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# Nonlinear System of Fractional Dynamic Equations Involving Initial and Boundary Conditions

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ABSTRACT: This paper presents a comprehensive investigation into the existence and uniqueness of solutions for nonlinear fractional dynamic equations defined on time scales. Both initial and boundary value problems are considered, and the solvability of the equations is examined through the application of fixed point theory. The theoretical framework is developed using fundamental concepts, lemmas, and propositions associated with Riemann–Liouville and Caputo-type fractional derivatives. To illustrate the validity and applicability of the established results, two representative examples are provided.

Keywords: Time scales, Riemann-Liouville integral, Riemann-Liouville derivative, Caputo derivative, Fixed point theorems, Green's function.

## Contents

## 1. Introduction

The exploration of dynamic equations based on time scale theory furnishes versatility for integrating continuous and discrete dynamical processes into a single mathematical structure. In recent times, the unification of dynamical system with fractional calculus on domain of time scale calculus has received much significance due to its competence to model processes with memory and hereditary characteristics. Time scales, denoted by $\mathbb{T}$ - a relatively fresh topic for researchers, was first introduced by Stefan Hilger along with Bernd Aulbach, his supervisor, in 1988, and thereafter, Hilger published more [22], [23] papers. Later on, the commencement of the concept of merging time scale calculus and fractional calculus was seen in N.R.O Bastos's Ph.D dissertation in 2012 [24]. Traditionally, researchers believed that dynamic processes can either be purely discrete or purely continuous and accordingly used either difference equations or differential equations respectively to illustrate the dynamic model mathematically. In due course of time, it has been realized that there are some unavoidable significant phenomena that not just exhibit continuous or discrete data but a mix of both, which led researchers to study further on this unification. Nowadays, fractional calculus has become more preferable method over integer-order calculus due to its wide advantages and precision, and thus it acts as a valuable tool in dynamic equations to solve dynamic models.

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Lately, many researchers are inclined towards this topic due to extensive applications of the dynamic equations in different areas of applied sciences. The concept of incorporating initial and boundary conditions in a dynamic equation can be seen in [20], [19], [16]. We refer readers to go through [19], [14], [13], [12], [11], [10], [8], [7], [9] and references therein for further information on existence and uniqueness of solutions to numerous dynamic equations incorporating initial and boundary conditions with different approaches.

## 2. Motivation of the Article

Based upon the prior work, we observe that dynamic processes or models occur under different conditions and hence, here we are motivated to study fractional dynamic equations subject to initial and boundary conditions defined over time scale by virtue of fixed point theorems. Also, the paper [6] has highly motivated to proceed with this paper.

Further, the existence of solutions to the following non-linear fractional dynamic equations involving initial and boundary conditions respectively and its uniqueness has been observed:

$$\begin{aligned} D^{\beta}h(\eta) &= \mathcal{X}(\eta, h(\eta), D^{\beta}h(\eta)) \\ h(\eta_0) &= h_0, \\ \eta_0, h_0 &\in \mathbb{R} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} D^{\beta}h(\eta) &= \mathcal{X}(\eta, h(\eta), D^{\beta}h(\eta)) \\ h(0) &= h(T) = 0, \\ \forall T &\in \mathbb{R}, \eta \in J \end{aligned} \tag{2.2}$$

where time scale interval $J$ is given by,

$$J = \{\eta \in J : 0 \leq \eta \leq T, T \in \mathbb{R}\}$$

The paper has been formatted in the following manner: in Sect.3, some definitions, important lemmas and propositions that are essential to solve fractional dynamic equations over time scales has been given. In Sect.4 and 5, we include that the solution to fractional dynamic equation consisting of initial and boundary conditions exist respectively and also its respective uniqueness has been shown. For more convenience to understand our theoritical work, two examples are given based on those theories in Sect.6. We have concluded the paper in Sect.7.

## 3. Preliminaries

[2] The time scale, represented as $\mathbb{T}$, is a closed subset ($\neq \phi$) of the real numbers $\mathbb{R}$.

A function $\rho : \mathbb{T} \to \mathbb{T}$, given by $\rho(\eta) = \sup \{\mathrm{e} \in \mathbb{T} : \mathrm{e} < \eta\}$, for $\eta \in \mathbb{T}$, is referred as the backward jump operator.

A function $\sigma : \mathbb{T} \to \mathbb{T}$, given by $\sigma(\eta) = \inf \{\mathrm{e} \in \mathbb{T} : \mathrm{e} > \eta\}$, for $\eta \in \mathbb{T}$, is referred as the forward jump operator.

If $\rho(\eta) < \eta$, then $\eta$ is known as left scattered and if $\eta > \inf \mathbb{T}$ and $\rho(\eta) = \eta$, then $\eta$ is known as left dense. If $\sigma(\eta) > \eta$, then $\eta$ is known as right scattered and if $\eta < \sup \mathbb{T}$ and $\sigma(\eta) = \eta$, then $\eta$ is known as right dense. If $\eta$ is left dense as well as right dense simultaneously, then $\eta$ is called dense.

$\nabla$-derivative of any ld-continuous function does not exist [5], and thus, we define the kappa operator denoted by $\mathbb{T}_\kappa$ of $\mathbb{T}$. This kappa operator is defined as:

$$\mathbb{T}_\kappa = \begin{cases} \mathbb{T} - \{inf\mathbb{T}, \sigma(inf\mathbb{T})\} & \text{, if } \inf \mathbb{T} > -\infty \\ \mathbb{T} & \text{, otherwise} \end{cases}$$

**Definition 3.1** *[24] Consider a function $\mathcal{H} : J \to \mathbb{R}$, continuous at all the left dense point of $J$ and if in the right dense point, its right sided limit exist, then it is considered to be left dense(ld) continuous. The family of all functions from $J$ to $\mathbb{R}$, denoted as $C(J, \mathbb{R})$, are considered as a space of ld continuous function.*
*The space of the function $C(J, \mathbb{R}) = \mathcal{K}$ equipped with the norm $||.||$ forms a Banach space which is given by:*

$$||h|| = \sup_{\eta \in J} |h(\eta)| \tag{3.1}$$

**Definition 3.2** *[8] [$\nabla$-integration] Assume a $\nabla$-integrable function $h(\eta)$ with domain in $J$. Then for any $\mu \in J$, we acquire*

$$\int_0^T h(\mu)\nabla\mu = \int_0^\eta h(\mu)\nabla\mu + \int_\eta^T h(\mu)\nabla\mu$$

**Definition 3.3** *[4] [$\nabla$- Power function] Assume $g_\beta : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\beta \geq 0$, which is a co-ordinate wise ld-continuous functions such that $g_0(\eta, \eta_0) = 1$ and*

$$g_{\beta+1}(\eta, \eta_0) = \int_{\eta_0}^{\eta} g_\beta(\mu, \eta_0)\nabla\mu, \eta, \mu \in \mathbb{T} \tag{3.2}$$

*Further, for $\beta$, $\zeta > 1$, we have*

$$\int_{\rho(\mu)}^{\eta} g_{\beta-1}(\eta, \rho(\theta)) g_{\zeta-1}(\theta, \rho(\eta))\nabla\theta = g_{\beta+\zeta-1}(\eta, \rho(\mu)) \tag{3.3}$$

*for all $\eta$,$\mu \in \mathbb{T}$.*
*On the condition that, $\mathbb{T} = \mathbb{R}$ then $\rho(\mu) = \mu$, and thus, we have*

$$g_{\beta-1}(t, \rho(\mu)) = \frac{(t-\mu)^{\beta-1}}{\Gamma(\beta)} \tag{3.4}$$

*However, in the case that, $\mathbb{T} = \mathbb{Z}$, then $\rho(\mu) = \mu - 1$, and thus, we have*

$$g_{\beta-1}(t, \rho(\mu)) = \frac{(t-\rho(\mu))^{\overline{\beta-1}}}{\Gamma(\beta)} = \binom{t-\rho(\mu)}{\beta} \tag{3.5}$$

**Definition 3.4** *[4] Consider a function which is ld-continuous, $h : \mathbb{T}_\kappa \to \mathbb{R}$ such that on the time scale interval $J$, $h$ is a Lebesgue $\nabla$ -integrable function, then in Riemann-Liouville sense, for $0 < \beta \leq 1$, the fractional $\nabla$- integration is defined by*

$$\mathcal{I}^{\beta}_{\eta_0} h(\eta) = \int_{\eta_0}^{\eta} g_{\beta-1}(\eta, \rho(\mu)) h(\mu)\nabla\mu, \mu \in \mathcal{U} \tag{3.6}$$

*where, $\mathcal{I}^0 h(\eta) = h(\eta)$ and $\mathcal{U}$ is a neighbourhood of $\eta$.*
*If $\mathbb{T} = \mathbb{R}$, then using the above equation 3.4,the equation 3.6 can be reduced to :*

$$\mathcal{I}^{\beta}_{\eta_0} h(\eta) = \int_{\eta_0}^{\eta} \frac{(\eta-\mu)^{\beta-1}}{\Gamma(\beta)} h(\mu) d\mu$$

*Again, if $\mathbb{T} = \mathbb{Z}$, then from above equation 3.5, the equation 3.6 can be reduced to :*

$$\begin{aligned}
\mathcal{I}^{\beta}_{0+} h(\eta) &= \int_0^\eta g_{\beta-1}(\eta, \rho(\mu)) h(\mu)\nabla\mu \\
&= \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - \rho(\mu))^{\overline{\beta-1}} h(\mu)\nabla\mu \\
&= \frac{1}{\Gamma(\beta)} \sum_{\eta=0}^{\eta-1} (\eta - (\mu-1))^{\overline{\beta-1}} h(\mu) \qquad (3.7)
\end{aligned}$$

*If $\mathbb{T} = q^{\mathbb{N}_0}$ , then $g_{\beta-1}(\eta, \rho(\mu)) = \Gamma_q(\beta)\frac{q^\beta-1}{q-1}(\eta - q\mu)_q^{\beta-1}$, where $\Gamma_q$ is a q-gamma function.*

**Definition 3.5** *[4] [Riemann-Liouville fractional $\nabla$- derivative] Consider a ld continuous function $h : \mathbb{T}_{\kappa^p} \to \mathbb{R}$, then the $\beta th(\geq 0) \in \mathbb{R}$ order Riemann-Liouville fractional $\nabla$ - derivative is defined by*

$$\mathcal{D}^{\beta}_{0+}h(\eta) = \mathcal{D}^{p}_{0+}\mathcal{I}^{p-\beta}_{0+}h(\eta), \eta \in J$$

*where, $p \in \mathbb{N}_0$ and $p = [\eta] + 1$. Here, $\mathbb{T}_{\kappa^p}$ is obtained by omission of $p$ left end points of $\mathbb{T}$ which are right scattered.*

**Definition 3.6** *[4] [Caputo fractional $\nabla$ - derivative] Suppose a ld continuous function $h(\eta)$ where $h : \mathbb{T}_{\kappa^p} \to \mathbb{R}$, then the Caputo fractional $\nabla$-derivative of $h(\eta)$ is given by*

$${}^{C}\mathcal{D}^{\beta}_{0+}h(\eta) = \int_{0}^{\eta} g_{p-\beta}(\eta, \rho(\eta))\nabla^{p}(h(\mu))\nabla\mu$$

*where, $\nabla^{p}h(\eta)$ exists in $\mathbb{T}_{k^p}$.*

**Definition 3.7** *[11] [Arzela-Ascoli theorem] Consider a set $\mathcal{G} \subset C(\mathbb{T}, \mathbb{R})$. If $\mathcal{G}$ is bounded and equicontinuous simultaneously, then it is relatively compact.*

**Definition 3.8** *[11] If for a bounded subset $Q \subseteq P$ , $\mathcal{D}(Q)$ is relatively compact in $P$, then the mapping $\mathcal{H} : P \to Q$ is completely continuous.*

**Definition 3.9** *[10] [Banach fixed point theorem] Consider $X = (X, d)$ be a metric space, where $X \neq \phi$. Let $T : X \to X$ be a contraction mapping on $X$, where $X$ is complete. Then $T$ has a unique fixed point.*

**Definition 3.10** *[6] [Schauder's Fixed Point Theorem] If a continuous mapping $f : P \to Q$ is relatively compact then the map $f$ contains a fixed point, where $P$ is a closed, convex and bounded subset of a Banach space $Q$.*

**Definition 3.11** *[21] [Krasnoselskii's fixed point theorem] Consider $Y\ (\neq \phi)$ be a closed and convex subset of a Banach space $X$. Assume that $M_1, M_2 : Y \to X$ be such that*
*1. $M_1$ is contractive.*
*2. $M_2$ is continuous and $M_2(Y)$ is relatively compact.*
*3. $M_1[\eta] + M_2[\mu] \in Y$, for all $\eta, \mu \in Y$.*
*Then there is a $c \in Y$ such that $M_1[c] + M_2[c] = c$.*

**Lemma 3.1** *[18] Let $\beta > 0$, then*

$$I^{\beta}_{0+}D^{\beta}_{0+}h(\eta) = h(\eta) + A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3 + ..... + A_n\eta^{\beta-n}$$

*for $A_a \in \mathbb{R}$ and $a = 1, 2, ...., n-1, n = [\beta] + 1$.*

**Remark 3.1** *From the definition of Riemann-Liouville fractional $\nabla$- derivative, we can also obtain ${}^{C}\mathcal{D}^{\beta}_{0+}f(\eta) = \mathcal{I}^{p-\beta}_{0+}D^{\beta}_{0+}$, where $p = [\beta] + 1$.*

**Proposition 3.1** *[3] Let $r, s \in \mathbb{T}$ such that $r \leq s$. Suppose a ld continuous function $h$ on $J$, then*

$$\begin{aligned}\int_{\rho(r)}^{s} h(\eta)\nabla\eta &= \int_{\rho(r)}^{r} h(\eta)\nabla\eta + \int_{r}^{\rho(s)} h(\eta)\nabla\eta + \int_{\rho(s)}^{s} h(\eta)\nabla\eta \\ &= [r - \rho(r)]h(r) + [s - \rho(s)]h(s) + \int_{r}^{\rho(s)} h(\eta)\nabla\eta\end{aligned}$$

**Proof:** Let $r, s \in \mathbb{T}$ such that $r \leq s$. Let h($\eta$) be a ld continuous function on [r,s] and let us partition the interval into three sub-intervals i.e $[\rho(r), r], [r, \rho(s)], [\rho(s), s]$ such that $\rho(r) < r < \rho(s) < s$.
Thus it can be written as :

$$\int_{\rho(r)}^{s} h(\eta)\nabla\eta = \int_{\rho(r)}^{r} h(\eta)\nabla\eta + \int_{r}^{\rho(s)} h(\eta)\nabla\eta + \int_{\rho(s)}^{s} h(\eta)\nabla\eta$$

Now, since h is ld continuous, thus we can obtain the result:

$$\int_{\rho(r)}^{r} h(\eta)\nabla\eta = [r - \rho(r)]h(r)$$

Similarly, we can acquire the other partitions and obtain the result

$$\int_{\rho(r)}^{s} h(\eta)\nabla\eta = [r - \rho(r)]h(r) + [s - \rho(s)]h(s) + \int_{r}^{\rho(s)} h(\eta)\nabla\eta.$$

□

**Proposition 3.2** *[6] If the function $\mathcal{H}$ is extension of $h$ in real interval $[r, s] \in \mathbb{T}$, where $h$ is a ld continuous function on the time scale interval, then*

$$\mathcal{H}(\mu) = \begin{cases} h(\mu) & , \textit{if } \mu \in \mathbb{T} \\ h(\eta) & , \textit{if } \eta \in (\rho(\eta), \eta) \notin \mathbb{T}, \end{cases}$$

*hence*

$$\int_{r}^{s} h(\eta)\nabla\eta \leq \int_{r}^{s} \mathcal{H}(\eta)d\eta.$$

**Proof:** The proposition has been proved in [6]. □

**Proposition 3.3** *[6] The $\nabla$- integral $\int_{\eta_0}^{\eta} g_{\beta-1}(\eta, \rho(\mu))\nabla\mu$ is bounded in the interval $[0, 2] \cap \mathbb{T}_{\kappa}$ such that $\int_{\eta_0}^{\eta} g_{\beta-1}(\eta, \rho(\mu))\nabla\mu \leq \frac{T^{\beta}}{\Gamma(\beta+1)}$*

**Proof:** The proposition has been proved in [6]. □

**Lemma 3.2** *A function $h \in \mathcal{K} \cap L_{\nabla}(J, \mathbb{R})$ solves the equation 2.1 iff for 0< $\beta \leq 1$, h satisfies the following integral equation :*

$$h(\eta) = h_0 + \int_{\eta_0}^{\eta} h_{\beta-1}(\eta, \rho(\mu))\mathcal{X}(\mu, h(\mu), D^{\beta}h(\mu))\nabla\mu \tag{3.8}$$

**Proof:** Proof of this lemma is given on [6]. □

To proceed further some assumptions has been made :
(K1) $\mathcal{X} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a ld continuous mapping.
(K2) $\exists$ constants $P > 0$ and $0 < Q < 1$ so that

$$|\mathcal{X}(\eta, \mu_1, \mu_2) - \mathcal{X}(\eta, \gamma_1, \gamma_2)| \leq P|\mu_1 - \gamma_1| + Q|\mu_2 - \gamma_2|$$

where, $\mu_a, \gamma_a \in \mathbb{R} \times \mathbb{R}$ for a=1,2 and $\eta \in J$.
(K3) $\exists$ a $\mathcal{B} \in \mathcal{K}$ and $\mathcal{M} > 0$ and $0 < \mathcal{N} < 1$, where $\mathcal{M}$ and $\mathcal{N}$ are two constants, so that

$$|\mathcal{X}(\eta, \mu_1, \mu_2)| \leq |\mathcal{B}(\eta)| + \mathcal{M}|\mu_1| + \mathcal{N}|\mu_2|$$

for $(\mu_1, \mu_2) \in \mathbb{R} \times \mathbb{R}$ and $\eta \in \mathbb{R}$.
(K4) Let us consider a bounded piecewise continuous function $\mathbb{G}(\eta, \mu)$, which is the Green's function, in each sub-interval of [0,T]. Also consider that $\mathbb{G}(\eta, \mu)$ is non-negative increasing function so that

$$\int_0^T \mathbb{G}(\eta, \mu)\nabla\mu = w$$

where $|w| \leq K$, for $K \in \mathbb{R}$.
Further, assume that for $0 < \eta < T$,

$$\int_0^\eta |\mathbb{G}(\eta, \mu)|\nabla\mu \leq x, \int_\eta^T |\mathbb{G}(\eta, \mu)|\nabla\mu \leq y$$

where $x, y \in \mathbb{R}$.
Moreover, let $Y = \{h \in \mathcal{K} : ||h|| \leq \chi\} \subseteq \mathcal{K}$ be a set.
Then an operator $E : Y \rightarrow Y$ is defined as

$$E(h(\eta)) = h_0 + \int_{\eta_0}^\eta h_{\beta-1}(\eta, \rho(\mu))g(\mu)\nabla\mu$$

where $g \in Y$. Thus, it is seen that Y $\subseteq \mathcal{K}$ is closed, convex and bounded.

## 4. Existence and Uniqueness of the solution to a fractional dynamic equation involving initial condition

**Definition 4.1** *Let $L_\nabla(J, \mathbb{R})$ be a lebesgue $\nabla$-integrable function from $J$ to $\mathbb{R}$. Then a function $h \in \mathcal{K} \cap L_\nabla(J, \mathbb{R})$ solves the equation 2.1 iff $h(\eta) \geq 0, \eta \in J$, and $h = h(\eta)$ satisfies equation 2.1.*

**Theorem 4.1** *The equation 2.1 must have a solution if the assumptions (K1)-(K3) holds.*

**Proof:** To proof the existence of the solution, we need to check if the operator $E : Y \rightarrow Y$ is continuous, bounded and equicontinuous.
Let $D^\beta h(\eta) = g(\eta)$.
To show that E is continuous, consider a sequence $< h_n > \in Y$, so that for $h \in Y$, $h_n \rightarrow h$ as $n \rightarrow \infty$.
Then for $\eta \in J$, we have

$$\begin{aligned} ||E(h_n) - E(h)|| &\leq |[h_0 + \int_{\eta_0}^\eta h_{\beta-1}(\eta, \rho(\mu))g_n(\mu)\nabla\mu] - [h_0 + \int_{\eta_0}^\eta h_{\beta-1}(\eta, \rho(\mu))g(\mu)\nabla\mu]| \\ &= |h_0 + \int_{\eta_0}^\eta h_{\beta-1}(\eta, \rho(\mu))g_n(\mu)\nabla\mu - h_0 - \int_{\eta_0}^\eta h_{\beta-1}(\eta, \rho(\mu))g(\mu)\nabla\mu| \\ &\leq \int_{\eta_0}^\eta h_{\beta-1}(\eta, \rho(\mu))|g_n(\mu) - g(\mu)|\nabla\mu \end{aligned} \tag{4.1}$$

Now, for $g_n, g \in Y$, $\mu \in J$, from the assumption (K2) we have

$$\begin{aligned} |g_n(\mu) - g(\mu)| &= |\mathcal{X}(\mu, h_n(\mu), g_n(\mu)) - \mathcal{X}(\mu, h(\mu), g(\mu))| \\ &\leq P|h_n(\mu) - h(\mu)| + Q|g_n(\mu) - g(\mu)| \\ &\leq \frac{P}{1-Q}|h_n(\mu) - h(\mu)| \end{aligned} \tag{4.2}$$

Using Eq.3.1 and Eq.4.2 in Eq.4.1, we have

$$||E(h_n) - E(h)|| \leq \frac{P||h_n - h||}{1-Q} \int_{\eta_0}^\eta h_{\beta-1}(\eta, \rho(\mu))\nabla\mu \tag{4.3}$$

Hence, when $h_n \to h$ implies $||E(h_n) - E(h)|| \to 0$.
Thus the operator E is continuous.
Similarly, we need to now show that $E : Y \to Y$ is bounded.
Let $h \in Y$ and $\eta \in J$. Then by proposition 3.3 we have,

$$\begin{aligned} ||E(h)|| &\le |h_0 + \int_{\eta_0}^{\eta} h_{\beta-1}(\eta, \rho(\mu))\mathcal{X}(\mu, h(\mu), g(\mu))\nabla\mu| \\ &\le |h_0| + \int_{\eta_0}^{\eta} |h_{\beta-1}(\eta, \rho(\mu))g(\mu)|\nabla\mu \\ &\le |h_0| + \frac{T^\beta}{\Gamma(\beta+1)}|g(\mu)| \end{aligned} \tag{4.4}$$

where $g \in Y$. Thus, for $\mu \in J$, and using assumption (K3), we get

$$\begin{aligned} |g(\mu)| &= |\mathcal{X}(\mu, h(\mu), g(\mu))| \\ &\le |\mathcal{B}(\mu)| + \mathcal{M}|h(\mu)| + \mathcal{N}|g(\mu)| \\ &\le \frac{|\mathcal{B}(\mu)| + \mathcal{M}|h(\mu)|}{1-\mathcal{N}} \end{aligned} \tag{4.5}$$

Now, using Eq.4.5 in Eq.4.4 and applying Eq.3.1 we have,

$$\begin{aligned} ||E(h)|| &\le ||h_0|| + \frac{T^\beta(||\mathcal{B}|| + \mathcal{M}||(h)||)}{(1-\mathcal{N})\Gamma(\beta+1)} \\ &\le ||h_0|| + \frac{T^\beta(||\mathcal{B}|| + \mathcal{M}\chi)}{(1-(N))\Gamma(\beta+1)} \\ &< L \end{aligned} \tag{4.6}$$

where, $L = ||h_0|| + \frac{T^\beta(||\mathcal{B}||+\mathcal{M}\chi)}{(1-\mathcal{N})\Gamma(\beta+1)}$ and $L \in \mathbb{R}$.
Hence, the operator is bounded.
Now, to show that the operator $E : Y \to Y$ is equicontinuous.
Let $\eta_1, \eta_2 \in J$ such that $\eta_1 < \eta_2$, then for $h \in Y$, we have

$$\begin{aligned} ||E(h(\eta_1)) - E(h(\eta_2))|| &\le |\int_{\eta_0}^{\eta_1} h(\eta_1, \rho(\mu))\mathcal{X}(\mu, h(\mu), g(\mu))\nabla\mu - \int_{\eta_0}^{\eta_2} h(\eta_2, \rho(\mu))\mathcal{X}(\mu, h(\mu), g(\mu))\nabla\mu| \\ &\le |\int_{\eta_0}^{\eta_1} h(\eta_1, \rho(\mu))g(\mu)\nabla\mu - \int_{\eta_0}^{\eta_2} h(\eta_2, \rho(\mu))g(\mu)\nabla\mu| \end{aligned} \tag{4.7}$$

Since, $h_{\beta-1}(\eta, \rho(\mu))$ is continuous, hence when $\eta_1 \to \eta_2$, the right hand side of Eq. 4.7 approaches to zero. Thus the operator is equicontinuous.
Therefore, by Arzela-Ascoli theorem, the operator $E : Y \to Y$ is completely continuous and thus, by means of Schauder's fixed point theorem, we can conclude that ∃ a fixed point in the operator E, which solves Eq.2.1. □

**Theorem 4.2** *(Uniqueness) The equation 2.1 contains a unique solution if the assumptions (K1)-(K3) holds. Given that,* $\frac{T^\beta\mathcal{P}}{(1-\mathcal{Q})\Gamma(\beta+1)} < 1$.

**Proof:** Let $D^\beta h(\eta) = g(\eta)$, then for $h_1, h_2 \in Y$, we have from Eq.4.1

$$||E(h_1) - E(h_2)|| \le \int_{\eta_0}^{\eta} h_{\beta-1}(\eta, \rho(\mu))|g_1(\mu) - g_2(\mu)|\nabla\mu \tag{4.8}$$

where $g_1, g_2 \in Y$. Now for $\mu \in J$ and from Eq.4.2 we get,

$$|g_1(\mu) - g_2(\mu)| \leq \frac{\mathcal{P}}{1 - \mathcal{Q}} |h_1(\mu) - h_2(\mu)|. \tag{4.9}$$

Thus, from Eq.4.3 and applying proposition 3.3, in Eq. 4.8, we have

$$||E(h_1) - E(h_2)|| \leq \frac{T^\beta \mathcal{P}}{(1 - \mathcal{Q})\Gamma(\beta + 1)} ||h_1 - h_2|| \tag{4.10}$$

Since it is given that, $\frac{T^\beta \mathcal{P}}{(1-\mathcal{Q})\Gamma(\beta+1)} < 1$ , we can conclude that the operator E is contractive. Hence, by applying Banach fixed point theorem, the operator E contains a unique solution of the equation 2.1. □

## 5. Existence and Uniqueness of the solution to a fractional dynamic equation involving boundary conditions

**Definition 5.1** *A function $g$ solves the fractional dynamic equation 2.2 involving boundary conditions if $g = g(\eta)$ for $\eta \in J$ solves equation 2.2, where $g \in \mathcal{K} \cap L_\nabla(J, \mathbb{R})$.*

**Lemma 5.1** *Let $h$ solves the following integral equation, then $h$ also solves the equation 2.2 where $h \in \mathcal{K} \cap L_\nabla(J, \mathbb{R})$.*

$$h(\eta) = \int_0^T \mathbb{G}(\eta, \mu) \mathcal{X}(\mu, h(\mu), D^\beta h(\mu)) \nabla \mu \tag{5.1}$$

where, the Green's function $\mathbb{G}(\eta, \mu)$ is defined by

$$\mathbb{G}(\eta, \mu) = \begin{cases} \frac{T h_{\beta-1}(\eta, \rho(\mu))}{T} + \frac{-\eta h_{\beta-1}(T, \rho(\mu))}{T} & , \text{if } 0 \leq \mu < \eta \\ \frac{[-\eta h_{\beta-1}(T, \rho(\mu))]}{T} & , \text{if } \eta \leq \mu < T \end{cases} \tag{5.2}$$

**Proof:** Let $D^\beta h(\eta) = g(\eta)$.
The following equivalent integral equation is obtained by using the Eq.2.2 with the Lemma 3.1:

$$h(\eta) = I^\beta h(\eta) - A_0 - A_1 \eta \tag{5.3}$$

for $A_0, A_1 \in \mathbb{R}$.
Employing the boundary conditions of Eq.2.2, we obtain

$$A_0 = 0, A_1 = \frac{1}{T} \int_0^T h_{\beta-1}(T, \rho(\mu)) g(\mu) \nabla \mu$$

Hence, from the Eq.5.3 and then using Eq.5.2, we have

$$\begin{aligned} h(\eta) &= \int_0^\eta h_{\beta-1}(\eta, \rho(\mu)) g(\mu) \nabla \mu - \frac{\eta}{T} \int_0^T h_{\beta-1}(T, \rho(\mu)) g(\mu) \nabla \mu \\ &= \int_0^\eta \Big( \frac{T h_{\beta-1}(\eta, \rho(\mu))}{T} + \frac{-\eta h_{\beta-1}(T, \rho(\mu))}{T} \Big) g(\mu) \nabla \mu + \int_\eta^T \frac{-\eta h_{\beta-1}(T, \rho(\mu))}{T} g(\mu) \nabla \mu \\ &= \int_0^T \mathbb{G}(\eta, \mu) g(\mu) \nabla \mu \\ &= \int_0^T \mathbb{G}(\eta, \mu) D^\beta h(\mu) \nabla \mu \end{aligned}$$

Further, using Eq.2.2, we obtain Eq.5.1. □

Now, in order to establish the existence and uniqueness results of the Eq.2.2, let us consider the following key points.
Suppose a subset of $\mathcal{K}$ is given by,

$$Z_a = \{h : J \to \mathbb{R} : h(\eta) \in \mathcal{K}, ||h|| \leq a, a > 0\} \tag{5.4}$$

Evidently, $Z_a$ is a Banach subspace of $\mathcal{K}$. Now, consider two operators:
$M_1 : Z_a \to \mathcal{K}$ and $M_2 : Z_a \to \mathcal{K}$ as

$$M_1[h](\eta) = \int_0^\eta \mathbb{G}(\eta,\mu)\mathcal{X}(\mu, h(\mu), D^\beta h(\mu))\nabla\mu \tag{5.5}$$

and

$$M_2[h](\eta) = \int_\eta^T \mathbb{G}(\eta,\mu)\mathcal{X}(\mu, h(\mu), D^\beta h(\mu))\nabla\mu \tag{5.6}$$

**Lemma 5.2** *Assume that (K1), (K2) and (K4) holds. Given that* $\frac{Px}{1-Q} < 1$ *then* $M_1 : Z_a \to \mathcal{K}$ *defined in Eq.5.5 is contractive.*

**Proof:** Let $D^\beta h_i(\eta) = g(\eta)$ where $h_i \in Z_a$, for $\eta \in J$ and i = 1,2. Then,

$$\begin{aligned}|M_1[h_1](\eta) - M_1[h_2](\eta)| &= \left|\int_0^\eta \mathbb{G}(\eta,\mu)\mathcal{X}(\mu, h_1(\mu), D^\beta h_1(\mu))\nabla\mu - \int_0^\eta \mathbb{G}(\eta,\mu)\mathcal{X}(\mu, h_2(\mu), D^\beta h_2(\mu))\nabla\mu\right| \\ &= \left|\int_0^\eta \mathbb{G}(\eta,\mu)[\mathcal{X}(\mu, h_1(\mu), D^\beta h_1(\mu)) - \mathcal{X}(\mu, h_1(\mu), D^\beta h_1(\mu))]\nabla\mu\right| \\ &\leq \int_0^\eta |\mathbb{G}(\eta,\mu)||\mathcal{X}(\mu, h_1(\mu), g_1(\mu)) - \mathcal{X}(\mu, h_1(\mu), g_2(\mu))|\nabla\mu \end{aligned} \tag{5.7}$$

where, $g_1, g_2 \in Z_a$.
From assumption (K2) we get,

$$\begin{aligned}|g_1(\mu) - g_2(\mu)| &= |\mathcal{X}(\mu, h_1(\mu), g_1(\mu)) - \mathcal{X}(\mu, h_2(\mu), g_2(\mu))| \\ &\leq P|h_1(\mu) - h_2(\mu)| + Q|g_1(\mu) - g_2(\mu)| \\ &\leq \frac{P}{1-Q}|h_1(\mu) - h_2(\mu)| \end{aligned} \tag{5.8}$$

Thus, using Eq.5.8 in Eq.5.7 and applying the norm of Eq.3.1, we get

$$||M_1[h_1](\eta) - M_1[h_2](\eta)|| \leq \frac{P}{1-Q}\int_0^\eta |\mathbb{G}(\eta,\mu)|||h_1(\mu) - h_2(\mu)||$$

In view of assumption (K4), we obtain

$$||M_1[h_1](\eta) - M_1[h_2](\eta)|| \leq \frac{Px}{1-Q}||h_1(\mu) - h_2(\mu)|| \tag{5.9}$$

Given that, $\frac{Px}{1-Q} < 1$, thus the mapping $M_1 : Z_a \to \mathcal{K}$ is contractive. □

**Theorem 5.1** *Assume that (K1)–(K4) holds. Then,* $M_2 : Z_a \to \mathcal{K}$ *defined in Eq.5.6 is continuous and relatively compact.*

**Proof:** Let $D^\beta h(\eta) = g(\eta)$.
To prove the theorem we need to check if it is continuous, bounded and equicontinuous.
Step 1 : To show that $M_2 : Z_a \to \mathcal{K}$ is continuous, suppose a sequence $< h_n > \in Z_a, n \in \mathbb{N}$ so that $h_n \to h$

for $h \in Z_a$.
Then, for $0 < \eta < T$, we have

$$\begin{aligned} |M_2[h_n](\eta) - M_2[h](\eta)| &= \left| \int_\eta^T \mathbb{G}(\eta,\mu)\mathcal{X}(\mu, h_n(\mu), D^\beta h_n(\mu))\nabla\mu - \int_\eta^T \mathbb{G}(\eta,\mu)\mathcal{X}(\mu, h(\mu), D^\beta h(\mu))\nabla\mu \right| \\ &= \left| \int_\eta^T \mathbb{G}(\eta,\mu)[\mathcal{X}(\mu, h_n(\mu), D^\beta h_n(\mu)) - \mathcal{X}(\mu, h(\mu), D^\beta h(\mu))]\nabla\mu \right| \\ &\leq \int_\eta^T |\mathbb{G}(\eta,\mu)||\mathcal{X}(\mu, h_n(\mu), g_n(\mu)) - \mathcal{X}(\mu, h(\mu), g(\mu))|\nabla\mu \end{aligned} \tag{5.10}$$

where $g_n, g \in Z_a$.
From assumption (K2) we get,

$$\begin{aligned} |g_n(\mu) - g(\mu)| &= |\mathcal{X}(\mu, h_n(\mu), g_n(\mu)) - \mathcal{X}(\mu, h(\mu), g(\mu))| \\ &\leq P|h_n(\mu) - h(\mu)| + Q|g_n(\mu) - g(\mu)| \\ &\leq \frac{P}{1-Q}|h_n(\mu) - h(\mu)| \end{aligned} \tag{5.11}$$

Thus, using Eq.5.11 in Eq.5.10 and applying the norm of Eq.3.1, we get

$$||M_2[h_n](\eta) - M_2[h](\eta)|| \leq \frac{P}{1-Q}\int_\eta^T |\mathbb{G}(\eta,\mu)|||h_n(\mu) - h(\mu)||$$

In view of assumption (K4), we obtain

$$||M_2[h_n](\eta) - M_2[h](\eta)|| \leq \frac{Py}{1-Q}||h_n(\mu) - h(\mu)|| \tag{5.12}$$

Thus, whenever $h_n \to h$ the right hand side of Eq.5.12 approaches to zero.
Hence, $M_2 : Z_a \to \mathcal{K}$ is continuous.
Step 2 : To show that $M_2 : Z_a \to \mathcal{K}$ is bounded, let us consider Eq.5.6. Then for $\eta \in J$,

$$\begin{aligned} |M_2[h](\eta)| &\leq \int_\eta^T |\mathbb{G}(\eta,\mu)||\mathcal{X}(\mu, h(\mu), D^\beta h(\mu))|\nabla\mu \\ &= \int_\eta^T |\mathbb{G}(\eta,\mu)||g(\mu)|\nabla\mu \end{aligned} \tag{5.13}$$

where $g \in Z_a$. Thus, for $\mu \in J$, and using assumption (K3), we get

$$\begin{aligned} |g(\mu)| &= |\mathcal{X}(\mu, h(\mu), g(\mu))| \\ &\leq |\mathcal{B}(\mu)| + \mathcal{M}|h(\mu)| + \mathcal{N}|g(\mu)| \\ &\leq \frac{|\mathcal{B}(\mu)| + \mathcal{M}|h(\mu)|}{1-\mathcal{N}} \end{aligned} \tag{5.14}$$

Now, using Eq.5.14 in Eq.5.13 and applying the norm 3.1, we get

$$||M_2[h](\eta)|| \leq \int_\eta^T |\mathbb{G}(\eta,\mu)|\frac{||\mathcal{B}(\mu)|| + \mathcal{M}||h(\mu)||}{1-\mathcal{N}}\nabla\mu$$

From (K4), we have

$$||M_2[h](\eta)|| \leq \frac{y||\mathcal{B}|| + \mathcal{M}\chi}{1-\mathcal{N}} \tag{5.15}$$

Thus, $M_2 : Z_a \to \mathcal{K}$ is bounded.
Step 3: To show that $M_2 : Z_a \to \mathcal{K}$ is equicontinuous, let us consider $\eta_1, \eta_2 \in J$ such that $\eta_1 < \eta_2$. Then for $h \in Z_a$, we have

$$\begin{aligned}
|M_2[h](\eta_1) - M_2[h](\eta_2)| &= \left| \int_{\eta_1}^{T} \mathbb{G}(\eta_1,\mu)\mathcal{X}(\mu,h(\mu),D^\beta h(\mu))\nabla\mu - \int_{\eta_2}^{T} \mathbb{G}(\eta_2,\mu)\mathcal{X}(\mu,h(\mu),D^\beta h(\mu))\nabla\mu \right| \\
&\leq \left| \int_{\eta_1}^{T} \mathbb{G}(\eta_1,\mu)\nabla\mu - \int_{\eta_2}^{T} \mathbb{G}(\eta_2,\mu)\nabla\mu \right| \left| \mathcal{X}(\mu,h(\mu),D^\beta h(\mu)) \right| \\
&= \left| \int_{\eta_1}^{T} \mathbb{G}(\eta_1,\mu)\nabla\mu - \int_{\eta_2}^{T} \mathbb{G}(\eta_2,\mu)\nabla\mu \right| \left| g(\mu) \right|
\end{aligned}$$

From Eq.5.15, we obtain

$$\begin{aligned}
|M_2[h](\eta_1) - M_2[h](\eta_2)| &\leq \left| \int_{\eta_1}^{T} \mathbb{G}(\eta_1,\mu)\nabla\mu - \int_{\eta_2}^{T} \mathbb{G}(\eta_2,\mu)\nabla\mu \right| \left( \frac{|\mathcal{B}(\mu)| + \mathcal{M}|h(\mu)|}{1-\mathcal{N}} \right) \\
&\leq \frac{||\mathcal{B}|| + \mathcal{M}\chi}{1-\mathcal{N}} \left| \int_{\eta_1}^{T} \mathbb{G}(\eta_1,\mu)\nabla\mu - \int_{\eta_2}^{T} \mathbb{G}(\eta_2,\mu)\nabla\mu \right| \qquad (5.16)
\end{aligned}$$

Now, for $\eta_1, \eta_2 \in J$ using the Green's function from Eq.5.2 and applying Def. 3.3, we get

$$\begin{aligned}
\int_{\eta_1}^{T} \mathbb{G}(\eta_1,\mu)\nabla\mu &= \int_{\eta_1}^{T} \frac{-\eta_1 h_{\beta-1}(T,\rho(\mu))}{T}\nabla\mu \\
&= \frac{-\eta_1 h_\beta(T,\rho(\mu))}{T} \qquad (5.17)
\end{aligned}$$

and similarly,

$$\int_{\eta_2}^{T} \mathbb{G}(\eta_2,\mu)\nabla\mu = \frac{-\eta_2 h_\beta(T,\rho(\mu))}{T} \qquad (5.18)$$

Substituting, Eq.5.17 and Eq.5.18 in Eq.5.16, we obtain

$$||M_2[h](\eta_1) - M_2[h](\eta_2)|| \leq \left( \frac{||\mathcal{B}|| + \mathcal{M}\chi}{1-\mathcal{N}} \right) \frac{h_\beta(T,\rho(\mu)}{T}(\eta_2 - \eta_1) \qquad (5.19)$$

Thus, whenever $\eta_2 \to \eta_1$ then the right side of Eq.5.19 approaches to zero
i.e $||M_2[h](\eta_1) - M_2[h](\eta_2)|| \to 0$, which implies that the mapping $M_2 : Z_a \to \mathcal{K}$ is equicontinuous.
Hence, by Arzela-Ascoli theorem, $M_2 : Z_a \to \mathcal{K}$ is relatively compact. □

**Theorem 5.2** *Assume (K1)-(K4) holds. Consider* $Z_a = \{p : J \to \mathbb{R} : p(\eta) \in \mathcal{K}, ||p|| \leq a\}$, *where 'a' is such that* $\frac{(x+y)(||\mathcal{B}||+\mathcal{M}a)}{1-\mathcal{N}} \leq a$. *Then, Eq.2.2 has a solution in* $Z_a$.

**Proof:** Let $D^\beta p(\eta) = g(\eta)$ and $D^\beta q(\eta) = h(\eta)$, where $p, q \in Z_a$.
We have already proven above that $M_1 : Z_a \to \mathcal{K}$ is contractive and $M_2 : Z_a \to \mathcal{K}$ is continuous and relatively compact. So we can write,

$$\begin{aligned}
|M_1[p](\eta) + M_2[q](\eta)| &\leq \int_0^\eta |\mathbb{G}(\eta,\mu)||\mathcal{X}(\theta,p(\mu),D^\beta p(\mu))|\nabla\mu + \int_\eta^T |\mathbb{G}(\eta,\mu)||\mathcal{X}(\mu,q(\mu),D^\beta q(\mu))|\nabla\mu \\
&\leq \int_0^\eta |\mathbb{G}(\eta,\mu)||g(\mu)|\nabla\mu + \int_\eta^T |\mathbb{G}(\eta,\mu)||h(\mu)|\nabla\mu \qquad (5.20)
\end{aligned}$$

Thus, for $\mu \in J$, and using assumption (K3), we get

$$\begin{aligned}
|g(\mu)| &= |\mathcal{X}(\mu,p(\mu),g(\mu))| \\
&\leq |\mathcal{B}(\mu)| + \mathcal{M}|(p(\mu))| + \mathcal{N}|(g(\mu))| \\
&\leq \frac{|\mathcal{B}(\mu)| + \mathcal{M}|p(\mu)|}{1-\mathcal{N}} \qquad (5.21)
\end{aligned}$$

Similarly,

$$|h(\mu)| \leq \frac{|\mathcal{B}(\mu)| + \mathcal{M}|(q(\mu))|}{1-\mathcal{N}} \tag{5.22}$$

Now, substituting Eq.5.21 and Eq.5.22 in Eq.5.20 and applying assumption (K4), we obtain

$$\begin{aligned} ||M_1[p](\eta) + M_2[q](\eta)|| &\leq \int_0^\eta |\mathbb{G}(\eta,\mu)| \left( \frac{||\mathcal{B}(\mu)|| + \mathcal{M}||p||}{1-\mathcal{N}} \right) \nabla\mu + \int_\eta^T |\mathbb{G}(\eta,\mu)| \left( \frac{||\mathcal{B}(\mu)|| + \mathcal{M}||q||}{1-\mathcal{N}} \right) \nabla\mu \\ &\leq x \left( \frac{||\mathcal{B}(\mu)|| + \mathcal{M}a}{1-\mathcal{N}} \right) + y \left( \frac{||\mathcal{B}(\mu)|| + \mathcal{M}a}{1-\mathcal{N}} \right) \\ &= \frac{(x+y)(||\mathcal{B}|| + \mathcal{M}a)}{1-\mathcal{N}} \\ &\leq a \end{aligned} \tag{5.23}$$

This shows that $M_1[p] + M_2[q] \in Z_a$, for $p, q \in Z_a$.
Thus, according to Krasnoselskii's fixed point theorem, all the conditions has been satisfied which yields that $\exists$ a fixed point $p \in Z_a$ so that $p = M_1[p] + M_2[p]$, which solves the equation 2.2. □

**Theorem 5.3** *(Uniqueness) Assume that (K1)-(K4) holds then the Eq.2.2 has a unique solution. Given that* $\frac{Pw}{1-Q} < 1$.

**Proof:** For $\eta \in J$, let $D^\beta h_i(\eta) = g_i(\eta)$, where i = 1,2. Consider $\mathcal{V}(h) \in Z_a$ such that $\mathcal{V}(h) = \int_0^T \mathbb{G}(\eta,\mu)\mathcal{X}(\mu, h(\mu), D^\beta h(\mu))$. Then we have,

$$\begin{aligned} |\mathcal{V}(h_1) - \mathcal{V}(h_2)| &\leq \int_0^T |\mathbb{G}(\eta,\mu)||\mathcal{X}(\mu, h_1(\mu), D^\beta h_1(\mu)) - \mathcal{X}(\mu, h_2(\mu), D^\beta h_2(\mu))|\nabla\mu \\ &\leq \int_0^T |\mathbb{G}(\eta,\mu)||\mathcal{X}(\mu, h_1(\mu), g_1(\mu)) - \mathcal{X}(\mu, h_2(\mu), g_2(\mu))|\nabla\mu \end{aligned} \tag{5.24}$$

Now, for $\mu \in J$ applying assumption (K2), we have

$$\begin{aligned} |g_1(\mu) - g_2(\mu)| &= |\mathcal{X}(\mu, h_1(\mu), g_1(\mu)) - \mathcal{X}(\mu, h_2(\mu), g_2(\mu))| \\ &\leq P|h_1(\mu) - h_2(\mu)| + Q|g_1(\mu) - g_2(\mu)| \\ &\leq \frac{P}{1-Q}|h_1(\mu) - h_2(\mu)| \end{aligned} \tag{5.25}$$

Substituting Eq.5.25 in Eq.5.24 and from (K4) we get,

$$\begin{aligned} |\mathcal{V}(h_1) - \mathcal{V}(h_2)| &\leq \frac{P}{1-Q} \int_0^T |\mathbb{G}(\eta,\mu)||h_1(\mu) - h_2(\mu)|\nabla\mu \\ &\leq \frac{Pw}{1-Q}|h_1(\mu) - h_2(\mu)| \end{aligned} \tag{5.26}$$

Now, given that $\frac{Pw}{1-Q} < 1$, hence $\mathcal{V}(h) \in Z_a$ is contractive. Thus by means of Banach fixed point theorem, $\exists$ a unique fixed point in $Z_a$ that solves the Eq.2.2. □

## 6. Example

**Example 6.1** *Suppose a fractional dynamic equation subject to initial conditions on time scale* $\mathbb{T} = [0, 2]$ *with* $\beta = \frac{1}{2}$ *such that*

$$D^\beta h(\eta) = \frac{h(\eta) + D^\beta h(\eta)}{5 + h(\eta)} + \frac{e^{-2\eta}}{20}, h(0) = 0.5 \tag{6.1}$$

*where,* $\eta \in [0,2] \cap \mathbb{T}_\kappa$*. Also, h(*$\eta$*) is a ld-continuous function on* $\mathbb{T}$*.*
*For* $\mu_1, \mu_2 \in \mathbb{R}$*, we obtain*

$$\mathcal{X}(\eta, \mu_1, \mu_2) = \frac{\mu_1 + \mu_2}{5 + \mu_1} + \frac{e^{-2\eta}}{20}$$

*then*

$$|\mathcal{X}(\eta, \mu_1, \mu_2)| \leq \frac{1}{20} + \frac{1}{5}|\mu_1| + \frac{1}{5}|\mu_2| \tag{6.2}$$

*Hence, comparing assumption (K3) and Eq.6.2 we obtain* $\mathcal{B} = \frac{1}{20}$*,* $\mathcal{M} = \frac{1}{5}$*,* $\mathcal{N} = \frac{1}{5}$*.*
*Also,*

$$|\mathcal{X}(\eta, \mu_1, \mu_2) - \mathcal{X}(\eta, \gamma_1, \gamma_2)| \leq \frac{1}{5}|\mu_2 - \mu_1| + \frac{1}{5}|\gamma_2 - \gamma_1|. \tag{6.3}$$

*Thus, comparing assumption (K2) and Eq.6.3, we have* $P = \frac{1}{5}$*,* $Q = \frac{1}{5}$*. Considering Eq.6.2 and Eq.6.3, we can deduce that the assumptions (K1)-(K3) has been satisfied by the Eq.6.1.*
*Now, using the value of* $P$ *and* $Q$ *in Theorem 4.2, we get*

$$\begin{aligned}\frac{T^{\frac{1}{2}} P}{(1-Q)\Gamma(\frac{1}{2}+1)} &= \frac{2^{\frac{1}{2}}\frac{1}{5}}{\frac{4}{5}\Gamma(\frac{3}{2})} \\ &< 1\end{aligned}$$

*Therefore, we can obtain a unique solution for Eq.6.1 over the time scale interval* $[0,2] \cap \mathbb{T}_\kappa$*.*

**Example 6.2** *Suppose a fractional dynamic equation subject to boundary conditions on a time scale* $\mathbb{T} = [0,2]$ *with* $\beta = \frac{1}{3}$ *such that*

$$\begin{cases} D^\beta h(\eta) = \frac{e^{-5\eta}}{2+e^{3\eta}}[1 + \frac{1}{8}|h(\eta)| + \frac{1}{8}|D^\beta h(\eta)|] \\ h(0) = h(2) = 0 \end{cases} \tag{6.4}$$

*where* $\eta \in [0,2] \cap \mathbb{T}_\kappa$*.*
*Now, for* $\mu_1, \mu_2 \in \mathbb{R}$*, we obtain*

$$\mathcal{X}(\eta, \mu_1, \mu_2) = \frac{e^{-5\eta}}{2+e^{3\eta}}[1 + \frac{1}{8}|\mu_1| + \frac{1}{8}|\mu_2|]$$

*Then,*

$$\begin{aligned}|\mathcal{X}(\eta, \mu_1, \mu_2) - \mathcal{X}(\eta, \gamma_1, \gamma_2)| &= \left|\frac{e^{-5\eta}}{2+e^3\eta}[1 + \frac{|\mu_1|}{8} + \frac{|\mu_2|}{8}] - \frac{e^{-5\eta}}{2+e^3\eta}[1 + \frac{|\gamma_1|}{8} + \frac{|\gamma_2|}{8}]\right| \\ &= \left|\frac{e^{-5\eta}}{2+e^3\eta}[\frac{1}{8}|\mu_1 - \gamma_1| + \frac{1}{8}|\mu_2 - \gamma_2|]\right| \\ &\leq \frac{1}{16}|\mu_1 - \gamma_1| + \frac{1}{16}|\mu_2 - \gamma_2| \end{aligned} \tag{6.5}$$

*Hence, comparing assumption(K2) and Eq.6.5, we have* $P = Q = \frac{1}{16}$*. Also, we obtain*

$$|\mathcal{X}(\eta, \mu_1, \mu_2)| \leq \frac{1}{2} + \frac{1}{16}|\mu_1| + \frac{1}{16}|\mu_2| \tag{6.6}$$

*Thus, comparing assumption(K3) and Eq.6.6 we have,* $\mathcal{B} = \frac{1}{2}, \mathcal{M} = \mathcal{N} = \frac{1}{16}$*.*
*Now, substituting the above values in the inequality* $\frac{Px}{1-Q} < 1$ *we get* $x < 15$*. Again, using this value in* $\frac{(x+y)(||\mathcal{B}+\mathcal{M}a)||}{1-N} \leq a$*, where* $a > 0$ *, we get* $y < \frac{120}{8+a}$*.*
*Thus, Eq.6.4 has a solution for values we obtained for* $x$ *and* $y$*.*
*Now, the inequality* $\frac{Pw}{1-Q} < 1$*, yields* $w < 15$*, which concludes by virtue of theorem 5.6 that Eq.6.4 contains a unique solution when* $w < 15$*.*

## 7. Conclusion

This paper has analyzed the existence and uniqueness of solution to a nonlinear fractional dynamic equation involving initial as well as boundary conditions on time scales via fixed point theorems. The discussion of fractional dynamic equations consisting initial conditions is derived using Schauder's fixed point theorem with the help of Banach fixed point theorem. Additionally fractional dynamic equation consisting boundary conditions is discussed on the basis of Krasnoselskii's fixed point theorem with the help of Banach fixed point theorem. Further, we have also provided two examples based on the theory we have reviewed. A vast application of the concept of nonlinear fractional dynamic equation consisting of initial as well as boundary conditions is evident in various fields such as Engineering, Mathematics, Physics, Economics, etc.

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