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On the Class of p-M-Weakly Demicompact Operators with Numerical Application

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ABSTRACT: In this paper, we introduce a new class of operators, called p-M-weakly demicompact operators (PMWD), within the framework of lattice-normed vector lattices. This new concept generalizes classical notions such as weak compactness and demicompactness by incorporating both the lattice structure and a vector-valued lattice norm. We establish relationships with the classes of p-compact and p-M-weakly compact operators, as well as their stability under perturbations. We use mixed-norm techniques to relate PMWD operators to M-weakly demicompact operators in mixed-normed settings. We give a numerical application of the stability of hidden states in neural networks.

Key Words: Weakly demicompact operator, p-M-weakly demicompact operator, lattice-normed, vector lattice, p-convergence, mixed-normed space, stability analysis, recurrent neural networks.

Contents

1. Introduction

A vector lattice X with a norm $\|\cdot\|$ is called a normed lattice if, whenever $|x| \le |y|$ in X, it follows that $\|x\| \le \|y\|$. If a normed lattice is complete with respect to its norm, then it is referred to as a Banach lattice. The idea of lattice-normed spaces was firstly investigated in [21]. Subsequently, this theory was further explored and significantly advanced by Semën Kutateladze and Anatoly Kusraev.

Many results from probability and ergodic theories have been generalized to lattice-normed vector lattices, as shown in [10,17,18]. Another results of the lattice-normed theory has been established with the help of the decomposability of the lattice norm (see [8,11,23,24]). In this paper, we give a more general study of lattice-normed vector lattices without requiring the lattice norm to be decomposable. Our characterization is based on the demicompactness and the weak demicompactness properties. The class of demicompactness was first used in [26] to study the structure of fixed point sets for nonlinear operators. Later, it was extended to the weak topology, leading to the notion of weak demicompactness [22]. In this setting, an operator $A : D \subset X \to X$ is weakly demicompact if, for every bounded sequence $(x_n)_n \in D$ with $(x_n - Ax_n)_n$ converging weakly to an element of X, $(x_n)_n$ has a subsequence that also converges weakly. Further, several results focused on this class which containing compact operators [9,26] and its important role in spectral theory (see [12,13,15]). More information can be found in [2,16,14]. Recently, the authors in [6] presented the M-weakly demicompact operator in the framework of Banach lattice and the class of weak demicompactness.

In this paper, we introduce the p-M-weakly demicompact (PMWD) operator notion. Then, we use the theory of lattice-normed vector lattices to study the concept of weak demicompactness and the characterization of p-M-weakly compact operators between lattice-normed vector lattices. This kind of operators include amount of well-known classes, such as compact, p-compact, weakly and order weakly compact

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operators. Note that, a p-bounded operator A between two lattice-normed vector lattices (X, p, E) and (Y, m, F) is called a p-M-weakly compact operator if, for every p-bounded disjoint sequence $(x_\alpha)_\alpha$, then $m(Ax_\alpha) \xrightarrow{o} 0$.

This paper is organized in the following way. In Section 2, we recall some mathematical tools and preliminary. In Section 3, we present the new class of PMWD operators (Definition 3.1). Furthermore, we give relations between M-weakly demicompact operator acting on mixed norm and PMWD operators. In section 4, we derive a characterizations of the new class by some interesting properties. Finally, in Section 5, we give a numerical application of the stability of hidden states in neural networks.

2. Preliminaries

In this section, we give basic properties of vector lattices, functional analysis and Banach lattice. For further information on lattice structures, we refer the reader to standard references [1,3,4,8,23,25].

Definition 2.1 *A space X is an ordered vector space with the order relation $\leq$ if it satisfies*

- (i) *If $x \leq y$, then $x + z \leq y + z$ for all $z \in X$.*
- (ii) *If $x \leq y$, then $\lambda x \leq \lambda y$ for all $\lambda \in \mathbb{R}^+$.*

For any $x, y \in X$, if $x \wedge y := \inf\{x, y\}$ and $x \vee y := \sup\{x, y\}$ exist, , then X is called vector lattice. In addition, X is order complete, if for any nonempty $A \subset X$, $\inf(A)$ exists whenever A is bounded below.

Definition 2.2 *Let X be a vector lattice. A net $(x_\alpha)_\alpha \in X$ converges in order to $x \in X$ ($x_\alpha \xrightarrow{o} x$ for short) if there is $(y_\beta)_\beta \in X$ that goes down to 0, and for each β, there is an index $\alpha_\beta \leq \alpha$ such that $|x_\alpha - x| \leq y_\beta$.*

Definition 2.3 *[23] Let $(X, \|\cdot\|)$ be a normed lattice.*

- (i) *X is order continuous if, whenever a net $x_\alpha \xrightarrow{o} 0$ in X implies that $\|x_\alpha\| \to 0$. If this condition holds for sequences, then X is said to be σ-order continuous.*
- (ii) *X is called a KB-space if, for every increasing net (x_α) with $0 \leq x_\alpha \uparrow$ and $\sup_\alpha \|x_\alpha\| < \infty$, the net (x_α) converges in norm.*

Remark 2.1 *Every order continuous normed lattice is automatically σ-order continuous.*

Let $E^+ := \{x \in E : x \geq 0\}$ be the positive cone of a vector lattice E.

Definition 2.4 *[23] Consider a vector space X together with a vector lattice E. A map $p : X \to E^+$ is a lattice norm if it follows*

- (i) *$p(x) = 0$ if and only if $x = 0$;*
- (ii) *$p(\lambda x) = |\lambda|\, p(x)$, for all vectors $x \in X$ and all scalars $\lambda \in \mathbb{R}$;*
- (iii) *$p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.*

When these rules hold, (X, p, E) is called a lattice-normed space (shortly LNS).

In LNS (X, p, E), the norm p is decomposable, if whenever $p(x) = e_1 + e_2$ for some $e_1, e_2 \in E^+$ and $x \in X$, we can find $x_1, x_2 \in X$ with $x = x_1 + x_2$ and $p(x_k) = e_k$ for $k = 1, 2$. In this case, (X, p, E) is called a decomposable lattice-normed space.

Definition 2.5 *Consider the LNS (X, p, E). If the norm p is monotone, that is, whenever $|x| \leq |y|$ it follows that $p(x) \leq p(y)$ and X is vector lattice, then (X, p, E) is called a LNVL space.*

Definition 2.6 *Let $(x_\alpha) \in X$, a subset $A \subseteq X$ with (X, p, E) a LNS.*

- *The sequence (x_α) is p-converges to some $x \in X$, denoted by $x_\alpha \xrightarrow{p} x$, if $p(x_\alpha - x) \xrightarrow{o} 0$ in E.*

- *A is p-bounded if there is an element $e \in E$ such that $p(a) \leq e$ for every $a \in A$.*

Definition 2.7 *[23] Let $X = (X, p, E)$ be a lattice-normed vector lattice. The space X is said to be*

(i) op-continuous if, whenever $x_\alpha \xrightarrow{o} 0$, it follows that $p(x_\alpha) \xrightarrow{o} 0$.

(ii) p-KB-space if every p-bounded, increasing net in X^+ is p-convergent.

Let $\mathcal{L}(X, Y)$ be the space of all linear operators between vector spaces X and Y and $\mathcal{R}(A)$ be the range of $A \in \mathcal{L}(X, Y)$.

Definition 2.8 *[23] Let (X, p_1, E), (Y, p_2, F) two LNSs and $A \in \mathcal{L}(X, Y)$. A is dominated if there exists a positive operator $S : E \to F$ such that $p_2(A(x)) \leq S(p_1(x))$ for all $x \in X$. The operator S is called a dominant of A.*
We also denote by $M(X, Y)$ the set of all dominated operators from X to Y.

Definition 2.9 *[23] Consider the LNS (X, p, E) and the normed lattice $(E, \|\cdot\|_E)$. The normed space $(X, p\text{-}\|\cdot\|_E)$ is defined as a mixed-normed space, where $p\text{-}\|x\|_E := \|p(x)\|_E$, for all $x \in X$.*

If, for every $x \in [0, a]$, there exists a scalar $\lambda \in \mathbb{R}$ such that $x = \lambda a$, then $a \in X$ is called an atom.

Definition 2.10 *Consider a vector lattice X and an atom $a \in X$. If the band generated by its atoms is X, then X is atomic.*

Two elements x and y of a vector lattice X are disjoint written as $x \perp y$ if $|x| \wedge |y| = 0$. Hence, a sequence (x_n) is called disjoint if $x_n \perp x_m$ for all $n \neq m$.

Definition 2.11 *Let E be a Banach lattice. If $x \wedge y = 0$, implies $\|x \vee y\| = \max\{\|x\|, \|y\|\}$, for all $x, y \in E$, then E is an AM-space.*

3. Characterizations of a p-M-Weakly Demicompact Operator

We first define the class of p-M-weak demicompactness.

Definition 3.1 *Consider the LNS X and let $A : (X, p, E) \to (X, p, E)$ be a p-bounded operator. A is called p-M-weakly demicompact (PMWD), if for any disjoint p-bounded sequence $(x_n) \in X$ satisfying $p(x_n - Ax_n) \xrightarrow{o} 0$, we have $x_n \xrightarrow{p} 0$.*

Example 3.1 *Let $X = E = \ell^1$ with the lattice norm $p(x) = |x|$. Consider the operator $A : (X, p, E) \to (X, p, E)$ defined by*

$$A(x_1, x_2, x_3, \dots) = \left(\frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \dots\right)$$

For any $x \in \ell^1$, we have

$$p(Ax) = \left(\frac{|x_1|}{2}, \frac{|x_2|}{3}, \frac{|x_3|}{4}, \dots\right) \leq p(x)$$

Thus, A is p-bounded.
Now, let (x_n) be a p-bounded disjoint sequence in ℓ^1 such that $p(x_n - Ax_n) \xrightarrow{o} 0$. Then $Ax_n \xrightarrow{\|\cdot\|_1} 0$. Since $(l^1, \|\cdot\|_1)$ is Banach lattice, it follows that $p(Ax_n) \xrightarrow{o} 0$. Now, we have

$$\begin{aligned} p(x_n) &= p(x_n - Ax_n + Ax_n) \\ &\leq p(x_n - Ax_n) + p(Ax_n) \xrightarrow{o} 0. \end{aligned}$$

Hence, we obtain that $p(x_n) \xrightarrow{o} 0$, consequently A is PMWD.

Let $L^{\sim}(E,F)$ be the space of all order bounded operators between E and F. Let (X,p_1,E) be a decomposable LNS and (Y,p_2,F) a LNS with F is order complete.

For any $T\in M(X,Y)$ then $|T|$ is the unique exact dominant for T. Consequently, the triple $(M(X,Y),p,L^{\sim}(E,F))$ forms LNS, with p is given by $p(T)=|T|$.

Definition 3.2 *Let $(M(X,Y),p,L^{\sim}(E,F))$ be LNS and a sequence of operators $(T_\alpha)\in M(X,Y)$. The p-convergence in $M(X,Y)$ is defined by*

$$\text{If } |T_\alpha - T| \xrightarrow{o} 0 \text{ in } L^{\sim}(E,F) \text{ then } T_\alpha \xrightarrow{p} T \text{ in } M(X,Y).$$

In the theorem that follows, we establish that if a net of PMWD dominated operators converges in p to some dominated operator, then the limit operator is also PMWD.

Theorem 3.1 *Consider the decomposable LNS (X,p,E) with E is order complete, and let $(A_n)_n\in M(X,Y)$. If for each n, A_n is PMWD and $A_n\xrightarrow{p}A$ in $M(X,Y)$, then A is PMWD operator.*

Proof: Take a p-bounded and disjoint sequence x_α satisfying $p(x_\alpha-Ax_\alpha)\xrightarrow{o}0$. From the p-boundedness of x_α, it yields that for all α, there exists $e\in E_+$ with $p(x_\alpha)\leq e$.
Now, we can write

$$x_\alpha - A_n x_\alpha = x_\alpha - A_n x_\alpha + Ax_\alpha - Ax_\alpha,$$

therefore, we get

$$\begin{aligned} p(x_\alpha - A_n x_\alpha) &= p(x_\alpha - A_n x_\alpha + Ax_\alpha - Ax_\alpha)\\ &\leq p(x_\alpha - Ax_\alpha) + p(A_n x_\alpha - Ax_\alpha). \end{aligned}$$

Since A_n is dominated, we infer that

$$p(A_n x_\alpha - Ax_\alpha) \leq |A_n - A|(p(x_\alpha)) \leq |A_n - A|(e).$$

Since $A_n\xrightarrow{p}A$, by applying Theorem VIII.2.3 in [7], we obtain that $|A_n-A|(e)\xrightarrow{o}0$ in E as $n\to\infty$. On the other hand, by hypothesis, we have $p(x_\alpha-Ax_\alpha)\xrightarrow{o}0$. Thus, we deduce that $p(x_\alpha-A_nx_\alpha)\xrightarrow{o}0$. Now, from the p-M-weak demicompactness of A_n, we infer that $x_\alpha\xrightarrow{p}0$. Consequently, A is PMWD. □

Remark 3.1 *Every order bounded sequence in an atomic KB-space X admits a subsequence order convergent. In fact, take an order bounded sequence (x_α). It follows immediately that (x_α) is also norm bounded. Hence, from [20, Theorem 7.5], there exists a subsequence (x_{α_β}) converging to some $x\in X$ in the unbounded topology. Moreover, unbounded convergence coincides with pointwise convergence in atomic order continuous Banach lattices (see [20, Corollary. 4.14]). Consequently, using Lemma 4 in [5], we obtain that x_{α_β} is unbounded order convergent to x. The order boundedness of the sequence (x_α) implies that $x_{\alpha_\beta}\xrightarrow{o}x$.*

Now, we give conditions under that an operator in $L^{\sim}(X)$ is PMWD.

Proposition 3.1 *Consider a vector lattice X which is an atomic KB-space and an op-continuous LNVL (X,p,E). An operator $A:(X,|\cdot|,X)\longrightarrow(X,p,E)$ is PMWD, if A is in $L^{\sim}(X)$.*

Proof: Take a p-bounded disjoint sequence (x_α) in $(X,|\cdot|,X)$, so order bounded such that $p(x_\alpha-Ax_\alpha)\xrightarrow{o}0$. Now, the fact that A is in $L^{\sim}(X)$, allows us to get the order boundedness of $(Ax_\alpha)_\alpha$. As X is an atomic KB-space, Remark 3.1 ensures the existence of a subsequence (x_{α_β}) satisfying $Ax_{\alpha_\beta}\xrightarrow{o}0$. Furthermore, since the LNVL (X,p,E) is op-continuous, we obtain that $p(Ax_{\alpha_\beta})\xrightarrow{o}0$. Finally, by hypothesis, we

have $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$.
Now, we can write $x_{\alpha\beta} = x_{\alpha\beta} - Ax_{\alpha\beta} + Ax_{\alpha\beta}$. Thus

$$\begin{aligned} p(x_{\alpha\beta}) &= p(x_{\alpha\beta} - Ax_{\alpha\beta} + Ax_{\alpha\beta}) \\ &\leq p(x_{\alpha\beta} - Ax_{\alpha\beta}) + p(Ax_{\alpha\beta}) \xrightarrow{o} 0. \end{aligned}$$

Hence, we deduce that $p(x_{\alpha\beta}) \xrightarrow{o} 0$. Thus, $x_{\alpha\beta} \xrightarrow{p} 0$, that is $x_\alpha \xrightarrow{p} 0$. Consequently, A is a PMWD operator. ☐

Example 3.2 *Take $X = E = \ell^1$ with a norm lattice $p(x) = |x|$ and let $A : \ell^1 \to \ell^1$ be the diagonal operator $A(x_1, x_2, \dots) = (a_1x_1, a_2x_2, \dots)$, with $\sup_n |a_n| < \infty$. Then (X, p, E) is an op-continuous LNVL, ℓ^1 is an atomic and a KB-space.*
In this LNVL, p-bounded for operators coincides with usual boundedness, hence A is p-bounded, this implies that it is order-bounded in $(\ell^1, |\cdot|, \ell^1)$. Now, since X is an atomic KB space, applying Proposition 3.1, we obtain that $A : (X, p, E) \to (X, p, E)$ is PMWD.

In the following result, we establish that, under certain assumptions, a p-bounded operator turns out to be PMWD.

Proposition 3.2 *Consider an atomic p-KB-space X and an LNVL (X, p, E). If $A : (X, p, E) \longrightarrow (X, |\cdot|, E)$ is p-bounded operator, then A is PMWD.*

Proof: Take a p-bounded disjoint sequence (x_α) such that $|x_\alpha - Ax_\alpha| \xrightarrow{o} 0$. From the p-boundedness of A and X is an atomic p-KB-space, it follows that $Ax_\alpha \xrightarrow{p} 0$. Then, we get $Ax_\alpha \xrightarrow{o} 0$, so we obtain that $|Ax_\alpha| \xrightarrow{o} 0$.
Now, we have $x_\alpha = x_\alpha - Ax_\alpha + Ax_\alpha$, which implies that

$$\begin{aligned} |x_\alpha| &= |x_\alpha - Ax_\alpha + Ax_\alpha| \\ &\leq |x_\alpha - Ax_\alpha| + |Ax_\alpha| \xrightarrow{o} 0. \end{aligned}$$

Hence, $|x_\alpha| \xrightarrow{o} 0$. Thus, $x_\alpha \xrightarrow{p} 0$. Consequently, A is PMWD. ☐

Proposition 3.3 *Consider an LNS (X, p, E). The following operators are PMWD:*

1. *Operators $A : (X, p, E) \to (X, p, E)$ in which $(I - A)$ has a bounded inverse.*
2. *The family $(\widetilde{A}_\alpha)_{\alpha \neq 1}$ of operators from $(X \times X, p, E \oplus E)$ into itself, given by the block matrix*

$$\begin{pmatrix} 0 & 0 \\ A & \alpha I \end{pmatrix},$$

where $A : (X, p, E) \to (X, p, E)$.

Proof: (1) Consider a p-bounded and disjoint sequence (x_α) satisfying $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$
From the existence and boundedness of $(I - A)^{-1}$, and the inequality

$$p(x_\alpha) = p\big((I - A)^{-1}(I - A)x_\alpha\big) \leq \|(I - A)^{-1}\|\, p((I - A)x_\alpha),$$

we get $x_\alpha \xrightarrow{p} 0$.

(2) Let $\alpha \neq 1$ and consider a p-bounded disjoint sequence $\{\tilde{x}_n = (x_n, y_n)\}$ in $X \times X$ satisfying $p(\tilde{x}_n - A\tilde{x}_n) \xrightarrow{o} 0$. As we have $p(\tilde{x}_n) = p(x_n) + p(y_n)$, it is enough to show that $x_n \xrightarrow{p} 0$ and $y_n \xrightarrow{p} 0$. For each n,

$$\begin{aligned} p(\tilde{x}_n - \tilde{A}\tilde{x}_n) &= p((x_n, y_n) - A(x_n, y_n)) \\ &= p((x_n, y_n) - (0, Ax_n + \alpha y_n)) \\ &= p((x_n, (1-\alpha)y_n - Ax_n)) = p(x_n) + p((1-\alpha)y_n - Ax_n) \xrightarrow{o} 0. \end{aligned}$$

Thus, $p(x_n) \longrightarrow 0$, that is $x_n \xrightarrow{p} 0$ and $p((1-\alpha)y_n - Ax_n) \xrightarrow{o} 0$.
Now, observe that

$$\begin{aligned} |1-\alpha| p(y_n) &= p((1-\alpha)y_n - Ax_n + Ax_n) \\ &\leq p((1-\alpha)y_n - Ax_n) + p(Ax_n), \end{aligned}$$

and since $\alpha \neq 1$, we get $p(y_n) \xrightarrow{o} 0$. □

Example 3.3 *Consider $E = \ell^1 \oplus \ell^\infty$ and an operator $A : \ell^1 \to \ell^\infty$. The matrix operator $\mathcal{A}$ defined on E as follows:*

$$\mathcal{A} = \begin{pmatrix} 0 & 0 \\ A & I \end{pmatrix},$$

is not PMWD. To see this, let $\tilde{x}_n = (0, e_n)$, where e_n denotes the sequence in ℓ^1 with 1 in the n-th entry and 0 elsewhere. Then $(\tilde{x}_n)$ is norm-bounded in E. We have (e_n) is weakly null in c_0, as well as in ℓ^∞, thus $(\tilde{x}_n)$ is weakly null in E. Moreover, $\|\tilde{x}_n - \mathcal{A}\tilde{x}_n\|_E = 0$. However, $\|\tilde{x}_n\|_E = \|e_n\|_\infty = 1 \neq 0$.

Now, we establish a connection between operators that are M-weakly demicompact with respect to mixed norms and those that are PMWD.

Proposition 3.4 *Consider an LNS (X, p, E) with a σ-order continuous Banach lattice $(E, \|\cdot\|_E)$. If an operator A from $(X, p, \|\cdot\|_E)$ into itself is M-weakly demicompact, then $A : (X, p, E) \to (X, p, E)$ is PMWD.*

Proof: Take a p-bounded disjoint sequence (x_α) with $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$. This implies that $p(x_\alpha - Ax_\alpha) \to 0$ in E. Since $(E, \|\cdot\|_E)$ is σ-order continuous, it follows that $\|p(x_\alpha - Ax_\alpha)\|_E \xrightarrow{o} 0$. From the M-weak demicompactness of A, we deduce that $\|p(x_\alpha)\|_E \to 0$. Now, using hypothesis that $(E, \|\cdot\|_E)$ is a Banach lattice, we get $p(x_\alpha) \xrightarrow{o} 0$. Then by [7, Theorem VII.2.1], we obtain $x_\alpha \xrightarrow{p} 0$. Hence, A is a PMWD operator. □

A positive element e of a vector lattice E is called a strong unit if, for each $x \in E$, one can find a constant $\lambda_x > 0$ such that $|x| \leq \lambda_x e$. It is known that in an AM-space possessing a strong unit, every norm-bounded subset is also order bounded.

Theorem 3.2 *Let (X, p, E) be a LNVL such that the AM-space $(E, \|\cdot\|_E)$ has a σ-order continuous norm and a strong unit. If $A : (X, p, E) \to (X, p, E)$ is PMWD, then $A : (X, p, \|\cdot\|_E) \to (X, p, \|\cdot\|_E)$ is M-weakly demicompact.*

Proof: From [5, Proposition 3], the operator A from $(X, p, \|\cdot\|_E)$ into itself is norm-continuous. Take a disjoint and normed-bounded sequence (x_α) in $(X, p, \|\cdot\|_E)$, i.e., for all α $\|p(x_\alpha)\|_E \leq k < 1$, and satisfies $\|x_\alpha - Ax_\alpha\|_E \longrightarrow 0$.
Now, the AM-space $(E, \|\cdot\|_E)$ has a strong unit, so the sequence $p(x_\alpha)$ is order bounded in E. Hence, we deduce the p-boundedness of (x_α) in (X, p, E). Thus, we have $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$.
By the assumption that A is PMWD, it follows that $p(x_\alpha) \xrightarrow{o} 0$ in E. As E has a σ-order continuous norm, we deduce that $\|p(x_\alpha)\|_E \to 0$, which in turn yields $\|x_\alpha\|_E \to 0$. □

Theorem 3.3 *Consider an order continuous Banach lattice* $(E, \|\cdot\|_E)$ *and an LNS* (X, p, E)*. Then* $A : (X, p, E) \to (X, p, E)$ *is PMWD if, and only if, A is M-weakly demicompact.*

Proof: Suppose first that A is PMWD. Consider a p-bounded disjoint sequence (x_α) with $\|x_\alpha - Ax_\alpha\|_E \to 0$. The fact that $(E, \|\cdot\|_E)$ is a Banach lattice, gives that $x_\alpha - Ax_\alpha \xrightarrow{o} 0$. From [7, Theorem VII.2.1], it follows that $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$. Now, taking into account that A is PMWD, it yields that $x_\alpha \xrightarrow{p} 0$. We can write $Ax_\alpha = Ax_\alpha - x_\alpha + x_\alpha$, we obtain

$$\begin{aligned} p(Ax_\alpha) &= p(Ax_\alpha - x_\alpha + x_\alpha) \\ &\leq p(Ax_\alpha - x_\alpha) + p(x_\alpha) \to 0, \end{aligned}$$

which implies that $p(Ax_\alpha) \to 0$. So, $Ax_\alpha \xrightarrow{o} 0$. Thus, using hypothesis that $(E, \|.\|_E)$ is an order continuous, $\|Ax_\alpha\|_E \to 0$. Further, we have $x_\alpha = x_\alpha - Ax_\alpha + Ax_\alpha$. Thus,

$$\begin{aligned} \|x_\alpha\|_E &= \|x_\alpha - Ax_\alpha + Ax_\alpha\|_E \\ &\leq \|x_\alpha - Ax_\alpha\|_E + \|Ax_\alpha\|_E \to 0. \end{aligned}$$

Hence, we deduce that $\|x_\alpha\|_E \to 0$. Consequently, A is M-weakly demicompact.
Conversely, assume that A is M-weakly demicompact. Take a p-bounded disjoint sequence (x_α) with $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$. This implies that $x_\alpha - Ax_\alpha \xrightarrow{o} 0$. When we use that $(E, \|\cdot\|_E)$ is an order continuous Banach lattice, we obtain $\|x_\alpha - Ax_\alpha\|_E \to 0$. Now, from the M-weak demicompactness of A, it follows that $\|x_\alpha\|_E \to 0$. As $(E, \|\cdot\|_E)$ is a Banach lattice, we get $x_\alpha \xrightarrow{o} 0$, and therefore $x_\alpha \xrightarrow{p} 0$. Consequently, A is PMWD.

□

4. Stability Results for p-M-Weakly Demicompact Operator

Definition 4.1 *Let E and F be two vector lattices , and let* $A : E \to F$ *be a linear operator. If* $|A(x)| \wedge |A(y)| = 0, \quad$ *whenever* $|x| \wedge |y| = 0,\ x, y \in E$, *then A is disjointness preserving operator.*

We now provide a sufficient condition ensuring that the modulus of a PMWD operator is itself PMWD.

Theorem 4.1 *Consider a LNVL* (X, p, E) *and a PMWD operator* $A : (X, p, E) \to (X, p, E)$*. If A is p-bounded and disjointness preserving, then* $|A|$ *is PMWD.*

Proof: Since A is a p-bounded, it yields that it is order bounded. Thus, A is an order disjointness preserving operator. By using a result of Meyer Nieberg ([25], Theorem 3.1.4), we get the existence of $|A|$ satisfying for all $x \in E^+$ $|A|(x) = |A(x)|$.
Now, take a disjoint and p-bounded sequence (x_α) with $p(x_\alpha - |A|x_\alpha) \xrightarrow{o} 0$. This implies that $p(x_\alpha - |Ax_\alpha|) \xrightarrow{o} 0$. Moreover, we have

$$\begin{aligned} p(x_\alpha - Ax_\alpha) &= p(|x_\alpha - Ax_\alpha|) \\ &\leq p(x_\alpha - |Ax_\alpha|) \xrightarrow{o} 0. \end{aligned}$$

Thus, we obtain that $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$. The fact that A is PMWD, gives that $x_\alpha \xrightarrow{p} 0$. Consequently, $|A|$ is PMWD.

□

Theorem 4.2 *Consider an LNS* (X, p, E) *be a lattice-normed space. If A is p-M-weakly compact from* (X, p, E) *into itself, then A is also PMWD.*

Proof: Take a sequence (x_α) which is p-bounded and disjoint in X satisfying $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$. We can write $x_\alpha = x_\alpha - Ax_\alpha + Ax_\alpha$. Thus, we get

$$\begin{aligned} p(x_\alpha) &= p(x_\alpha - Ax_\alpha + Ax_\alpha) \\ &\leq p(x_\alpha - Ax_\alpha) + p(Ax_\alpha). \end{aligned}$$

Now, from the p-M-weak compactness of A, we get $p(Ax_\alpha) \xrightarrow{o} 0$. Hence, it yields that $p(x_\alpha) \xrightarrow{o} 0$, so $x_\alpha \xrightarrow{p} 0$. Consequently, A is a PMWD operator. □

Remark 4.1 *The converse of Theorem 4.2 is not true. For example, let $I : \ell^\infty \to \ell^\infty$ be the identity operator. Clearly, $-I$ is PMWD. However, since the norm in ℓ^∞ is not order continuous, $-I$ fails to be p-M-weakly compact.*

The next result shows that adding a p-M-weakly compact operator to a PMWD operator preserves the property of p-M-weak demicompactness.

Theorem 4.3 *Consider (X, p, E) an LNS. Let A be PMWD from (X, p, E) into itself and $S : (X, p, E) \to (X, p, E)$ be p-M-weakly compact. Then $A + S$ is PMWD.*

Proof: Consider a p-bounded and disjoint sequence (x_α) satisfying $p\big(x_\alpha - (A+S)x_\alpha\big) \xrightarrow{o} 0$. Now, from the p-M-weak compactness of S, we have $p(Sx_\alpha) \xrightarrow{o} 0$. Thus, we obtain that

$$\begin{aligned} p(x_\alpha - Ax_\alpha) &= p(x_\alpha - Ax_\alpha + Sx_\alpha - Sx_\alpha) \\ &\leq p(x_\alpha - (A+S)x_\alpha) + p(Sx_\alpha) \xrightarrow{o} 0. \end{aligned}$$

Therefore, since A is PMWD and using the above inequality, we infer that $x_\alpha \xrightarrow{p} 0$. Consequently, $A+S$ is a PMWD operator. □

Definition 4.2 *Let (X, p, E) be a LNS and $A : (X, p, E) \to (X, p, E)$. If there exists a positive integer m such that A^m is p-M-weakly compact , then A is power p-M-weakly compact.*

The next result extends Theorem 4.2 to this broader setting.

Theorem 4.4 *Consider an LNS (X, p, E) be a lattice-normed space. If A is a power p-M-weakly compact operator from (X, p, E) into itself , then A is PMWD.*

Proof:

Consider a p-bounded and disjoint sequence (x_α) with $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$.
By hypothesis , there is $k \in \mathbb{N}^*$ satisfying A^k is p-M-weakly compact. This implies that $p(A^k x_\alpha) \xrightarrow{o} 0$. Now, we have

$$\begin{aligned} p(x_\alpha) &= p(x_\alpha - A^k x_\alpha + A^k x_\alpha) \\ &\leq p(x_\alpha - A^k x_\alpha) + p(A^k x_\alpha) \\ &= p((x_\alpha - Ax_\alpha)(I + A + \cdots + A^{k-1})) + p(A^k x_\alpha) \\ &\leq p(x_\alpha - Ax_\alpha).\|I + A + \cdots + A^{k-1}\| + p(A^k x_\alpha). \end{aligned}$$

Hence, we obtain the order convergence of $p(x_\alpha)$, consequently $x_\alpha \xrightarrow{p} 0$. □

Theorem 4.5 *Let (X, p, E) be a LNVL, where $(E, \|\cdot\|_E)$ is an atomic normed lattice with a σ-order continuous norm. Suppose that*

(*i*) *A is p-bounded operator from (X, p, E) into itself.*

(*ii*) *A is M-weakly compact from $(X, p\text{ -}\|.\|_E)$ into itself.*

Hence, $A : (X,p,E) \to (X,p,E)$ is a PMWD operator.

Proof: Consider a p-bounded and disjoint sequence (x_α) in (X,p,E). Thus, it is normed- bounded and disjoint in $(X, p\text{ -}\|.\|_E)$. Using the condition (ii), we get $p\text{ -}\|Ax_\alpha\|_E \to 0$.
Now, using the fact that (x_α) and the operator A are both p-bounded, we get the order boundedness of $p(Ax_\alpha)$ in E.
Consider an atom $a \in E$, we have

$$\begin{aligned}|f_a(p(x_\alpha))| &\leq |f_a(p(x_\alpha - Ax_\alpha))| + |f_a(p(Ax_\alpha))| \\ &\leq \|f_a\| \left[\|p(x_\alpha - Ax_\alpha)\|_E + \|p(Ax_\alpha)\|_E\right] \to 0.\end{aligned}$$

Since E is atomic, we deduce that x_α p-converges to 0. Therefore A is PMWD. □

Theorem 4.6 *Let an LNS (X,p,E) and A be an operator from (X,p,E) into itself. Assume that the norms of E and its dual E' are order continuous. The following statements are equivalent:*

(*i*) *A is PMWD.*

(*ii*) *for a sequence x_α in E_+ which is p-bounded, if x_α converges weakly to 0 and $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$, we have $x_\alpha \xrightarrow{p} 0$.*

Proof: $(i) \Longrightarrow (ii)$ Take a p-bounded sequence (x_α) in E_+ satisfying x_α converges weakly to 0 and $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$.
Now, for each subsequence (y_α) of (x_α) a subsequence (z_α) of (y_α) exists, along with a disjoint sequence (w_α) for which $p(w_\alpha - z_\alpha) \xrightarrow{o} 0$.

$$\begin{aligned}p(w_\alpha - Aw_\alpha) &= p(w_\alpha - Az_\alpha + Az_\alpha - Aw_\alpha) \\ &\leq p(w_\alpha - Az_\alpha) + p(Az_\alpha - Aw_\alpha) \\ &\leq p(w_\alpha - z_\alpha + z_\alpha - Az_\alpha) + p(Az_\alpha - Aw_\alpha) \\ &\leq p(w_\alpha - z_\alpha) + p(z_\alpha - Az_\alpha) + p(Az_\alpha - Aw_\alpha) \xrightarrow{o} 0.\end{aligned}$$

Thus, from the p-M-weak demicompactness of A, we get $w_\alpha \xrightarrow{p} 0$.
Using the following inequality

$$p(z_\alpha) \leq p(z_\alpha - w_\alpha) + p(w_\alpha),$$

we obtain that

$$p(z_\alpha) \to 0, \text{ that is } \; z_\alpha \xrightarrow{p} 0.$$

As (y_α) is arbitrary, it yields that $x_\alpha \xrightarrow{p} 0$.
$(ii) \Longrightarrow (i)$ Assume that $p(x_\alpha - Ax_\alpha) \xrightarrow{o} 0$, for a p-bounded disjoint sequence (x_α).
Using [6, Theorem 2.4.14], the disjointness of x_α together with the order continuity of the norm on E', gives that $x_\alpha \xrightarrow{p} 0$. Consequently, A is PMWD. □

Proposition 4.1 *Let $E = \bigoplus_{i=1}^{m} E_i$, where $(E_i)_{1\leq i\leq m}$ are a Banach lattices, and for all $1 \leq i,j \leq m$ an operator $A_{i,j} : (X,p,E_j) \to (X,p,E_i)$. Assume that the following assumptions hold:*

(*i*) *$A_{i,i}$ is PMWD.*

(*ii*) *$A_{i,j}$ is p-M-weakly compact.*

Thus, the operator $\mathcal{A} : (X, p, E) \to (X, p, E)$ *defined as*

$$\mathcal{A} = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} \end{pmatrix}$$

is PMWD.

Proof: Consider a p-bounded and disjoint sequence $\{X_n = (x_n^1, x_n^2, \dots, x_n^m); n \in \mathbb{N}\}$. Suppose that $X_n - \mathcal{A}X_n \xrightarrow{p} 0$ and we prove the p-convergence of X_n. Since

$$p(X_n) = p(x_n^1) + \cdots + p(x_n^m),$$

for each $1 \le i \le m$, it suffices to show that $x_n^i \xrightarrow{p} 0$. Therefore, we obtain

$$X_n - \mathcal{A}X_n = \begin{pmatrix} x_n^1 - A_{1,1}x_n^1 - A_{1,2}x_n^2 - \cdots - A_{1,m}x_n^m \\ x_n^2 - A_{2,1}x_n^1 - A_{2,2}x_n^2 - \cdots - A_{2,m}x_n^m \\ \vdots \\ x_n^m - A_{m,1}x_n^1 - A_{m,2}x_n^2 - \cdots - A_{m,m}x_n^m \end{pmatrix}.$$

Then $p(x_n^i - A_{i,1}x_n^1 - A_{i,2}x_n^2 - \cdots - A_{i,m}x_n^m) \to 0$. Let us first take $i = 1$. The disjointness of (x_n^j) and the p-M-weak compactness of $A_{1,j}$ ensure that

$$A_{1,j}x_n^j \xrightarrow{p} 0 \quad \text{for each } 2 \le j \le m.$$

When using the inequality below

$$p(x_n^1 - A_{1,1}x_n^1) \le p(x_n^1 - A_{1,1}x_n^1 - A_{1,2}x_n^2 - \cdots - A_{1,m}x_n^m) + p(A_{1,2}x_n^2) + \cdots + p(A_{1,m}x_n^m),$$

for each n, we infer that $p(x_n^1 - A_{1,1}x_n^1) \to 0$. Consequently, from the p-M-weak demicompactness of $A_{1,1}$, we get $x_n^1 \xrightarrow{p} 0$.

Now, for $i < j \le m$, $A_{i,j}$ is p-M-weakly compact, which implies that $p(A_{i,j}x_n^j) \to 0$. Thus, from the use of the following inequality

$$\begin{aligned} p(x_n^i - A_{i,i}x_n^i) \le\ & p(A_{i,1}x_n^1) + \cdots + p(A_{i,i-1}x_n^{i-1}) \\ & + p(x_n^i - A_{i,1}x_n^1 - \cdots - A_{i,i-1}x_n^{i-1} - A_{i,i+1}x_n^{i+1} - \cdots - A_{i,m}x_n^m) \\ & + p(A_{i,i+1}x_n^{i+1}) + \cdots + p(A_{i,m}x_n^m), \end{aligned}$$

we get $p(x_n^i - A_{i,i}x_n^i) \to 0$. Further, since $A_{i,i}$ is PMWD, we obtain the p-convergence of x_n^i to 0.

Finally, the last case $i = m$. We can write

$$\begin{aligned} p(x_n^m - A_{m,m}x_n^m) \le\ & p(A_{m,1}x_n^1) + p(A_{m,2}x_n^2) + \cdots + p(A_{m,m-1}x_n^{m-1}) \\ & + p(x_n^m - A_{m,1}x_n^1 - A_{m,2}x_n^2 - \cdots - A_{m,m}x_n^m), \end{aligned}$$

then $x_n^m - A_{m,m}x_n^m \xrightarrow{p} 0$. Now, taking into account that $A_{m,m}$ satisfies the assumption (i), we get $x_n^m \xrightarrow{p} 0$. □

Example 4.1 *Let* $X = E = \ell^1$ *with the lattice norm* $p((x, y)) = |x| + |y|, \quad (x, y) \in \ell^1 \oplus \ell^1$. *We define the following operators on* ℓ^1 *by*

$$\begin{aligned} A_{11}(x_1, x_2, \dots) &= \Big(\tfrac{1}{2}x_1, \tfrac{1}{4}x_2, \tfrac{1}{8}x_3, \dots\Big). \\ A_{22}(x_1, x_2, \dots) &= (0, x_2, 0, \dots). \\ A_{12}(x_1, x_2, \dots) &= (x_1, 0, 0, \dots). \\ A_{21}(x_1, x_2, \dots) &= (0, x_1, 0, \dots), \end{aligned}$$

Now, the operator A_{22} *is a finite-rank projection, so it is a compact operator. Thus, we get* A_{22} *is a* p-M*-weakly compact. Hence, from Theorem 4.2, we deduce that* A_{22} *is PMWD.*
Further, the operators A_{12} *and* A_{21}*, we also are finite-rank operators, which are compact as well. Consequently,* A_{12} *and* A_{21} *are* p-M*- weakly compact. Next, when we use the example 3.2, we infer that* A_{11} *is PMWD. Now, we consider the following matrix operator* $\mathcal{A}$ *defined by*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : X \oplus X \to X \oplus X$$

The operator A *acts on* $(x, y) \in \ell^1 \oplus \ell^1$ *according to*

$$A(x, y) = \Big((\tfrac{1}{2}x_1 + y_1, \tfrac{1}{4}x_2, \tfrac{1}{8}x_3, \dots),\ (x_1 + y_2, 0, 0, \dots) \Big).$$

By construction, all diagonal and off-diagonal blocks satisfy the necessary conditions of Proposition 4.1 , and therefore the block operator $\mathcal{A}$ *is PMWD on* $(\ell^1 \oplus \ell^1, p)$*.*

5. Numerical Application

In this section, we present a numerical application of the new class PMWD operators in the context of neural networks. The aim is to show how the abstract results obtained in Sections 3 and 4 can be used to explain the stability of hidden states in recurrent models. Recurrent neural networks (RNNs) and related iterative models generate a sequence of hidden states $(h_t)_{t \geq 0}$ by the update rule

$$h_{t+1} = \sigma(Ah_t + b),$$

where $A \in \mathbb{R}^{n \times n}$ is the weight matrix, $b \in \mathbb{R}^n$ is a bias vector, and σ is a nonlinear activation function. The long-term stability of (h_t) is essential for the success of training and prediction. If the operator A amplifies oscillations, the hidden states may diverge or become unstable. To align with the lattice-normed framework, we consider $\mathbb{R}^n$ equipped with the componentwise order (positive cone $\mathbb{R}^n_+$) and lattice norm $p(h) = |h|$ (componentwise absolute value), where $\mathrm{E} = \mathbb{R}^n$ is a Banach lattice under the sup norm. In this setting, p-convergence corresponds to componentwise convergence to zero. If we decompose the matrix A as

$$A = \alpha I + K,$$

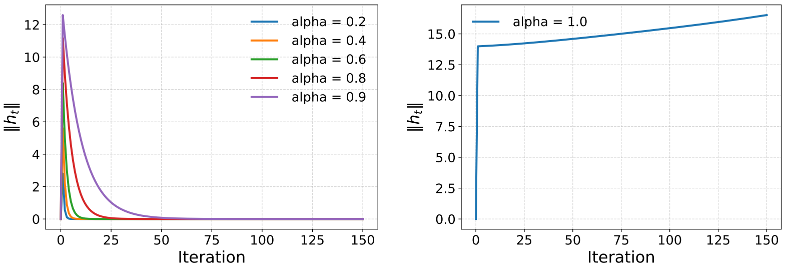
where I is the identity, $\alpha \in (0, 1)$ is a scalar, and K is a low-rank operator. Then, this structure is natural in low-rank recurrent models and in efficient deep learning architectures. From the theoretical point of view, the scaled identity αI is PMWD. The low-rank K is compact-like; in finite dimensions with the lattice norm $p(\cdot)$, K is also PMWD. By Theorem 4.3 A is PMWD. Consequently, the sequence (h_t) is guaranteed to exhibit p-convergence to a stable point, controlling oscillations in the dynamics.

We illustrate this application with simple linear RNN (i.e, $\sigma(x) = x$). In this case we write

$$h_{t+1} = Ah_t + b, \qquad A = \alpha I + UV^\top,$$

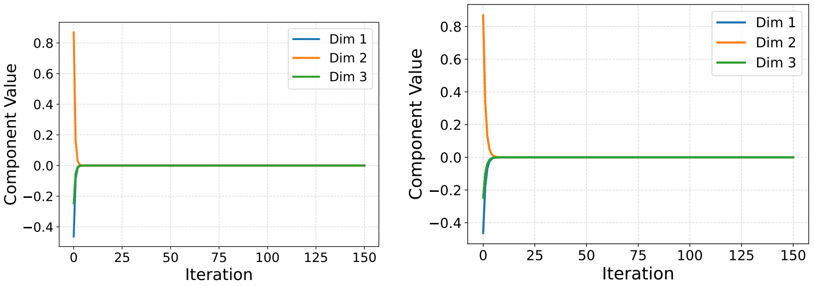
where $U, V \in \mathbb{R}^{n \times r}$ with $r \ll n$. The operator A is the sum of a scaled identity and a compact-like low-rank term. We take $n = 200$, $r = 5$. The low-rank $K = UV^\top$ is scaled so its maximum singular value is 0.1, ensuring a small perturbation. The initial state h_0 is chosen randomly from a standard normal distribution, and the iteration is run for $T = 150$ steps. For comparison, we take different values of α.

We record the ℓ_2-norm $\|h_t\|_2$ of the hidden states and the singular values of the low-rank part $K = UV^\top$ and eigenvalue distribution in complex plane of structured and random matrices, and Hidden state trajectories for first three dimensions of structured matrix.

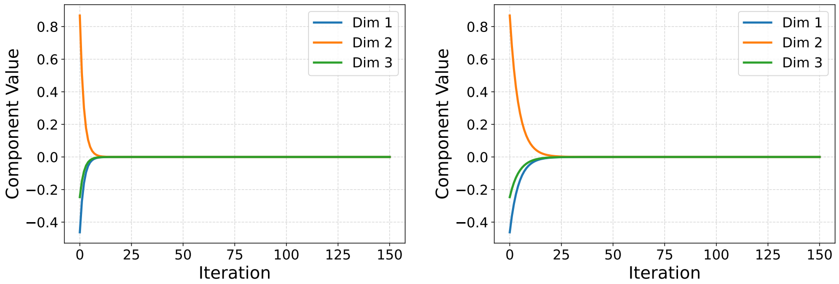


(a) $\alpha \in \{0.2, \dots, 0.8, 0.9\}$ (b) $\alpha = 1.0$

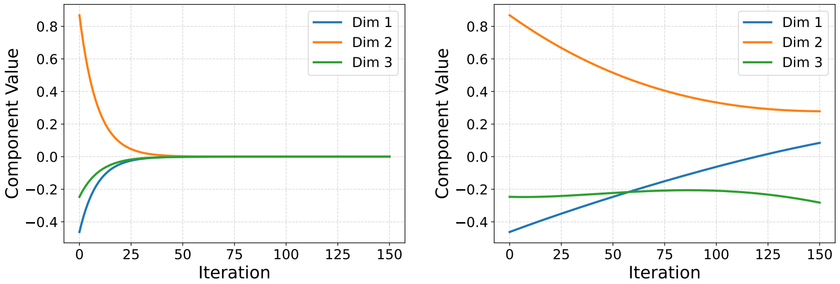
Figure 1: Graphs of hidden state norm over iterations for a structured matrix $A = \alpha I + K$ for different α.



(a) $\alpha = 0.2$ (b) $\alpha = 0.4$



(c) $\alpha = 0.6$ (d) $\alpha = 0.8$



(e) $\alpha = 0.9$ (f) $\alpha = 1.0$

Figure 2: Graphs of the hidden state trajectories for first 3 dimensions over iterations for a structured matrix $A = \alpha I + K$ for different α.

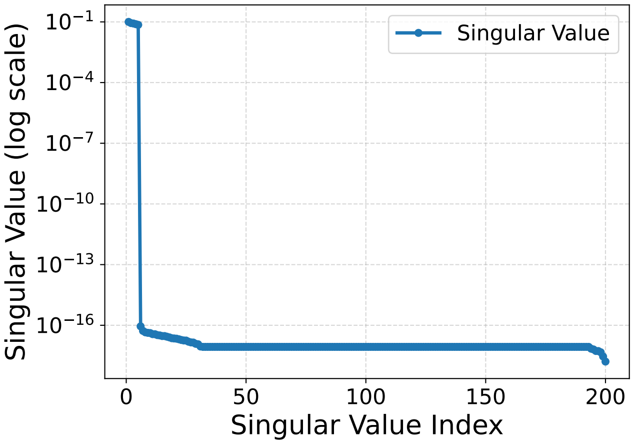


Figure 3: Singular value decay of $K = UV^\top$.

In Figure 1, we tested a simple type of neural network for a structured matrix $A = \alpha I + K$. We checked how the hidden states $(h_t)_t$ change over time for different values of $\alpha \in \{0.2, \dots, 1.0\}$. When $\alpha \in (0.2, 0.8)$ is small, the $\|h_t\|_2$ drops fast to almost zero. This means the system becomes stable quickly, with no big ups and downs. For $\alpha = 1$, the norm stays around the same level, about 14, and grows a little bit slowly as in Table 1. But for $\alpha = 1.5$, it grows a lot, showing the system can become unstable if α is too big. In Figure 2, we show the paths of the first three dimensions $(h_t[0], h_t[1], h_t[2])$ of h_t over iteration steps $t = 0, 1, \dots, T$. It shows smooth changes without wild moves when α is small. Figure 3 shows the singular values of K drop fast. This proves K acts like a small change that does not hurt stability much.

Our tests connect well to the theoretical part. PMWD operators are meant to handle disjoint sequences and stop weak shakes. In the tests, this shows as stable hidden states in RNNs. The structured matrix follows Theorem 4.3, where adding a small compact part keeps the PMWD property. This helps in real uses, like making neural networks that do not blow up or shake.

Remark 5.1 *In deep learning, this approach is useful but has limitations. Stable states help avoid large errors during training. Low-rank structures, like our K, make models faster and use less memory, a common practice in large AI systems today. However, if norms drop to zero too quickly, it can trigger the* vanishing gradients *problem. This hinders the network's ability to learn from distant past data, a well-known issue in vanilla RNNs. That said, we can view PMWD stability as a guiding framework for RNN applications. In this RNN context, tuning parameters to stabilize the norm of h_t around 1 can prevent vanishing gradients [19,28,27]. This suggests that PMWD operators may serve as a rigorous tool for designing stable neural architectures, potentially extending to nonlinear cases with positive activations to preserve the lattice structure.*

Table 1: Table of $\|h_t\|$ for different values of α

Iterations	$\alpha = 0.4$	$\alpha = 0.8$	$\alpha = 1.0$	$\alpha = 1.5$
0	13.985	13.985	13.985	13.985
1	5.601	11.194	13.991	20.984
2	2.243	8.961	13.998	31.485
3	0.899	7.173	14.005	47.243
4	0.360	5.743	14.012	70.888
5	0.144	4.597	14.020	106.369
6	0.058	3.681	14.028	159.610
7	0.023	2.947	14.037	239.503
8	0.009	2.359	14.045	359.390
9	0.004	1.889	14.054	539.293
10	0.001	1.513	14.064	809.259
20	0.000	0.164	14.170	4.68e4
40	0.000	0.018	14.298	2.717e6
80	0.000	0.002	14.440	1.576e8
120	0.000	0.000	14.594	9.146e9
150	0.000	0.000	16.526	4.008e27

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