(s)\big(\mathfrak{G}(t)-\mathfrak{G}(s)\big)^{\Lambda-1}\Big(\|y(t)\|+\widehat{J}_{\mathcal{F}}\left[\|y\|+\|\mathfrak{B}y\|\right]\Big)\,ds\right]\\
&\leq \varpi\widehat{J}_{\mathcal{F}}(1+\mathfrak{B}^*)+\overline{L}_{\mathcal{H}}\left[\left|\overline{\Delta}\right|(\mathfrak{G}(d)-\mathfrak{G}(c))^{\Lambda+1}+\frac{\varpi\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Upsilon+\Lambda}}{\xi^{\Upsilon+\Lambda}\,\Gamma(\Upsilon+\Lambda+1)}\widehat{K}_{\mathcal{M}}(1+\mathfrak{B}^*)\right.\\
&\left.+\frac{\varpi\Theta^*\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Lambda}}{\xi^{\Lambda}\,\Gamma(\Lambda+1)}\Big(1+\widehat{J}_{\mathcal{F}}(1+\mathfrak{B}^*)\Big)\right]\\
&\leq \overline{L}_{\mathcal{H}}\left|\overline{\Delta}\right|(\mathfrak{G}(d)-\mathfrak{G}(c))^{\Lambda+1}+\varpi\left[\widehat{J}_{\mathcal{F}}(1+\mathfrak{B}^*)+\frac{\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Upsilon+\Lambda}}{\xi^{\Upsilon+\Lambda}\,\Gamma(\Upsilon+\Lambda+1)}\overline{L}_{\mathcal{H}}\widehat{K}_{\mathcal{M}}(1+\mathfrak{B}^*)\right.\\
&\left.+\frac{\overline{L}_{\mathcal{H}}\Theta^*\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Lambda}}{\xi^{\Lambda}\,\Gamma(\Lambda+1)}\Big(1+\widehat{J}_{\mathcal{F}}(1+\mathfrak{B}^*)\Big)\right]\\
&=:\overline{\Pi}+\varpi\widehat{\Pi}\leq\varpi.
\end{aligned}
$$

Then, $\mathfrak{W}$ maps $\mathfrak{S}_\varpi$ into itself.
Step 2. The operator $\mathfrak{W}$ is continuous.
Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of $\mathfrak{S}_\varpi$ with $y_n\to y$ in $\mathfrak{S}_\varpi$ as $n\to+\infty$. Then, we have

$$
\begin{aligned}
&\|(\mathfrak{W}y_n)(t)-(\mathfrak{W}y)(t)\|\\
&\leq\Big\|\mathcal{F}(t,y_n(t),\mathfrak{B}y_n(t))-\mathcal{F}(t,y(t),\mathfrak{B}y(t))\Big\|+\Big\|\mathcal{H}(t,y_n(t))\left[\overline{\Delta}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1}\right.\\
&\left.+\ {}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}\mathcal{M}(t,y_n(t),\mathfrak{B}y_n(t))+\ {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}\Theta(t)\left(y_n(t)-\mathcal{F}(t,y_n(t),\mathfrak{B}y_n(t))\right)\right]\\
&-\mathcal{H}(t,y(t))\left[\overline{\Delta}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1}+\ {}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}\mathcal{M}(t,y_n(t),\mathfrak{B}y_n(t))\right.\\
&\left.+\ {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}\Theta(t)\left(y_n(t)-\mathcal{F}(t,y_n(t),\mathfrak{B}y_n(t))\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathcal{H}(t,y(t)) \Big[\overline{\Delta}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1} + {}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}} \mathcal{M}(t,y_n(t),\mathfrak{B}y_n(t)) \\
&+ {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}} \Theta(t) \left(y_n(t) - \mathcal{F}(t,y_n(t),\mathfrak{B}y_n(t))\right)\Big] \\
&- \mathcal{H}(t,y(t)) \Big[\overline{\Delta}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1} +{}_{\xi} I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}} \mathcal{M}(t,y(t),\mathfrak{B}y(t)) \\
&+ {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}} \Theta(t) \left(y(t) - \mathcal{F}(t,y(t),\mathfrak{B}y(t))\right)\Big]\Big\| \\
&\leq \Big\| \mathcal{F}(t,y_n(t),\mathfrak{B}y_n(t)) - \mathcal{F}(t,y(t),\mathfrak{B}y(t)) \Big\| + \Big\| \mathcal{H}(t,y_n(t)) - \mathcal{H}(t,y(t)) \Big\| \\
&\times \Bigg[\left|\overline{\Delta}\right| (\mathfrak{G}(d)-\mathfrak{G}(c))^{\Lambda+1} + \frac{\varpi\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Upsilon+\Lambda}}{\xi^{\Upsilon+\Lambda}\,\Gamma(\Upsilon+\Lambda+1)} \widehat{K}_{\mathcal{M}}(1+\mathfrak{B}^*) \\
&+ \frac{\varpi\Theta^*\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Lambda}}{\xi^{\Lambda}\,\Gamma(\Lambda+1)} \Big(1+\widehat{J}_{\mathcal{F}}(1+\mathfrak{B}^*)\Big)\Bigg] \\
&+ \overline{L}_{\mathcal{H}} \Big[{}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}} \left\|\mathcal{M}(t,y_n(t),\mathfrak{B}y_n(t)) - \mathcal{M}(t,y(t),\mathfrak{B}y(t))\right\| \\
&+ \Theta^*\, {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}} \left\|(y_n(t) - \mathcal{F}(t,y_n(t),\mathfrak{B}y_n(t))) - (y(t)-\mathcal{F}(t,y(t),\mathfrak{B}y(t)))\right\|\Big].
\end{aligned}
$$

Using continuity of the functions $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{M}$ and the Lebesgue dominated convergence theorem, from the above inequality, we get

$$\|(\mathfrak{W}y_n)(t) - (\mathfrak{W}y)(t)\| \to 0 \text{ as } n \to +\infty.$$

This implies that the operator $\mathfrak{W}$ is continuous.

Step 3. The operator $\mathfrak{W}$ is compact.

(i) The operator $\mathfrak{W}$ is uniformly bounded: Let $t \in \mathfrak{T}$ and $y \in \mathfrak{S}_{\varpi}$. Based on the same arguments of the Step 1 we find

$$
\begin{aligned}
&\|(\mathfrak{W}y)(t)\| \\
&\leq \overline{L}_{\mathcal{H}} \left|\overline{\Delta}\right| (\mathfrak{G}(d)-\mathfrak{G}(c))^{\Lambda+1} + \varpi \Bigg[\widehat{J}_{\mathcal{F}}(1+\mathfrak{B}^*) + \frac{\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Upsilon+\Lambda}}{\xi^{\Upsilon+\Lambda}\,\Gamma(\Upsilon+\Lambda+1)} \overline{L}_{\mathcal{H}} \widehat{K}_{\mathcal{M}}(1+\mathfrak{B}^*) \\
&+ \frac{\overline{L}_{\mathcal{H}}\Theta^*\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Lambda}}{\xi^{\Lambda}\,\Gamma(\Lambda+1)} \Big(1+\widehat{J}_{\mathcal{F}}(1+\mathfrak{B}^*)\Big)\Bigg] \\
&:= \overline{\Pi} + \varpi\widehat{\Pi} \leq \varpi.
\end{aligned}
$$

Therefore, the operator $\mathfrak{W}$ is uniformly bounded.

(ii) The operator $\mathfrak{W}$ is equicontinuous:

Let $t_1, t_2 \in \mathfrak{T}$, $t_1 < t_2$, and $y \in \mathfrak{S}_{\varpi}$. Then by the fact that $e^{\frac{\xi-1}{\xi}(\mathfrak{G}(\cdot)-\mathfrak{G}(\cdot))} < 1$, we get

$$
\begin{aligned}
&\|(\mathfrak{W}y)(t_2) - (\mathfrak{W}y)(t_1)\| \\
&\leq \Big\| \mathcal{F}(t_2,y(t_2),\mathfrak{B}y(t_2)) - \mathcal{F}(t_1,y(t_1),\mathfrak{B}y(t_1)) \Big\| + \Big\| \mathcal{H}(t_2,y(t_2)) \left[\overline{\Delta}(\mathfrak{G}(t_2)-\mathfrak{G}(c))^{\Lambda+1} \right. \\
&+ {}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}} \mathcal{M}(t_2,y(t_2),\mathfrak{B}y(t_2)) + {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}} \Theta(t_2) \left(y(t_2) - \mathcal{F}(t_2,y(t_2),\mathfrak{B}y(t_2))\right)\Big] \\
&- \mathcal{H}(t_1,y(t_1)) \Big[\overline{\Delta}(\mathfrak{G}(t_2)-\mathfrak{G}(c))^{\Lambda+1} + {}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}} \mathcal{M}(t_2,y(t_2),\mathfrak{B}y(t_2)) \\
&+ {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}} \Theta(t_2) \left(y(t_2) - \mathcal{F}(t_2,y(t_2),\mathfrak{B}y(t_2))\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathcal{H}(t_1, y(t_1)) \Big[\overline{\Delta}(\mathfrak{G}(t_2) - \mathfrak{G}(c))^{\Lambda+1} + {}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}} \mathcal{M}(t_2, y(t_2), \mathfrak{B}y(t_2)) \\
&+ {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}} \Theta(t_2) \left(y(t_2) - \mathcal{F}(t_2, y(t_2), \mathfrak{B}y(t_2)) \right) \Big] \\
&- \mathcal{H}(t_1, y(t_1)) \Big[\overline{\Delta}(\mathfrak{G}(t_1) - \mathfrak{G}(c))^{\Lambda+1} +_{\xi} I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}} \mathcal{M}(t_1, y(t_1), \mathfrak{B}y(t_1)) \\
&+ {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}} \Theta(t_1) \left(y(t_1) - \mathcal{F}(t_1, y(t_1), \mathfrak{B}y(t_1)) \right) \Big] \Big\| \\
&\le \Big\| \mathcal{F}(t_2, y(t_2), \mathfrak{B}y(t_2)) - \mathcal{F}(t_1, y(t_1), \mathfrak{B}y(t_1)) \Big\| + \Big\| \mathcal{H}(t_2, y(t_2)) - \mathcal{H}(t_1, y(t_1)) \Big\| \\
&\times \Bigg[\left| \overline{\Delta} \right| (\mathfrak{G}(d) - \mathfrak{G}(c))^{\Lambda+1} + \frac{\varpi \big(\mathfrak{G}(d) - \mathfrak{G}(c) \big)^{\Upsilon+\Lambda}}{\xi^{\Upsilon+\Lambda} \Gamma(\Upsilon+\Lambda+1)} \widehat{K}_{\mathcal{M}} (1 + \mathfrak{B}^*) \\
&+ \frac{\varpi \Theta^* \big(\mathfrak{G}(d) - \mathfrak{G}(c) \big)^{\Lambda}}{\xi^{\Lambda} \Gamma(\Lambda+1)} \Big(1 + \widehat{J}_{\mathcal{F}} (1 + \mathfrak{B}^*) \Big) \Bigg] \\
&+ \overline{L}_{\mathcal{H}} \Big[\overline{\Delta} \left[(\mathfrak{G}(t_2) - \mathfrak{G}(c))^{\Lambda+1} - (\mathfrak{G}(t_1) - \mathfrak{G}(c))^{\Lambda+1} \right] \\
&+ \frac{1}{\xi^{\Upsilon+\Lambda} \Gamma(\Upsilon+\Lambda)} \int_c^{t_1} \mathfrak{G}'(s) \Big[\big(\mathfrak{G}(t_2) - \mathfrak{G}(s) \big)^{\Upsilon+\Lambda-1} - \big(\mathfrak{G}(t_1) - \mathfrak{G}(s) \big)^{\Upsilon+\Lambda-1} \Big] \left\| \mathcal{M}(s, y(s), \mathfrak{B}y(s)) \right\| ds \\
&+ \frac{1}{\xi^{\Upsilon+\Lambda} \Gamma(\Upsilon+\Lambda)} \int_{t_1}^{t_2} \mathfrak{G}'(s) \big(\mathfrak{G}(t_2) - \mathfrak{G}(s) \big)^{\Upsilon+\Lambda-1} \left\| \mathcal{M}(s, y(s), \mathfrak{B}y(s)) \right\| ds \\
&+ \frac{\Theta^*}{\xi^{\Lambda} \Gamma(\Lambda)} \int_c^{t_1} \mathfrak{G}'(s) \Big[\big(\mathfrak{G}(t_2) - \mathfrak{G}(s) \big)^{\Lambda-1} - \big(\mathfrak{G}(t_1) - \mathfrak{G}(s) \big)^{\Lambda-1} \Big] \left\| (y(s) - \mathcal{F}(s, y(s), \mathfrak{B}y(s))) \right\| ds \\
&+ \frac{\Theta^*}{\xi^{\Lambda} \Gamma(\Lambda)} \int_{t_1}^{t_2} \mathfrak{G}'(s) \big(\mathfrak{G}(t_2) - \mathfrak{G}(s) \big)^{\Lambda-1} \left\| (y(s) - \mathcal{F}(s, y(s), \mathfrak{B}y(s))) \right\| ds \Big].
\end{aligned}
$$

Thanks to the continuity of the function $\mathfrak{G}(t)$ and Lebesgue-dominated convergence theorem, from the above inequality, we obtain

$$
\|(\mathfrak{W}y)(t_2) - (\mathfrak{W}y)(t_1)\| \to 0 \quad \text{as } t_1 \to t_2.
$$

Then, the operator $\mathfrak{W}$ is equicontinuous. From points (i) and (ii), and by applying the Arzelà-Ascoli theorem, we conclude that $\mathfrak{W}(\mathfrak{S}_{\varpi})$ is relatively compact. Based on steps 2 and 3, it follows that the operator $\mathfrak{W}(\mathfrak{S}_{\varpi})$ is both continuous and compact. Therefore, by Theorem 2, the operator $\mathfrak{W}$ has at least a fixed point in $\mathfrak{S}_{\varpi}$. This, in turn, implies that the nonlinear boundary value Langevin hybrid fractional integro-differential system (1.3) has at least a solution in $\mathfrak{S}_{\varpi}$. □

We will now show that the solution to the boundary value Langevin hybrid fractional integro-differential system (1.3) is unique, as stated in the following theorem:

Theorem 3.2 *Let assumptions $(\mathcal{A}_1) - (\mathcal{A}_3)$ hold. Then the nonlinear boundary value Langevin hybrid fractional integro-differential system (1.3) has a unique solution $C(\mathfrak{T}, \mathbb{R})$ provided that:*

$$
\begin{aligned}
\Sigma := \Bigg\{ & J_{\mathcal{F}}(1 + \mathfrak{B}^*) + \overline{L}_{\mathcal{H}} \Bigg[\frac{\big(\mathfrak{G}(d) - \mathfrak{G}(c) \big)^{\Upsilon+\Lambda}}{\xi^{\Upsilon+\Lambda} \Gamma(\Upsilon+\Lambda+1)} K_{\mathcal{M}} (1 + \mathfrak{B}^*) \\
& + \Theta^* \frac{\big(\mathfrak{G}(d) - \mathfrak{G}(c) \big)^{\Lambda}}{\xi^{\Lambda} \Gamma(\Lambda+1)} (1 + J_{\mathcal{F}}(1 + \mathfrak{B}^*)) \Bigg] \Bigg\} < 1. \qquad (3.1)
\end{aligned}
$$

Proof: Let $t \in \mathfrak{T}$ and $x, y \in \mathfrak{S}_{\varpi}$. Thanks to the assumptions $(\mathcal{A}_1, (i))$, $(\mathcal{A}_2, (i))$, $(\mathcal{A}_3)$, and the fact that $e^{\frac{\xi-1}{\xi}(\mathfrak{G}(\cdot)-\mathfrak{G}(\cdot))}<1$, we get

$$\begin{aligned}
&\|(\mathfrak{W}x)(t)-(\mathfrak{W}y)(t)\| \\
&\leq \left\|\mathcal{F}(t,x(t),\mathfrak{B}x(t))-\mathcal{F}(t,y(t),\mathfrak{B}y(t))\right\| + \Big\|\mathcal{H}(t,x(t))\left[\overline{\Delta}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1}\right. \\
&+ {}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}\mathcal{M}(t,x(t),\mathfrak{B}x(t)) + {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}\Theta(t)\left(x(t)-\mathcal{F}(t,x(t),\mathfrak{B}x(t))\right)\Big] \\
&- \mathcal{H}(t,y(t))\Big[\overline{\Delta}(\mathfrak{G}(t)-\mathfrak{G}(c))^{\Lambda+1} +{}_{\xi} I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}\mathcal{M}(t,y(t),\mathfrak{B}y(t)) + {}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}\Theta(t)\left(y(t)-\mathcal{F}(t,y(t),\mathfrak{B}y(t))\right)\Big]\Big\| \\
&\leq \left\|\mathcal{F}(t,x(t),\mathfrak{B}x(t))-\mathcal{F}(t,y(t),\mathfrak{B}y(t))\right\| + \overline{L}_{\mathcal{H}}\Big[{}_{\xi}I_{c^+}^{\Upsilon+\Lambda,\mathfrak{G}}\|\mathcal{M}(t,x(t),\mathfrak{B}x(t))-\mathcal{M}(t,y(t),\mathfrak{B}y(t))\| \\
&+ \Theta^*\,{}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}\|x(t)-y(t)\| + \Theta^*\,{}_{\xi}I_{c^+}^{\Lambda,\mathfrak{G}}\|\mathcal{F}(t,x(t),\mathfrak{B}x(t))-\mathcal{F}(t,y(t),\mathfrak{B}y(t))\|\Big] \\
&\leq J_{\mathcal{F}}(1+\mathfrak{B}^*)\parallel x-y\parallel + \overline{L}_{\mathcal{H}}\left[\frac{\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Upsilon+\Lambda}}{\xi^{\Upsilon+\Lambda}\Gamma(\Upsilon+\Lambda+1)}K_{\mathcal{M}}\parallel x-y\parallel(1+\mathfrak{B}^*)\right. \\
&\left.+ \Theta^*\frac{\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Lambda}}{\xi^{\Lambda}\Gamma(\Lambda+1)}\parallel x-y\parallel(1+J_{\mathcal{F}}(1+\mathfrak{B}^*))\right] \\
&\leq \parallel x-y\parallel\left\{J_{\mathcal{F}}(1+\mathfrak{B}^*)+\overline{L}_{\mathcal{H}}\left[\frac{\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Upsilon+\Lambda}}{\xi^{\Upsilon+\Lambda}\Gamma(\Upsilon+\Lambda+1)}K_{\mathcal{M}}(1+\mathfrak{B}^*)\right.\right. \\
&\left.\left.+ \Theta^*\frac{\big(\mathfrak{G}(d)-\mathfrak{G}(c)\big)^{\Lambda}}{\xi^{\Lambda}\Gamma(\Lambda+1)}(1+J_{\mathcal{F}}(1+\mathfrak{B}^*))\right]\right\} \\
&:= \Sigma\parallel x-y\parallel.
\end{aligned}$$

□

Due to condition (3.1), the operator $\mathfrak{W}$ is a contraction. Therefore, $\mathfrak{W}$ possesses a unique fixed point $y\in C(\mathfrak{T},\mathbb{R})$, which corresponds to the unique solution of the nonlinear boundary value Langevin hybrid fractional integro-differential system (1.3).

4. Example

In this section, we present a practical example to demonstrate the application of our main findings. Let $\mathfrak{T}=[0,1]$, $\Lambda=\xi=\frac{1}{2}$, $\Upsilon=\frac{3}{2}$, $\Theta(t)=\frac{e^t}{35}$, $\mathfrak{G}(t)=t$,

$$\begin{aligned}
\mathcal{F}(t,y(t),\mathfrak{B}y(t)) &= \frac{e^t|y(t)|}{25(t^2+2)(1+|y(t)|)}+\frac{e^t}{50}\mathfrak{B}y(t), \\
\mathcal{M}(t,y(t),\mathfrak{B}y(t)) &= \frac{t^2}{5(t^4+5)}\sin\left(\frac{\pi}{3}y(t)\right)+\frac{\pi}{3(e^t+5)^2}\mathfrak{B}y(t), \\
\mathcal{H}(t,y(t)) &= \frac{\sin(y(t))}{54}+\frac{1}{18},
\end{aligned}$$

where

$$\mathfrak{B}y(t)=\int_0^t t^3e^s|y(s)|ds.$$

$$\mathfrak{B}^*=\max_{t\in[0,1]}\int_0^t t^3e^sds=e-1\simeq 1,718.$$

We consider the following nonlinear boundary value Langevin hybrid fractional integro-differential system:

$$\begin{cases} {}^{C}_{\frac{1}{2}}D^{\frac{3}{2},t}_{0^+}\left({}^{C}_{\frac{1}{2}}D^{\frac{1}{2},t}_{0^+} + \frac{e}{35}\left(\frac{\sin(y(t))}{54}+\frac{1}{18}\right)\right)\frac{y(t)-\left(\frac{e^t|y(t)|}{25(t^2+2)(1+|y(t)|)}+\frac{e^t}{50}\mathfrak{B}y(t)\right)}{\frac{\sin(y(t))}{54}+\frac{1}{18}} \\ = \frac{t^2}{5(t^4+5)}\sin\left(\frac{\pi}{3}y(t)\right)+\frac{\pi}{3(e^t+5)^2}\mathfrak{B}y(t), \quad \tau\in\mathfrak{T}=[0,1], \\ \left[\frac{y(t)-\left(\frac{e^t|y(t)|}{25(t^2+2)(1+|y(t)|)}+\frac{e^t}{50}\mathfrak{B}y(t)\right)}{\frac{\sin(y(t))}{54}+\frac{1}{18}}\right]_{t=0} = 0, \\ {}^{C}_{\frac{1}{2}}D^{\frac{1}{2},t}_{0^+}\left[\frac{y(t)-\left(\frac{e^t|y(t)|}{25(t^2+2)(1+|y(t)|)}+\frac{e^t}{50}\mathfrak{B}y(t)\right)}{\frac{\sin(y(t))}{54}+\frac{1}{18}}\right]_{t=0} = 0, \\ \left[y(t)-\left(\frac{e^t|y(t)|}{25(t^2+2)(1+|y(t)|)}+\frac{e^t}{50}\mathfrak{B}y(t)\right)\right]_{t=1} = 0, \\ {}^{C}_{\frac{1}{2}}D^{\frac{1}{2},t}_{0^+}\left[\frac{y(t)-\left(\frac{e^t|y(t)|}{25(t^2+2)(1+|y(t)|)}+\frac{e^t}{50}\mathfrak{B}y(t)\right)}{\frac{\sin(y(t))}{54}+\frac{1}{18}}\right]_{t=1} = \nu\in\mathbb{R}. \end{cases} \tag{4.1}$$

First, let us check assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, and $(\mathcal{A}_3)$:
For all $t\in[0,1]$ and $x,\ y\in\mathbb{R}$ we have:

$$\begin{aligned} \|\mathcal{F}(t,x,\mathfrak{B}x)-\mathcal{F}(t,y,\mathfrak{B}y)\| &\le \frac{e^t}{25(t^2+2)}\frac{\|x-y\|}{(1+\|x\|)(1+\|y\|)}+\frac{e^t}{50}\|\mathfrak{B}x-\mathfrak{B}y\| \\ &\le \frac{e}{50}\left[\|x-y\|+\|\mathfrak{B}x-\mathfrak{B}y\|\right]. \\ \|\mathcal{F}(t,y,\mathfrak{B}y)\| &= \left\|\frac{e^t|y(t)|}{25(t^2+2)(1+|y(t)|)}+\frac{e^t}{50}\mathfrak{B}y(t)\right\| \\ &\le \frac{e}{50}\left[\|y\|+\|\mathfrak{B}y\|\right]. \end{aligned}$$

Hence, assumption $(\mathcal{A}_1)$ holds with $J_{\mathcal{F}}=\widehat{J}_{\mathcal{F}}=\frac{e}{50}$.

$$\begin{aligned} \|\mathcal{M}(t,x,\mathfrak{B}x)-\mathcal{M}(t,y,\mathfrak{B}y)\| &\le \frac{\pi}{75}\|x-y\|+\frac{\pi}{3(e^t+5)^2}\|\mathfrak{B}x-\mathfrak{B}y\| \\ &\le \frac{\pi}{75}\left[\|x-y\|+\|\mathfrak{B}x-\mathfrak{B}y\|\right]. \end{aligned}$$

$$\begin{aligned} \|\mathcal{M}(t,y,\mathfrak{B}y)\| &= \left\|\frac{t^2}{5(t^4+5)}\sin\left(\frac{\pi}{3}y(t)\right)+\frac{\pi}{3(e^t+5)^2}\mathfrak{B}y(t)\right\| \\ &\le \frac{\pi}{75}\left[\|y\|+\|\mathfrak{B}y\|\right]. \end{aligned}$$

Then, the assumption $(\mathcal{A}_2)$ holds, with $K_{\mathcal{M}}=\widehat{K}_{\mathcal{M}}=\frac{\pi}{75}$.

$$\|\mathcal{H}(t,y)\| \le \frac{1}{18}+\frac{1}{54}=\frac{2}{27}.$$

Therefore, the assumption $(\mathcal{A}_3)$ holds, with $\overline{L}_{\mathcal{H}}=\frac{2}{27}$.
We note that all the conditions of Theorem 3.1 are fulfilled. Consequently, the nonlinear boundary value Langevin hybrid fractional integro-differential system (4.1) admits at least one solution $y\in C(\mathfrak{T},\mathbb{R})$.

The uniqueness of the solution to problem (4.1) is guaranteed by condition (3.1), which holds as follows:

$$\begin{aligned}\Sigma &:= \left\{ J_{\mathcal{F}}(1+\mathfrak{B}^*) + \overline{L}_{\mathcal{H}} \left[\frac{\big(\mathfrak{G}(1)-\mathfrak{G}(0)\big)^{\Upsilon+\Lambda}}{\xi^{\Upsilon+\Lambda}\,\Gamma(\Upsilon+\Lambda+1)} K_{\mathcal{M}}(1+\mathfrak{B}^*) \right.\right. \\ &\left.\left. + \ \Theta^* \frac{\big(\mathfrak{G}(1)-\mathfrak{G}(0)\big)^{\Lambda}}{\xi^{\Lambda}\,\Gamma(\Lambda+1)} \left(1 + J_{\mathcal{F}}(1+\mathfrak{B}^*)\right) \right] \right\} \\ &= 0,1477 + \frac{2}{27}\left[\frac{0,1138}{\left(\frac{1}{2}\right)^2 \Gamma(3)} + \frac{e}{35} \times \frac{1,1477}{\left(\frac{1}{2}\right)^{\frac{1}{2}} \Gamma(\frac{3}{2})} \right] \\ &\simeq 0,1750 < 1.\end{aligned}$$

Then, the nonlinear boundary value Langevin hybrid fractional integro-differential system (4.1) has a unique solution $y \in C(\mathfrak{T}, \mathbb{R})$.

5. Conclusion

In this study, we examined the existence and uniqueness of solutions for a new class of nonlinear boundary value Langevin hybrid fractional integro-differential systems involving the (Υ, Λ)-order Caputo generalized proportional derivative. The originality of the problem lies in its integration of the Langevin equation with a hybrid framework under the generalized Caputo proportional fractional derivative. By applying Schauder's and Banach's fixed point theorems, we established the existence and uniqueness of solutions, respectively. A representative example is included to effectively illustrate our main results. These findings highlight the versatility and applicability of fixed point theorems in tackling sophisticated mathematical models. This research not only contributes to the growing body of work in fractional calculus and differential equations but also paves the way for further studies on more complex systems such as p-Laplacian and dual-hybrid models with fractional and nonlinear characteristics.

Looking ahead, a promising direction for future research involves extending our results to systems governed by the ψ-Hilfer generalized proportional derivative, along with an investigation into Ulam–Hyers stability. Furthermore, we aim to broaden our study to encompass a novel class of p-Laplacian equations.

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