



On the Existence of Solutions for Hybrid Caputo-Type q -Fractional Differential Equations in Banach Algebra

Khurshida Parvin and Bipan Hazarika

ABSTRACT: In this paper, we investigate the existence of solutions for q -fractional hybrid differential equations in Caputo-type (q -FHDEs)

$$\begin{cases} {}^c\mathfrak{D}_q^\alpha \left[\frac{\phi(\eta) - g(\eta, \phi(\eta))}{h(\eta, \phi(\eta))} \right] = k(\eta, \varphi(\eta)), & \eta \in \mathcal{J}, \\ {}^c\mathfrak{D}_q^\alpha \left[\frac{\varphi(\eta) - g(\eta, \varphi(\eta))}{h(\eta, \varphi(\eta))} \right] = k(\eta, \phi(\eta)), & \eta \in \mathcal{J}, \quad q \in (0, 1), \\ \phi(0) = 0, \quad \varphi(0) = 0, \end{cases}$$

where ${}^c\mathfrak{D}_q^\alpha$ represents the fractional q -derivative in Caputo type with order $0 < \alpha < 1$, let $g : \mathcal{J} \times \mathbb{R} \rightarrow \mathbb{R} - \{0\}$, $h : \mathcal{J} \times \mathbb{R} \rightarrow \mathbb{R}$, $g(0, 0) = 0$ and $k : \mathcal{J} \times \mathbb{R} \rightarrow \mathbb{R}$ be the functions that satisfy certain prescribed conditions. The main results are established by employing a coupled fixed point theorem. An illustrative example is presented to demonstrate the applicability of the theoretical findings.

Keywords: Fractional q -differential equations, Hybrid differential equations, coupled fixed point theorem, Banach algebras.

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1. Introduction

Quantum calculus sometimes called calculus without limits, has attracted growing interest in recent years. Its origins date back to 1908, stemming from the foundational work of Jackson [11]. Unlike classical calculus, quantum calculus eliminates the reliance on limits, offering a broader mathematical framework. A notable branch of this field is known as q -calculus.

It is important to highlight that perturbation methods play a significant role in nonlinear analysis, particularly for examining the behavior of dynamical systems governed by nonlinear differential and integral equations. For certain dynamical systems, the corresponding differential equations may not admit exact analytical solutions; in such cases, applying perturbation methods can provide valuable approximations. Perturbed differential equations can be classified into several categories, among which a notable type is the hybrid differential equation, which combines continuous dynamics with discrete events or structures.

In recent years, fractional hybrid differential equations have attracted considerable interest from researchers. For fundamental contributions and previous studies in this area, see references [3,5,7,8,9,14]. The developments in the study of the existence results for FHDE's have been addressed in [13,15,16].

In 2009, Gafiychuk et al. [10] studied the nonlinear reaction-diffusion process described by fractional differential equations, while Su [17] examined the two-point boundary value problem associated with a coupled system of fractional differential equations. In recent years, coupled systems of fractional differential equations have garnered significant attention from mathematical scientists due to their wide range of practical applications. A wide range of problems related to this topic has been explored in the literature. For further details, the readers are referred to [2,8].

2020 *Mathematics Subject Classification*: 26A33, 34A38, 32A65.

Submitted September 12, 2025. Published March 12, 2026

Further generalization of the results to the coupled system in the following was carried out by A. Ali, et al. in [3].

$$\begin{cases} {}^c\mathfrak{D}^x \left(\frac{a(\varsigma) - l(\varsigma, a(\varsigma), b(\varsigma))}{l_1(\varsigma, a(\varsigma), b(\varsigma))} \right) = \nu(\varsigma, a(\varsigma), b(\varsigma)), & a.e. \varsigma \in K = [0, 1], \\ {}^c\mathfrak{D}^y \left(\frac{b(\varsigma) - m(\varsigma, a(\varsigma), b(\varsigma))}{m_1(\varsigma, a(\varsigma), b(\varsigma))} \right) = \mu(\varsigma, a(\varsigma), b(\varsigma)), & a.e. \varsigma \in K = [0, 1], \\ a(0) = u, \quad a(1) = v, \quad b(0) = w, \quad b(1) = z, \end{cases}$$

where $\varsigma \in K = [0, 1]$, and the fractional orders x, y lie in the interval $(1, 2]$. The operator ${}^c\mathfrak{D}$ denotes the Caputo fractional derivative. The constants u, v, w, z are real numbers, while the functions $l_1, m_1 : K \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+ - \{0\}$ and $l, m, \nu, \mu : K \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are assumed to be continuous.

The study by T. Bashiri et. al. [5] in 2016 focused on proving the existence of solutions for certain FHDE in Banach algebra:

$$\begin{cases} \mathfrak{D}^\delta \left(\frac{a(\xi) - u(\xi, a(\xi))}{v(\xi, a(\xi))} \right) = \mu(\xi, b(\xi)), \\ \mathfrak{D}^\delta \left(\frac{b(\xi) - u(\xi, b(\xi))}{v(\xi, b(\xi))} \right) = \mu(\xi, a(\xi)), \quad \xi \in \mathbb{K}, \quad \delta \in (0, 1), \\ a(0) = 0, \quad b(0) = 0. \end{cases}$$

Here, \mathfrak{D}^δ represents the RL-fractional derivative with order δ , $\mathbb{K} = [0, 1]$ and the functions $f : \mathbb{K} \times \mathbb{R} \rightarrow \mathbb{R} - \{0\}$, $v : \mathbb{K} \times \mathbb{R} \rightarrow \mathbb{R}$ with $v(0, 0) = 0$, $\mu : \mathbb{K} \times \mathbb{R} \rightarrow \mathbb{R}$ are assumed to satisfy specific conditions.

In [12], the authors studied the coupled systems of the following hybrid fractional differential equations

$$\begin{cases} \mathfrak{D}^\beta \left(\frac{a(\varsigma) - x(\varsigma, a(\varsigma))}{y(\varsigma, a(\varsigma))} \right) = \mu(\varsigma, b(\varsigma)), \\ \mathfrak{D}^\beta \left(\frac{b(\varsigma) - x(\varsigma, b(\varsigma))}{y(\varsigma, b(\varsigma))} \right) = \mu(\varsigma, a(\varsigma)), \end{cases}$$

with the boundary conditions $a(\varsigma) = u$, $b(0) = v$, $\varsigma \in \mathcal{J} = [0, 1]$, where \mathfrak{D}^β is the Riemann-Liouville fractional derivative with order $\beta \in (0, 1)$, $u, v \in \mathbb{R}$.

To best of our knowledge, no research has been conducted for the existence of solutions for Hybrid Caputo q -FDEs in Banach Algebra. Also q -FHDEs are studied due to their ability to describe complex dynamical behaviors that exhibit memory effects, discrete patterns, and sudden transitions. In this work, we investigate the existence of solutions for a system of Caputo q -fractional hybrid differential equations

$$\begin{cases} {}^c\mathfrak{D}_q^\alpha \left[\frac{\phi(\eta) - g(\eta, \phi(\eta))}{h(\eta, \phi(\eta))} \right] = k(\eta, \varphi(\eta)), \quad \eta \in \mathcal{J}, \\ {}^c\mathfrak{D}_q^\alpha \left[\frac{\varphi(\eta) - g(\eta, \varphi(\eta))}{h(\eta, \varphi(\eta))} \right] = k(\eta, \phi(\eta)), \quad \eta \in \mathcal{J}, \quad q \in (0, 1), \\ \phi(0) = 0, \quad \varphi(0) = 0, \end{cases} \quad (1.1)$$

This problem (1.1) represents a generalization of the problem discussed by T. Bashiri in [5].

This paper is organized as follows: Some fundamental definitions and notations about q -calculus are presented in the first section. Afterward, the core result is derived using a couple fixed point approach. To conclusion, an approach example is provided to illustrate the main result.

2. Preliminaries

In this section, we consider the space $\mathfrak{S} = \mathfrak{C}(\mathcal{J}, \mathbb{R})$, consisting of all continuous real valued functions defined on the interval \mathcal{J} , endowed with the supremum norm. It is well known that \mathfrak{S} , equipped with point-wise addition and multiplication, constitute a Banach algebra.

Let \mathfrak{S} be a subset of $\mathfrak{C}([0, 1], \mathbb{R})$, and consider the norm defined by

$$\|\phi\| = \sup\{|\phi(\eta)| : \eta \in [0, 1]\}.$$

Then, \mathfrak{S} endowed with pointwise operations and the norm $\|\cdot\|$, forms a Banach algebra. Specifically, for any $\phi, \varphi \in \mathfrak{S}$, the operation \cdot is defined pointwise as follows

$$(\phi \cdot \varphi)(\eta) = \phi(\eta) \cdot \varphi(\eta), \quad \forall \eta \in [0, 1],$$

under which \mathfrak{S} remains a Banach algebra.

The normed product space $\mathfrak{S} = \mathfrak{X} \times \mathfrak{X}$ is defined as

$$\|(\phi, \varphi)\| = \|\phi\| + \|\varphi\|$$

is also a Banach space, whenever \mathfrak{X} is itself a Banach space.

Define addition and scalar multiplication in the product space $\mathfrak{X} \times \mathfrak{X}$ by the following rules

$$(\phi_1, \varphi_1) + (\phi_2, \varphi_2) = (\phi_1 + \phi_2, \varphi_1 + \varphi_2),$$

$$\lambda(\phi, \varphi) = (\lambda\phi, \lambda\varphi), \text{ for } \lambda \in \mathbb{R}.$$

We present several precise definitions and lemmas that are essential for establishing the main results in the following sections. For further details, the reader may refer to [1,4].

Let $q \in (0, 1)$ be a real parameter and $\alpha \in \mathbb{R}$. The q -number of α , denoted by $[\alpha]_q$ is defined as

$$[\alpha]_q = \frac{1 - q^\alpha}{1 - q}, \quad \alpha \in \mathbb{R}.$$

For $n \in \mathbb{N}$, the q -analogue of the expression $(a - b)^n$ is defined as follows

$$(u - v)^{(0)} = 1,$$

and

$$(x - y)^{(n)} = \prod_{i=0}^{n-1} (x - q^i y), \quad \forall n \in \mathbb{N}, x, y \in \mathbb{R}.$$

Again, if $\delta \in \mathbb{R}$, then

$$(x - y)^{(\delta)} = x^\delta \prod_{j=0}^{\infty} \frac{x - q^j y}{x - q^{\delta+j} y}.$$

We can easily see that

$$[x(a - b)]^{(\delta)} = x^\delta (a - b)^{(\delta)}.$$

Moreover, note that if $y = 0$ then $x^{(\delta)} = x^\delta$.

Definition 2.1 [11] The q -gamma function, denoted by $\Gamma_q(w)$ is defined for $w \in \mathbb{R} - \{0, 1, 2, \dots\}$ and $0 < q < 1$ by

$$\Gamma_q(w) = \frac{(1 - q)^{(w-1)}}{(1 - q)^{w-1}},$$

and it fulfills the recurrence relation

$$\Gamma_q(w + 1) = [w] \Gamma_q(w).$$

The function x has a q -derivative at v denoted by $\mathfrak{D}_q x(v)$ and defined as

$$(\mathfrak{D}_q x)(v) = \frac{x(qu) - x(u)}{(1 - q)u}, \quad u \neq 0, \quad (\mathfrak{D}_q x)(0) = \lim_{u \rightarrow 0} (\mathfrak{D}_q x)(u)$$

and higher order q -derivatives are defined as

$$\mathfrak{D}_q^0 x(u) = x(u),$$

$$\mathfrak{D}_q^m x(u) = \mathfrak{D}(\mathfrak{D}_q^{m-1} x(u)), \quad m \in \mathbb{N}.$$

The integration of x with respect to the q -calculus over $[0, \eta]$, for $a \in [0, \eta]$ is defined as

$$(I_q x)(a) = \int_0^a x(\varsigma) d_q \varsigma = a(1 - q) \sum_{m=0}^{\infty} x(aq^m) q^m, \quad a \in [0, \eta].$$

Let y be a function defined in $[0, \eta]$. Then, its q -integral over the interval $[\xi, \eta]$ is given by

$$\int_{\xi}^{\eta} y(\zeta) d_q \zeta = \int_0^{\eta} y(\zeta) d_q \zeta - \int_0^{\xi} y(\zeta) d_q \zeta.$$

Just as we define higher order q -derivatives, we can define the m -th order q -integral operator, denoted I_q^m as follows

$$\begin{aligned} I_q^0 x(\zeta) &= x(\zeta), \\ I_q^m x(\zeta) &= I_q(I_q^{m-1} x(\zeta)), \quad m \in \mathbb{N}. \end{aligned}$$

The fundamental result in q -calculus that establishes the relationship involving the q -derivative \mathfrak{D}_q and the q -integral I_q is

$$\mathfrak{D}_q(I_q x(\zeta)) = x(\zeta).$$

Definition 2.2 [1] Consider a function x defined on the interval $[0, \eta]$, with $\beta > 0$. The q -fractional integral of order β , denoted by $I_q^\beta x(\zeta)$, is defined by

$$I_q^\beta x(\zeta) = \frac{1}{\Gamma_q(\beta)} \int_0^{\zeta} (\zeta - qs)^{\beta-1} x(s) d_q s, \quad \zeta \in [0, \eta],$$

with the special case $I_q^0 x(\zeta) = x(\zeta)$, where $\Gamma_q(\beta)$ denotes the q -Gamma function.

Definition 2.3 [4] The Caputo q -fractional derivative of order $\beta > 0$ is defined as

$${}^C \mathfrak{D}_q^\alpha x(\zeta) = I_q^{[m]-\alpha} \mathfrak{D}_q^{[m]} x(\zeta),$$

where $[m]$ is the smallest integer such that $[m] \geq \alpha$.

Lemma 2.1 [4] Let $\alpha > 0$ and let ϕ be a function defined on the interval $[0, \eta]$. Then, the following properties of the q -fractional integral and derivative hold:

1. $I_q^\alpha I_q^\alpha \phi(\eta) = I_q^{2\alpha} \phi(\eta)$;
2. $\mathfrak{D}_q^\alpha I_q^\alpha \phi(\eta) = \phi(\eta)$.

Definition 2.4 [6] We say that $(r, s) \in \mathfrak{X} \times \mathfrak{X}$ is said to be a coupled fixed point of the mapping $\mathfrak{F} \rightarrow \mathfrak{X} \times \mathfrak{X}$ if it satisfies $\mathfrak{F}(r, s) = r$ and $\mathfrak{F}(r, s) = s$.

Let ω represent the collection of all functions $\Omega_q : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\Omega_q(0) = 0$ and $\Omega_q(p) < p$ for $p > 0$.

A function $(x, y) \in AC(\mathcal{J}, \mathbb{R} \times \mathbb{R})$ is called a solution to the q -FHDE system (1.1) if it satisfies the system in the interval \mathcal{J}

- (i) For every $\phi \in \mathbb{R}$, the mapping $\eta \rightarrow \frac{\phi(\eta) - g(\eta, \phi(\eta))}{h(\eta, \phi(\eta))}$ is absolutely continuous.
- (ii) Equation (1.1) admits (ϕ, φ) as a solution.

We denote by $AC(\mathcal{J}, \mathbb{R} \times \mathbb{R})$ the collection of functions that are absolutely continuous on \mathcal{J} into $\mathbb{R} \times \mathbb{R}$.

Definition 2.5 [5] The mapping $\mathbb{P} : \mathfrak{X} \rightarrow \mathfrak{X}$ is said to be a δ -contraction when \exists a constant $\delta \in (0, 1)$ along with a function $\Psi_{\mathbb{P}} \in \Omega_q$ fulfills the required condition

$$\|\mathbb{P}\phi - \mathbb{P}\varphi\| = \delta \Psi_{\mathbb{P}}(\|\phi - \varphi\|), \quad \forall \phi, \varphi \in \mathfrak{X},$$

where $\Psi_{\mathbb{P}}$ is a nonlinear function associated with \mathfrak{X} .

Lemma 2.2 [5] Consider a subset \mathfrak{S} of the Banach algebra \mathfrak{X} that is non-empty, convex and bounded. Define the cartesian product set:

$$\mathfrak{S} = \tilde{\mathfrak{S}} \times \tilde{\mathfrak{S}}.$$

Consider the operators $\mathbb{P}, \mathcal{R} : \mathfrak{X} \rightarrow \mathfrak{X}$, and $\mathbb{Q} : \mathfrak{S} \rightarrow \mathfrak{X}$ satisfying the following conditions:

(H1) The operator \mathbb{P} is assumed to be a δ -nonlinear contraction associated with a nonlinear contraction function $\Psi_{\mathbb{P}}$, while the operator \mathbb{Q} is a τ -nonlinear contraction corresponding to a nonlinear contraction function $\Psi_{\mathbb{Q}}$.

(H2) The mapping \mathcal{R} possesses continuity.

(H3) $\phi = \mathbb{P}\phi\mathbb{Q}\varphi + \mathcal{R}\phi \implies \phi \in \mathfrak{S}, \forall \varphi \in \mathfrak{S}$,

(H4) Suppose the inequality $4\delta\|\mathbb{Q}\mathfrak{S}\| + \tau < 1$ holds, where $\|\mathbb{Q}\mathfrak{S}\| = \sup\{|\mathbb{Q}(\phi)| : \phi \in \mathfrak{S}\}$.

Then, the operator $\mathfrak{F}(\phi, \varphi) = \mathbb{P}\phi\mathbb{Q}\varphi + \mathcal{R}\phi$ possesses at least one coupled fixed point in \mathfrak{S} .

Lemma 2.3 Let $k \in \mathfrak{C}[0, 1]$, with $0 < \alpha < 1$, and suppose that $g, h \in \mathfrak{C}([0, 1] \times \mathbb{R}, \mathbb{R})$, where $g \in (0, 0) = 0$ and $h(0, 0) \neq 0$. Then, the unique solution to the following system is given by

$$\begin{cases} {}^c\mathfrak{D}_q^\alpha \left[\frac{\phi(\eta) - g(\eta, \phi(\eta))}{h(\eta, \phi(\eta))} \right] = k(\eta), \quad \eta \in (0, 1), \\ \phi(0) = 0, \end{cases} \quad (2.1)$$

is

$$\phi(\eta) = \frac{h(\eta, \phi(\eta))}{\Gamma_q(\alpha)} \int_0^\eta \frac{k(s)}{(\eta - qs)^{1-\alpha}} d_qs + g(\eta, \phi(\eta)), \quad \eta \in [0, 1].$$

Proof: Applying the operator I_q^α to equation ${}^c\mathfrak{D}_q^\alpha \left[\frac{\phi(\eta) - g(\eta, \phi(\eta))}{h(\eta, \phi(\eta))} \right] = k(\eta)$, $\eta \in (0, 1)$ and utilizing the boundary conditions, we have

$$\phi(\eta) = h(\eta, \phi(\eta))(I_q^\alpha k(\eta) + a + b\eta),$$

where a and b are arbitrary constants. Now, we use the boundary conditions and substitute the value of constants, we get

$$\frac{\phi(\eta) - g(\eta, \phi(\eta))}{h(\eta, \phi(\eta))} = \frac{1}{\Gamma_q(\alpha)} \int_0^\eta \frac{k(s)}{(\eta - qs)^{1-\alpha}} d_qs.$$

This implies

$$\phi(\eta) = \frac{h(\eta, \phi(\eta))}{\Gamma_q(\alpha)} \int_0^\eta \frac{k(s)}{(\eta - qs)^{1-\alpha}} d_qs + g(\eta, \phi(\eta)).$$

3. Main Results

We formulate the subsequent hypothesis to support our existence results

(E1) Assume $h : \mathcal{J} \times \mathbb{R} \rightarrow \mathbb{R} - \{0\}$, $g : \mathcal{J} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions with $g(0, 0) = 0$. We consider two non-negative functions Ω_q and χ_q , bounded by $\|\Omega_q\|$ and $\|\chi_q\|$, such that for all $\phi, \varphi \in \mathfrak{X}$ and $\eta \in \mathcal{J}$, we get

$$\begin{aligned} |h(\eta, \phi(\eta)) - h(\eta, \varphi(\eta))| &\leq \Omega_q(\eta)|\phi(\eta) - \varphi(\eta)|, \\ |g(\eta, \phi(\eta)) - g(\eta, \varphi(\eta))| &\leq \chi_q(\eta)|\phi(\eta) - \varphi(\eta)|. \end{aligned}$$

(E2) Let $\vartheta \in \mathfrak{C}(\mathcal{J}, \mathbb{R})$ be a continuous function such that for all $\eta \in \mathcal{J}$ and for every $e \in \mathbb{R}$, the inequality

$$\vartheta(\eta, e) \leq \vartheta(\eta).$$

(E3) We define the Banach space $\mathfrak{X} = \mathfrak{C}(\mathcal{J}, \mathbb{R})$ comprises all continuous functions from \mathcal{J} to \mathbb{R} , with the supremum norm as its norm structure. Define

$$\mathbb{S} = \{\phi \in \mathfrak{X} : \|\phi\| \leq \mathbb{M}_q\},$$

where \mathbb{M}_q is mathematically represented by:

$$\mathbb{M}_q = \frac{\mathbb{U}_q \nabla + \mathbb{V}_q}{1 - \|\Omega_q\| \nabla - \|\chi_q\|},$$

with the constants defined as $\mathbb{U}_q = \max_{\varsigma \in \mathcal{J}} |h(\eta, 0)|$, $\mathbb{V}_q = \max_{\varsigma \in \mathcal{J}} |g(\eta, 0)|$ and $\nabla = \frac{\|\vartheta\|}{\Gamma_q(\alpha+1)}$.

Theorem 3.1 *Assume that hypothesis (E1), (E2) and (E3) are satisfied. Then, the system of q -FHDEs (1.1) has at least one solution on the interval \mathcal{J} , providing that the following condition holds*

$$\frac{4}{\Gamma_q(\alpha+1)} \|\Omega_q\| \|\vartheta\| + \|\chi_q\| < 1.$$

Proof: We make use of a coupled fixed point result to show the operator \mathfrak{F} has a coupled fixed point. Since \mathfrak{X} is a closed, convex and bounded subset of the Banach algebra \mathfrak{S} . A function $\phi(\eta)$ is a solution of the system of q -FHDEs given by (1.1) if and only if $\phi(\eta)$ holds the associated system of integral equations of the form

$$\begin{cases} \phi(\eta) = \frac{h(\eta, \phi(\eta))}{\Gamma_q(\alpha)} \int_0^\eta \frac{k(s, \varphi(s))}{(\eta - qs)^{1-\alpha}} d_qs + g(\eta, \phi(\eta)), \\ \varphi(\eta) = \frac{h(\eta, \varphi(\eta))}{\Gamma_q(\alpha)} \int_0^\eta \frac{k(s, \phi(s))}{(\eta - qs)^{1-\alpha}} d_qs + g(\eta, \varphi(\eta)), \quad \eta \in \mathcal{J}. \end{cases} \quad (3.1)$$

Next, we construct the operators $\mathbb{P}, \mathcal{R} : \mathfrak{X} \rightarrow \mathfrak{X}$ and $\mathbb{Q} : \mathfrak{S} \rightarrow \mathfrak{X}$, given by:

$$\begin{cases} \mathbb{P}\phi(\eta) = h(\eta, \phi(\eta)), \\ \mathbb{Q}\phi(\eta) = \frac{1}{\Gamma_q(\alpha)} \int_0^\eta (\eta - qs)^{\alpha-1} k(s, \phi(s)) d_qs, \\ \mathcal{R}\phi(\eta) = g(\eta, \phi(\eta)), \quad \eta \in \mathcal{J}. \end{cases}$$

Thus, system (3.1) is thereby reduced to an equivalent formulation in terms of operator equations

$$\begin{cases} \phi(\eta) = \mathbb{P}\phi(\eta)\mathbb{Q}\varphi(\eta) + \mathcal{R}\phi(\eta), \\ \varphi(\eta) = \mathbb{P}\varphi(\eta)\mathbb{Q}\phi(\eta) + \mathcal{R}\varphi(\eta), \quad \eta \in \mathcal{J}. \end{cases} \quad (3.2)$$

We now verify that the operators \mathbb{P} , \mathbb{Q} and \mathcal{R} meet all the conditions outlined in Lemma 2.2 by proceeding through the steps below

Step-I: From the assumption (E1), let $\phi, \varphi \in \mathfrak{X}$, we get

$$\begin{aligned} |\mathbb{P}\phi(\eta) - \mathbb{P}\varphi(\eta)| &= |h(\eta, \phi(\eta)) - h(\eta, \varphi(\eta))| \\ &\leq \Omega_q(\eta) |\varphi(\eta) - \phi(\eta)|. \end{aligned}$$

Now, we take the supremum overall η ,

$$\|\mathbb{P}\phi(\eta) - \mathbb{P}\varphi(\eta)\| \leq \|\Omega_q\| \|\phi(\eta) - \varphi(\eta)\|.$$

Hence, the operator \mathbb{P} is Lipschitz continuous on \mathfrak{X} with a Lipschitz constant given by $\|\Omega_q\|$. In a similar manner, the operator \mathfrak{F} is also Lipschitz continuous on \mathfrak{X} , possessing $\|\chi_q\|$ as its Lipschitz constant.

Step-II: To establish that \mathbb{Q} is a continuous and compact operator on \mathfrak{S} , we proceed as follows. Let $\{\phi_m\}$ be any sequence in \mathfrak{S} which converges to a point $\phi \in \mathfrak{S}$, and by Lebesgue dominated convergence (LDC) theorem, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathbb{Q}\phi_m(\eta) &= \lim_{m \rightarrow \infty} \frac{1}{\Gamma_q(\alpha)} \int_0^\eta (\eta - qs)^{\alpha-1} \vartheta(s - \phi(s)) d_qs \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^\eta (\eta - qs)^{\alpha-1} \lim_{m \rightarrow \infty} \vartheta(s, \phi_m(s)) d_qs \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^\eta (\eta - qs)^{\alpha-1} \vartheta(s, \phi(s)) d_qs \\ &= \mathbb{Q}\phi(\eta), \text{ for every } \eta \in \mathcal{J}. \end{aligned}$$

As $m \rightarrow \infty$ so, $\mathbb{Q}\phi_m(\eta) \rightarrow \mathbb{Q}\phi(\eta)$ uniformly on \mathbb{R}^+ and it follows that \mathbb{Q} is continuous on \mathcal{J} .

Next, we demonstrate that $\mathbb{B}(s)$ is a uniformly bounded subset of \mathfrak{S} . Consider $\phi \in \mathfrak{S}$ by using assumption (E2), it follows $\forall \eta \in \mathcal{J}$ that

$$\begin{aligned} |\mathbb{Q}\phi(\eta)| &= \frac{1}{\Gamma_q(\alpha)} \left| \int_0^\eta (\eta - qs)^{\alpha-1} \vartheta(s, \phi(s)) d_qs \right| \\ &\leq \frac{1}{\Gamma_q(\alpha)} \int_0^\eta (\eta - qs)^{\alpha-1} |\vartheta(s, \phi(s))| d_qs \\ &\leq \frac{1}{\Gamma_q(\alpha)} \int_0^\eta (\eta - qs)^{\alpha-1} \vartheta(s) d_q(s) \\ &\leq \frac{\|\vartheta\|}{\Gamma_q(\alpha)} \int_0^\eta (\eta - qs)^{\alpha-1} d_qs \\ &\leq \frac{\|\vartheta\|}{\Gamma_q(\alpha + 1)}. \end{aligned}$$

If we take supremum, we get for all $\phi \in \mathfrak{S}$

$$\|\mathbb{Q}\phi(\eta)\| \leq \frac{\|\vartheta\|}{\Gamma_q(\alpha + 1)} = \nabla.$$

Therefore, \mathbb{Q} is uniformly bounded on \mathfrak{S} .

We begin by showing that the set $\mathbb{Q}(\mathfrak{S})$ is equicontinuous. To do this, we first establish that the operator \mathbb{Q} is uniformly bounded.

Consider $\eta_1, \eta_2 \in \mathcal{J}$ and $\phi \in \mathfrak{S}$,

$$\begin{aligned} |\mathbb{Q}\phi(\eta_1) - \mathbb{Q}\phi(\eta_2)| &= \frac{1}{\Gamma_q(\alpha)} \left| \int_0^{\eta_1} (\eta_1 - qs)^{\alpha-1} \vartheta(s, \phi(s)) d_qs - \int_0^{\eta_2} (\eta_2 - qs)^{\alpha-1} \vartheta(s, \phi(s)) d_qs \right| \\ &\leq \frac{1}{\Gamma_q(\alpha)} \left| \int_0^{\eta_1} (\eta_1 - qs)^{\alpha-1} \vartheta(s, \phi(s)) d_qs - \int_0^{\eta_1} (\eta_2 - qs)^{\alpha-1} \vartheta(s, \phi(s)) d_qs \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^{\eta_1} (\eta_2 - qs)^{\alpha-1} \vartheta(s, \phi(s)) d_qs - \int_0^{\eta_2} (\eta_2 - qs)^{\alpha-1} \vartheta(s, \phi(s)) d_qs \right| \\
& \leq \frac{\|\vartheta\|}{\Gamma_q(\alpha)} \left(\left| \int_0^{\eta_1} \{(\eta_1 - qs)^{\alpha-1} - (\eta_2 - qs)^{\alpha-1}\} d_qs \right| + \left| \int_{\eta_1}^{\eta_2} (\eta_2 - qs)^{\alpha-1} d_qs \right| \right) \\
& \leq \nabla (|\eta_1^\alpha - \eta_2^\alpha| + |\eta_2 - \eta_1|^\alpha).
\end{aligned}$$

Given that η^α is uniformly continuous on \mathcal{J} , for every $\epsilon > 0$, \exists an $\delta > 0$ such that whenever $|\eta_1 - \eta_2| < \delta$, we get

$$|\eta_1^\alpha - \eta_2^\alpha| < \frac{1}{2\nabla} \epsilon.$$

Setting

$$\delta = \min \left\{ \delta_1, \left(\frac{1}{2\nabla} \nabla \right)^{\frac{1}{\alpha}} \right\},$$

if $|\eta_2 - \eta_1| < \delta$, we get

$$|\mathbb{Q}\phi(\eta_1) - \mathbb{Q}\phi(\eta_2)| < \nabla \left(\frac{1}{2\nabla} \epsilon + \frac{1}{2\nabla} \epsilon \right) = \epsilon.$$

This confirms that $\mathbb{Q}(\mathfrak{S})$ is an equicontinuous set. Consequently, \mathbb{Q} is a completely continuous operator on \mathfrak{S} .

Step-III: To verify condition (H3) of Lemma 2.2, consider $\phi \in \mathbb{P}$ and $\varphi \in \mathfrak{S}$ be s.t., according to assumption (E1), we get:

$$\begin{aligned}
|\phi(\eta)| & = |\mathbb{P}\phi(\eta)\mathbb{Q}\varphi(\eta) + \mathbb{R}\phi(\eta)| \\
& \leq |\mathbb{P}\phi(\eta)| |\mathbb{Q}\varphi(\eta)| + |\mathbb{R}\phi(\eta)| \\
& \leq \frac{\|\vartheta\|}{\Gamma_q(\alpha)} \left(|h(\eta, \phi(\eta)) - h(\eta, 0)| + |h(\eta, 0)| \right) \int_0^\eta |(\eta - qs)^{\alpha-1}| d_qs + (|g(\eta, \phi(\eta)) - g(\eta, 0)| + |g(\eta, 0)|) \\
& \leq \frac{\|\vartheta\|}{\Gamma_q(\alpha+1)} (\Omega_q(\eta)(|\phi(\eta)| + |h(\eta, 0)|) + (\chi_q(\eta)(|\phi(\eta)|) + |g(\eta, 0)|) \\
& \leq (\|\Omega_q\|(\|\phi\|) + \mathbb{U}_q + (\|\chi_q\|(\|\phi\|) + \mathbb{V}_q)).
\end{aligned}$$

Now, taking the supremum over $\eta \in \mathcal{J}$ and utilizing (E3). Then, we have

$$\begin{aligned}
\|\phi(\eta)\| (1 - \nabla(\|\Omega_q\| - \|\chi_q\|)) & \leq \nabla \mathbb{U}_q + \mathbb{V}_q, \\
\|\phi(\eta)\| & \leq \frac{\nabla \mathbb{U}_q + \mathbb{V}_q}{1 - \nabla(\|\Omega_q\| - \|\chi_q\|)} = \mathbb{M}_q.
\end{aligned}$$

This implies that $\phi \in \mathfrak{S}$.

Step-IV: Finally, we establish that $4\delta\|\mathbb{Q}\mathfrak{S}\| + \tau \leq 1$, thereby confirming that condition (H4) of Lemma 2.2 is satisfied.

Since $\|\mathbb{Q}\mathfrak{S}\| = \sup\{|\mathbb{Q}(\phi)(\eta)|\} \leq \frac{\|\vartheta\|}{\Gamma_q(\alpha+1)}$. Also

$$4\delta\|\mathbb{Q}\mathfrak{S}\| + \tau \leq 4\|\Omega_q\| \left(\frac{\|\vartheta\|}{\Gamma_q(\alpha+1)} \right) + \|\chi_q\| \leq 4\|\Omega_q\|\nabla + \|\chi_q\| < 1,$$

where $\delta = \|\Omega_q\|$, $\nabla = \frac{\|\vartheta\|}{\Gamma_q(\alpha+1)}$, $\tau = \|\chi_q\|$.

Thus, assumption (E3) of Lemma 2.2 has been verified. Consequently, as all its assumptions of Lemma 2.2 are satisfied. It follows that the operator $\mathfrak{F}(\phi, \varphi) = \mathbb{P}\phi\mathbb{Q}\varphi + \mathbb{R}\phi$ admits a coupled fixed point in \mathbb{P} . Therefore, the q -fractional hybrid differential equation q -FHDEs given in (1.1) possesses a solution in the interval \mathcal{J} . This completes the proof. \square

This section illustrates the correctness of the results by presenting one example.

Example 3.1 Suppose the following q -FHDEs

$$\begin{cases} \mathfrak{D}_{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}}^{\frac{1}{2}} \left(\frac{\phi(\eta) - g(\eta, \phi(\eta))}{h(\eta, \phi(\eta))} \right) = \frac{\eta^2 |\phi(\eta)|}{(9 + e^{2\eta}) |\phi(\eta) + 3|}, & \eta \in \mathcal{J} \\ \mathfrak{D}_{\frac{1}{5}, \frac{1}{5}, \frac{1}{2}}^{\frac{1}{2}} \left(\frac{\varphi(\eta) - g(\eta, \varphi(\eta))}{h(\eta, \varphi(\eta))} \right) = \frac{\eta^2 |\varphi(\eta)|}{(9 + e^{2\eta}) |\varphi(\eta) + 3|}, & \eta \in \mathcal{J} \\ \phi(0) = 0, & \varphi(0) = 0, \end{cases} \quad (3.3)$$

where

$$\begin{aligned} h(\eta, \phi(\eta)) &= \frac{1}{14} \eta \left(\frac{1}{15} \cos \phi(\eta) + \frac{\phi^2(\eta) + |\phi(\eta)|}{|1 + \cos \phi(\eta)|} \right), \\ g(\eta, \phi(\eta)) &= \frac{1}{7} e^\eta \left(\frac{1}{3} \cos \phi(\eta) + \frac{4|\phi(\eta)|}{1 + |\cos \phi(\eta)|} \right). \end{aligned}$$

For any elements $\phi, \varphi \in \mathcal{J}$ and any $\eta \in [0, 1]$, it holds that

$$\begin{aligned} |h(\eta, \phi(\eta)) - h(\eta, \varphi(\eta))| &\leq \frac{1}{15} |\phi(\eta) - \varphi(\eta)|, \\ |g(\eta, \phi(\eta)) - g(\eta, \varphi(\eta))| &\leq \frac{1}{3} e^\eta |\phi(\eta) - \varphi(\eta)|, \\ |\vartheta(\eta, \phi(\eta)) - \vartheta(\eta, \varphi(\eta))| &\leq \frac{\eta^2}{9 + e^{2\eta}}. \end{aligned}$$

This implies that

$$\begin{aligned} \Omega_q(\eta) &= \frac{1}{15} \eta, \quad \chi_q(\eta) = \frac{e^\eta}{7}, \quad \|\Omega_q\| \approx 0.0667, \quad \|\chi_q\| \approx 0.3883, \quad \mathbb{U}_q = 0, \\ \mathbb{V}_q &= 0.3523, \quad \nabla = \frac{\|\vartheta\|}{\Gamma_q(\alpha + 1)} = 0.0647. \end{aligned}$$

It follows that

$$4\|\Omega_q\|\nabla + \|\chi_q\| \approx 0.4056 < 1.$$

Through Theorem 3.1, the existence of a solution to problem (3.1) is demonstrated.

Acknowledgment: The research of the first author (Khurshida Parvin) is supported by SERB, DST, New Delhi, India via the application reference number DST/INSPIRE/03/2022/001487.

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Khurshida Parvin,
Department of Mathematics,
Gauhati University,
India.
E-mail address: khurshidaparvin473@gmail.com

and

Bipan Hazarika,
Department of Mathematics,
Gauhati University,
India.
E-mail address: bh_rgu@yahoo.co.in