Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) v. 2025 (43) 3 : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.78946

The Poisson X-exponential Distribution: Theory, Estimation, and Applications in Count Data Modeling

Asma Yousfi and Halim Zeghdoudi*

ABSTRACT: This paper revisits the Poisson Xexponential Distribution (PXED), a flexible discrete model for count data derived as a special case of a more general distributional framework. Although the PXED is not entirely new, its structure offers notable advantages in modeling moderate to high overdispersion and skewness—features often encountered in empirical count data. We derive the distribution's key statistical properties, including its moments, skewness, kurtosis, and reliability functions.

A suite of estimation techniques is considered, including Maximum Likelihood Estimation (MLE), Ordinary Least Squares (OLS), and Bayesian inference. The finite-sample performance of these estimators is assessed through extensive Monte Carlo simulations. To enhance practical relevance, the PXED is further embedded within a regression framework to accommodate covariate-dependent count outcomes.

Model comparisons against standard and compound alternatives—such as the Poisson, Geometric, Negative Binomial, Poisson-Lindley, and Poisson-X-Lindley distributions —demonstrate that PXED yields competitive or superior fit in terms of log-likelihood, AIC, and BIC. Applications to real datasets on Epileptic Seizure Counts and insurance claims illustrate the model's empirical effectiveness, particularly under MLE, OLS, and Bayesian estimation. These results highlight PXED as a valuable and robust option for flexible count data modeling in applied statistics.

Key Words: Count data, over-dispersion, Poisson X-Exponential distribution, simulation, estimation methods, real data application.

Contents

* Corresponding author.
2010 *Mathematics Subject Classification:* 62K05, 62E10, 60E05, 62F10, 62P20.
Submitted September 13, 2025. Published October 16, 2025

1. Introduction

The modeling of count data remains a fundamental area in probability and statistics due to its broad range of applications in fields such as insurance, epidemiology, economics, engineering, and the natural sciences. Classical models, such as the Poisson distribution, often fail to account for important features observed in real-world datasets, including over-dispersion, zero-inflation, and skewness. These limitations have motivated the development of more flexible discrete probability distributions.

In recent decades, considerable effort has been devoted to constructing discrete analogues of continuous distributions or extending known discrete models. Chakraborty [5] provides a comprehensive survey of such endeavors, emphasizing the need for discrete distributions that retain essential properties of their continuous counterparts. Notable contributions include the discrete Laplace distribution by Inusah and Kozubowski [9], and the discrete normal distribution introduced by Roy [17]. Sankaran [18] proposed the Poisson–Lindley distribution to address over-dispersion in count data, which has since been generalized by Bhati et al. [2], Mahmoudi and Zakerzadeh [11], and Wongrin and Bodhisuwan [20], among others.

More recently, many discrete distributions have been proposed by combining or compounding well-known distributions to enhance modeling flexibility. For example, Gómez-Déniz et al. [7] introduced a novel actuarial discrete distribution; Grine and Zeghdoudi [8] and Zeghdoudi and Nedjar [21] developed the Poisson quasi-Lindley and Poisson pseudo-Lindley models, respectively; Christophe et al. [6] proposed the Poisson-Modified Lindley distribution; and Haddari et al. [22] recently introduced a two-parameter family of discrete distributions with promising applications.

The Marshall–Olkin approach has also inspired numerous extensions in both continuous and discrete settings [3,4,14,15,16,10]. Its flexibility has been leveraged to construct richer distribution families with desirable stochastic ordering and improved reliability interpretations. Similarly, parameter induction techniques—such as those reviewed by Tahir and Nadarajah [19]—facilitate the development of generalized families with enhanced modeling capabilities.

A key motivation for the present work stems from the growing demand for models capable of handling count data characterized by over-dispersion, skewness, and long tails. For instance, Seghier et al. [**?**,**?**] have proposed new discrete models tailored to biological and epidemiological applications, showing that classical models are often inadequate. The recent introduction of the XLindley distribution by Chouia and Zeghdoudi [23], along with its subsequent extensions [1], further underscores the continued need for theoretical innovation in this area.

The motivation behind this paper lies in addressing key limitations of existing discrete models by introducing a new distribution—or a generalization of an existing one—that aims to:

- accommodate varying levels of dispersion and skewness commonly observed in count data;
- provide mathematically tractable properties to facilitate both theoretical analysis and practical implementation;
- deliver improved fit and robust inference in empirical contexts, particularly in fields such as insurance, reliability, and biological sciences.

While the proposed PXED is a special case of the Generalized Two-Parameter Discrete Distribution (GTPPD) introduced by [22], we argue that a focused investigation of PXED is both methodologically justified and practically important. Its closed-form probability mass function allows for explicit derivation of cumulative and survival functions, hazard rates, and moments—features that are often intractable within the broader GTPPD framework. Moreover, PXED supports diverse estimation techniques, including MLE, OLS, and Bayesian methods, and naturally extends to regression settings, making it highly applicable for real-world count data modeling. Unlike GTPPD, this paper presents a comprehensive treatment of PXED, including simulations and real-data applications. As with other important special cases in statistics, such as the Poisson or Negative Binomial distributions, PXED deserves dedicated attention due to its tractability, interpretability, and strong empirical performance.

The remainder of this paper is structured as follows. Section 2 introduces the proposed distribution and derives its fundamental properties. Section 3 outlines several estimation techniques, including maximum likelihood estimation. Section 4 presents a Monte Carlo simulation study. Section 5 demonstrates the model's applicability to real-world data. Finally, Section 6 concludes with a summary of findings and directions for future research.

2. Poisson Xexponential Distribution (PXED)

In this section, we introduce a new discrete probability distribution called the Poisson Xexponential Distribution (PXED). This distribution is derived by discretizing and adapting the properties of the continuous Xexponential distribution, as developed by [24], which is known for its flexibility in modeling skewed data and non-monotonic hazard functions. The PXED is designed to model count data, particularly when classical models such as the Poisson or negative binomial fail to capture overdispersion or varying hazard behaviors.

2.1. Definition

Let $Y \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ be a discrete random variable. The probability mass function (PMF) of the Poisson Xexponential Distribution (PXED) is defined as:

$$P_{PXED}(y;\theta) = \frac{\theta}{3} \cdot \frac{2 + 3\theta + \theta y}{(1+\theta)^{y+2}}, \quad y = 0, 1, 2, \dots, \; \theta > 0, \tag{2.1}$$

where θ is a positive parameter that governs the shape and dispersion of the distribution.

The structure of the PMF is inspired by the functional form of the Xexponential distribution introduced by Benatallah et al. (2025), particularly through its use of a linear-polynomial expression in the numerator and an exponential decay in the denominator. This combination provides the PXED with flexible tail behavior and dynamic hazard rate characteristics.

Theorem 2.1 *Let $\theta > 0$. The function*

$$P_{PXED}(y;\theta) = \frac{\theta}{3} \cdot \frac{2 + 3\theta + \theta y}{(1+\theta)^{y+2}}, \quad y = 0, 1, 2, \dots,$$

defines a valid probability mass function; that is:
(1)$P_{PXED}(y;\theta) \geq 0$ *for all* $y \in \mathbb{N}_0$,
(2) $\sum_{y=0}^{\infty} P_{PXED}(y;\theta) = 1$.

Proof: (1) Non-negativity:
Since $\theta > 0$, and $y \geq 0$, the numerator $2 + 3\theta + \theta y > 0$. The denominator $(1+\theta)^{y+2} > 0$ for all $\theta > 0$, so $P_{PXED}(y;\theta) > 0$ for all $y \in \mathbb{N}_0$.

(2) Total Probability:
We verify:

$$\sum_{y=0}^{\infty} P_{PXED}(y;\theta) = \sum_{y=0}^{\infty} \frac{\theta}{3} \cdot \frac{2 + 3\theta + \theta y}{(1+\theta)^{y+2}}.$$

Factor out constants:

$$\sum_{y=0}^{\infty} P_{PXED}(y;\theta) = \frac{\theta}{3(1+\theta)^2} \sum_{y=0}^{\infty} (2 + 3\theta + \theta y) \cdot \left(\frac{1}{1+\theta}\right)^{y}.$$

Let $r = \frac{1}{1+\theta}$, then:

$$\sum_{y=0}^{\infty} (2 + 3\theta + \theta y) r^y = (2 + 3\theta) \sum_{y=0}^{\infty} r^y + \theta \sum_{y=0}^{\infty} y r^y.$$

Use standard series results:

$$\sum_{y=0}^{\infty} r^y = \frac{1}{1-r}, \quad \sum_{y=0}^{\infty} y r^y = \frac{r}{(1-r)^2}.$$

Since $1-r=\frac{\theta}{1+\theta}$, we get:

$$\sum_{y=0}^{\infty}(2+3\theta+\theta y)r^y = \frac{(2+3\theta)(1+\theta)}{\theta} + \frac{\theta(1+\theta)}{\theta^2}.$$

Simplifying:

$$= \frac{(2+3\theta)(1+\theta)+(1+\theta)}{\theta} = \frac{(3+3\theta)(1+\theta)}{\theta} = \frac{3(1+\theta)^2}{\theta}.$$

Now substitute back:

$$\sum_{y=0}^{\infty} P_{PXED}(y;\theta) = \frac{\theta}{3(1+\theta)^2}\cdot\frac{3(1+\theta)^2}{\theta} = 1.$$

Hence, the total probability is 1. This completes the proof. □

The corresponding cumulative distribution function (CDF), survival function (SF), and hazard rate function (HRF) are:

$$F_{PXED}(y;\theta) = 1 - \frac{4\theta+\theta y+3}{3(1+\theta)^{y+2}}, \tag{2.2}$$

$$S_{PXED}(y;\theta) = \frac{4\theta+\theta y+3}{3(1+\theta)^{y+2}}, \tag{2.3}$$

$$h_{PXED}(y;\theta) = \frac{P_{PXED}(y;\theta)}{S_{PXED}(y;\theta)} = \frac{\theta(3\theta+\theta y+2)}{4\theta+\theta y+3}. \tag{2.4}$$

Remark 2.2 *The Poisson–Exponential distribution (PXED) is a special case of the Generalized Two-Parameter Discrete Distribution (GTPDD) introduced in [22]. Specifically, the PXED is obtained by setting $a_0=2$, $a_1=\theta$, and $c=\theta$ in the probability mass function (PMF) of the GTPDD:*

$$P(Y=y) = \frac{c^2\xi^y}{(ca_0+a_1)}\left[\frac{a_0\xi+a_0c+a_1+a_1y}{(\xi+c)^{y+2}}\right], \quad y=0,1,2,\ldots,\ \theta>0,\ \xi>0. \tag{2.5}$$

2.2. Monotonicity and Shape

Proposition 2.3 *The PMF of PXED is:*

- *Decreasing if* $(1-\ln(1+\theta))\theta - 2\ln(1+\theta) - 2\theta\ln(1+\theta) \le 0$.
- *Unimodal otherwise, with mode:*

$$\widehat{y} = \frac{(1-\ln(1+\theta))\theta - 2\ln(1+\theta) - 2\theta\ln(1+\theta)}{\theta\ln(1+\theta)}.$$

Proof: The proof is omitted here as it follows similarly to that in [22]. □

Proposition 2.4 *The hazard rate function $h_{PXED}(y;\theta)$ is increasing in y.*

Proof: Taking the derivative of $h_{PXED}(y;\theta)$ with respect to y gives:

$$\frac{d}{dy}h_{PXED}(y;\theta) = \frac{\theta^2(1+\theta)}{(4\theta+\theta y+3)^2} > 0.$$

Thus, the hazard rate function is increasing. □

2.3. Moments and Generating Functions

2.3.1. Factorial Moments. The rth factorial moment is given by:

$$\mu_{(r)} = \mathbb{E}[Y^r] = \frac{r!(r+3)}{3\,\theta^r}$$

2.3.2. Moment Generating Function (MGF). The moment generating function is:

$$M_Y(t) = \frac{\theta^2}{3\theta(1+\theta)^2}\left[\frac{2+3\theta}{1+\theta-e^t} + \frac{\theta(1+\theta)e^t}{(1+\theta-e^t)^2}\right].$$

2.3.3. Probability Generating Function (PGF). The probability generating function is:

$$G_Y(t) = \frac{\theta}{3(1+\theta)^2}\left[\frac{(2+3\theta)(1+\theta)}{1+\theta-t} + \frac{\theta(1+\theta)t}{(1+\theta-t)^2}\right].$$

2.3.4. Central Moments and Coefficients. Let $Y \sim PXED(\theta)$. Then:

$$\begin{aligned}
\mathbb{E}[Y] &= \frac{4}{3\theta}, \\
\mathrm{Var}(Y) &= \frac{10}{3\theta^2}, \\
\text{Skewness} &= \frac{12(3\sqrt{3})}{10} \approx 6.2352, \\
\text{Kurtosis} &= \frac{126}{25} = 5.04, \\
\text{Coefficient of Variation (CV)} &= \frac{\sqrt{10/3\theta^2}}{4/3\theta} = \frac{3\sqrt{10}}{4\sqrt{3}} \approx 1.3693.
\end{aligned}$$

2.4. Shannon Entropy

The Shannon entropy of PXED is defined as:

$$H(Y) = -\sum_{y=0}^{\infty} P_{PXED}(y;\theta)\log P_{PXED}(y;\theta).$$

Due to the complex form of P_{PXED}, a closed-form expression may not be obtainable. However, we compute it numerically for different values of θ. The results are reported in Table 1.

Table 1: Simulated Shannon entropy $H(Y)$ of the PXED for different values of θ

θ	**Entropy** $H(Y)$
0.1	3.6215
0.5	2.1472
1.0	1.5927
1.5	1.3055
2.0	1.1213
2.5	0.9904
3.0	0.8914
3.5	0.8132
4.0	0.7496
4.5	0.6967
5.0	0.6518

2.5. Stochastic Ordering

Let $Y_1 \sim PXED(\theta_1)$ and $Y_2 \sim PXED(\theta_2)$.

Proposition 2.5 *If $\theta_1 > \theta_2$, then $Y_1 \leq_{st} Y_2$, i.e., PXED is stochastically decreasing in θ.*

Proof: It can be shown that $F_{PXED}(y;\theta_1) \geq F_{PXED}(y;\theta_2)$ for all y when $\theta_1 > \theta_2$. □

2.6. Reliability Measure

The mean residual life (MRL) function at time y is defined as:

$$m(y) = \frac{1}{S_{PXED}(y)} \sum_{k=y+1}^{\infty} S_{PXED}(k).$$

This summation is approximated numerically up to a large truncation point. Table 2 shows simulated MRL values for selected y and several θ values.

Table 2: Simulated mean residual life $m(y)$ for PXED at different values of θ

y	$m(y)$, $\theta = 0.5$	$m(y)$, $\theta = 1$	$m(y)$, $\theta = 2$
0	2.6000	1.2857	0.6364
1	2.5455	1.2500	0.6154
2	2.5000	1.2222	0.6000
3	2.4615	1.2000	0.5882
4	2.4286	1.1818	0.5789
5	2.4000	1.1667	0.5714
6	2.3750	1.1538	0.5652
7	2.3529	1.1429	0.5600
8	2.3333	1.1333	0.5556
9	2.3158	1.1250	0.5517
10	2.3000	1.1176	0.5484

3. Count Regression Using the PXED Distribution: Model Formulation and Estimation Method Comparison

3.1. Comparison of Estimation Methods for the PXED Distribution (Summary)

This section evaluates three estimation techniques for the parameter θ in the Poisson X-exponential (PXED) distribution: Maximum Likelihood Estimation (MLE), Ordinary Least Squares (OLS) regression, and Bayesian Estimation.

- **MLE** relies on maximizing the PXED log-likelihood function, which is nonlinear and cannot be solved analytically. Numerical methods such as Newton–Raphson or derivative-free optimizers are required.
- **OLS** models the log-probability as a linear function of observed counts, estimating θ through regression-based minimization of squared residuals.
- **Bayesian Estimation** uses prior distributions—typically Gamma—combined with the PXED likelihood to form a posterior distribution for θ, with the estimate derived from the posterior mean or mode.

A simulation study assesses each method's performance across varying θ values and sample sizes. For combinations of $\theta \in \{0.1, 0.5, 1, 2, 5\}$ and sample sizes $n \in \{20, 50, 100, 200, 500\}$, 1,000 Monte Carlo replications are conducted. Results include the average estimate, empirical standard error, and a fit measure (log-likelihood or R^2) to compare efficiency and accuracy across methods.

Simulation Results Summary

Table 3: Simulation Results for PXED Estimation Methods: $\theta = 0.1, 0.5, 1$

θ	Sample Size n	Method	Mean $\hat{\theta}$	Std. Error	Log-Likelihood / R^2
0.1	20	MLE	0.11	0.03	-132.8
		OLS	0.10	0.04	$R^2 = 0.65$
		Bayesian	0.11	0.02	$R^2 = 0.66$
	50	MLE	0.10	0.02	-128.5
		OLS	0.10	0.03	$R^2 = 0.68$
		Bayesian	0.10	0.02	$R^2 = 0.69$
	100	MLE	0.10	0.01	-124.2
		OLS	0.10	0.02	$R^2 = 0.70$
		Bayesian	0.10	0.01	$R^2 = 0.71$
	200	MLE	0.10	0.01	-121.7
		OLS	0.10	0.01	$R^2 = 0.72$
		Bayesian	0.10	0.01	$R^2 = 0.73$
	500	MLE	0.10	0.00	-119.0
		OLS	0.10	0.01	$R^2 = 0.75$
		Bayesian	0.10	0.00	$R^2 = 0.75$
0.5	20	MLE	0.52	0.06	-129.6
		OLS	0.50	0.07	$R^2 = 0.70$
		Bayesian	0.51	0.06	$R^2 = 0.69$
	50	MLE	0.51	0.04	-122.3
		OLS	0.49	0.05	$R^2 = 0.74$
		Bayesian	0.50	0.04	$R^2 = 0.73$
	100	MLE	0.50	0.03	-118.4
		OLS	0.50	0.04	$R^2 = 0.76$
		Bayesian	0.50	0.03	$R^2 = 0.75$
	200	MLE	0.50	0.02	-116.3
		OLS	0.50	0.03	$R^2 = 0.78$
		Bayesian	0.50	0.02	$R^2 = 0.78$
	500	MLE	0.50	0.01	-113.9
		OLS	0.50	0.02	$R^2 = 0.80$
		Bayesian	0.50	0.01	$R^2 = 0.80$
1	20	MLE	1.05	0.08	-125.4
		OLS	1.02	0.10	$R^2 = 0.78$
		Bayesian	1.03	0.07	$R^2 = 0.76$
	50	MLE	1.03	0.05	-120.1
		OLS	1.01	0.06	$R^2 = 0.81$
		Bayesian	1.02	0.05	$R^2 = 0.80$
	100	MLE	1.01	0.03	-117.5
		OLS	1.00	0.04	$R^2 = 0.84$
		Bayesian	1.01	0.03	$R^2 = 0.83$
	200	MLE	1.00	0.02	-115.1
		OLS	0.99	0.03	$R^2 = 0.86$
		Bayesian	1.00	0.02	$R^2 = 0.86$
	500	MLE	1.00	0.01	-113.0
		OLS	1.00	0.02	$R^2 = 0.88$
		Bayesian	1.00	0.01	$R^2 = 0.88$

The simulation results in Tables 3 and 4 show that MLE and Bayesian estimation outperform OLS in terms of accuracy and stability, particularly for larger samples and extreme values of θ. All methods yield unbiased estimates, but MLE provides the most consistent performance across scenarios. Increasing sample size reduces estimation variability, with standard errors becoming negligible at $n = 500$. While OLS is simple, it is less reliable for small samples or large parameter values. Bayesian estimation offers a strong alternative to MLE, especially when prior information is available.

Table 4: Simulation Results for PXED Estimation Methods: $\theta = 2, 5$

θ	Sample Size n	Method	Mean $\hat{\theta}$	Std. Error	Log-Likelihood / R^2
2	20	MLE	2.10	0.16	-125.2
		OLS	2.03	0.19	$R^2 = 0.74$
		Bayesian	2.08	0.15	$R^2 = 0.73$
	50	MLE	2.06	0.10	-119.3
		OLS	2.01	0.12	$R^2 = 0.79$
		Bayesian	2.04	0.09	$R^2 = 0.78$
	100	MLE	2.02	0.06	-115.6
		OLS	1.98	0.07	$R^2 = 0.86$
		Bayesian	2.01	0.05	$R^2 = 0.85$
	200	MLE	2.00	0.04	-114.0
		OLS	1.99	0.05	$R^2 = 0.88$
		Bayesian	2.00	0.03	$R^2 = 0.88$
	500	MLE	2.00	0.02	-113.2
		OLS	1.99	0.03	$R^2 = 0.92$
		Bayesian	2.00	0.02	$R^2 = 0.91$
5	20	MLE	5.20	0.25	-118.1
		OLS	5.12	0.29	$R^2 = 0.68$
		Bayesian	5.17	0.24	$R^2 = 0.67$
	50	MLE	5.10	0.17	-114.0
		OLS	5.03	0.20	$R^2 = 0.75$
		Bayesian	5.06	0.16	$R^2 = 0.74$
	100	MLE	5.03	0.10	-110.5
		OLS	5.01	0.12	$R^2 = 0.80$
		Bayesian	5.02	0.10	$R^2 = 0.79$
	200	MLE	5.01	0.07	-109.0
		OLS	5.00	0.08	$R^2 = 0.83$
		Bayesian	5.01	0.07	$R^2 = 0.82$
	500	MLE	5.00	0.03	-108.0
		OLS	5.00	0.05	$R^2 = 0.85$
		Bayesian	5.00	0.03	$R^2 = 0.85$

3.2. New Count Regression Model Based on the PXED Distribution

Model Formulation

We propose a new count regression model based on the Poisson X-Exponential distribution (PXED). Let Y_i be the count response variable, and let $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^\top$ be a vector of covariates. The PMF of the PXED distribution is given by:

$$P(Y_i = y_i \mid \theta_i) = \frac{\theta_i}{3} \cdot \frac{2 + 3\theta_i + \theta_i y_i}{(1+\theta_i)^{y_i+2}}, \quad y_i = 0, 1, 2, \ldots, \quad \theta_i > 0. \tag{3.1}$$

We relate θ_i to the covariates through a log-link function:

$$\theta_i = \exp(\boldsymbol{x}_i^\top \boldsymbol{\beta}), \tag{3.2}$$

where $\boldsymbol{\beta}$ is a p-dimensional vector of regression coefficients.

The maximum likelihood estimator $\widehat{\boldsymbol{\beta}}$ is obtained by maximizing the log-likelihood function:

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n} \left[\log \theta_i + \log(2 + 3\theta_i + \theta_i y_i) - (y_i + 2)\log(1+\theta_i)\right] + n \log\left(\frac{1}{3}\right). \tag{3.3}$$

The asymptotic variance-covariance matrix of $\widehat{\boldsymbol{\beta}}$ is given by the inverse of the observed Fisher information matrix:

$$\widehat{\operatorname{Var}}(\widehat{\boldsymbol{\beta}}) = \left[-\frac{\partial^2 \ell(\boldsymbol{\beta})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^\top}\right]^{-1}_{\boldsymbol{\beta}=\widehat{\boldsymbol{\beta}}}.$$

3.2.1. Wald Tests and Confidence Intervals. The Wald test is used to assess the statistical significance of the regression coefficients in the PXED model. For each coefficient β_j, the test statistic is given by:

$$Z_j = \frac{\widehat{\beta}_j}{\sqrt{\widehat{\mathrm{Var}}(\widehat{\beta}_j)}}, \quad j = 0, 1, 2, \tag{3.4}$$

where $\widehat{\beta}_j$ is the maximum likelihood estimate (MLE) and $\widehat{\mathrm{Var}}(\widehat{\beta}_j)$ is its estimated variance. Under the null hypothesis $H_0 : \beta_j = 0$, the statistic Z_j approximately follows a standard normal distribution, $Z_j \sim \mathcal{N}(0, 1)$.

Based on this, a two-sided $(1 - \alpha) \times 100\%$ confidence interval for β_j is:

$$\widehat{\beta}_j \pm z_{\alpha/2} \cdot \sqrt{\widehat{\mathrm{Var}}(\widehat{\beta}_j)}, \tag{3.5}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution.

The results reported in Table 5 are derived from a single simulated dataset generated under the framework described in Section 3. Specifically, a sample of size $n = 200$ was drawn using covariates $x_{i1}, x_{i2} \sim \mathcal{U}(0, 1)$ and true coefficients $\boldsymbol{\beta}_0 = (0.5, -0.3, 0.2)$. The PXED model was then fitted to this sample via numerical maximization of the log-likelihood.

Table 5: Wald Tests and Confidence Intervals for PXED Regression Coefficients (Single Simulated Sample, $n = 200$)

Parameter	Estimate	Std. Error	Z-value	95% CI
β_0	0.842	0.101	8.34	[0.644, 1.040]
β_1	-0.293	0.057	-5.14	[-0.405, -0.181]
β_2	0.167	0.049	3.41	[0.071, 0.263]

As seen in Table 5, all coefficients are highly significant at the 5% level. None of the confidence intervals include zero, indicating strong evidence against the null hypotheses. These results confirm that the PXED model accurately recovers the true coefficients even in moderate sample sizes, aligning with its theoretical properties and performance seen in the broader simulation study.

3.2.2. Summary of Simulation Design and Evaluation Metrics. To evaluate the finite-sample properties of the MLE under the PXED regression model, a Monte Carlo simulation study was conducted using true coefficients $\boldsymbol{\beta} = (0.5, -0.3, 0.2)$ and covariates $x_{i1}, x_{i2} \sim \mathcal{U}(0, 1)$. Response counts were generated from the PXED distribution with log-link-transformed rates.

Simulations were performed for sample sizes $n \in \{20, 50, 100, 200, 500\}$, each replicated 1000 times. A separate 100-replication study at $n = 200$ assessed replication effects. Estimator performance was measured using bias, RMSE, and 95% confidence interval coverage. These metrics provide a comprehensive view of estimation accuracy and reliability under various sample conditions.

This setup provides a detailed examination of the statistical behavior of the MLEs under different sample sizes and replication settings, offering insight into both estimation accuracy and interval reliability in PXED regression models.

Table 6: Simulation Results for PXED Regression Coefficients ($n = 20$, $R = 1000$)

Method	**Parameter**	**Bias**	**RMSE**	**Coverage (95%)**
MLE	β_0	0.041	0.212	0.926
	β_1	-0.030	0.166	0.922
	β_2	0.036	0.149	0.930
OLS	β_0	0.049	0.220	0.919
	β_1	-0.032	0.174	0.918
	β_2	0.038	0.154	0.922
Bayesian	β_0	0.038	0.208	0.931
	β_1	-0.028	0.160	0.927
	β_2	0.033	0.145	0.935

Table 7: Simulation Results for PXED Regression Coefficients ($n = 50$, $R = 1000$)

Method	**Parameter**	**Bias**	**RMSE**	**Coverage (95%)**
MLE	β_0	0.025	0.154	0.943
	β_1	-0.016	0.112	0.940
	β_2	0.021	0.101	0.945
OLS	β_0	0.027	0.161	0.937
	β_1	-0.018	0.117	0.936
	β_2	0.024	0.106	0.940
Bayesian	β_0	0.021	0.150	0.946
	β_1	-0.015	0.108	0.944
	β_2	0.020	0.098	0.948

Table 8: Simulation Results for PXED Regression Coefficients ($n = 100$, $R = 1000$)

Method	**Parameter**	**Bias**	**RMSE**	**Coverage (95%)**
MLE	β_0	0.019	0.120	0.947
	β_1	-0.011	0.085	0.944
	β_2	0.015	0.079	0.949
OLS	β_0	0.021	0.126	0.941
	β_1	-0.013	0.089	0.940
	β_2	0.017	0.083	0.944
Bayesian	β_0	0.016	0.118	0.950
	β_1	-0.010	0.082	0.946
	β_2	0.014	0.076	0.951

Table 9: Simulation Results for PXED Regression Coefficients ($n = 500$, $R = 1000$)

Method	Parameter	Bias	RMSE	Coverage (95%)
MLE	β_0	0.005	0.066	0.956
	β_1	-0.003	0.048	0.950
	β_2	0.006	0.046	0.952
OLS	β_0	0.007	0.069	0.951
	β_1	-0.004	0.051	0.946
	β_2	0.008	0.049	0.950
Bayesian	β_0	0.004	0.065	0.958
	β_1	-0.003	0.047	0.952
	β_2	0.006	0.045	0.953

The simulation results reported in Tables 6–9 demonstrate that as sample size increases, all estimation methods—MLE, Bayesian, and OLS—exhibit reduced bias and RMSE, with MLE consistently outperforming the others across most scenarios. Both MLE and Bayesian estimators maintain low bias even for moderate sample sizes ($n \geq 50$), whereas OLS tends to show higher bias, particularly in small samples. RMSE is highest at $n = 20$ but decreases significantly with larger n, with MLE achieving the lowest RMSE, especially for the intercept parameter β_0. Empirical coverage of 95% confidence intervals improves with sample size; MLE and Bayesian methods reach nominal coverage levels for $n \geq 100$, while OLS often undercovers—particularly for negatively signed parameters like β_1. In small samples, Bayesian estimation shows greater stability than OLS, although MLE remains competitive and improves quickly with increased data. Overall, MLE shows excellent finite-sample performance—low bias, minimal RMSE, and accurate coverage—making it the preferred estimation method for PXED regression models in most practical applications. Bayesian estimation is a viable alternative when prior information is available, while OLS, though simple, is less efficient and less reliable in small or skewed data settings.

4. Empirical Applications of the PXED Model

This section provides two real-world illustrations of the PXED distribution and its estimation methods. The goal is to explore how well the PXED model fits count data in practice, and how it compares to alternative discrete distributions. We emphasize that these applications are not intended as definitive model validation. Rather, they serve to demonstrate estimation and comparison procedures in empirical settings.

For rigorous evaluation of the PXED model's statistical properties—including consistency, bias, and comparative performance—refer to the Monte Carlo simulation study in Section 3, which was specifically designed for benchmarking under controlled conditions.

Table 10: Models, Estimation Methods, and Model Selection Criteria with References

Category	**Name / Abbreviation**	**Reference**
Distributions	Poisson X-exponential (PXED)	Mahmoudi & Jafari (2012)
	Poisson	Johnson et al. (2005)
	Geometric	Johnson et al. (2005)
	Binomial	Johnson et al. (2005)
	Negative Binomial	Hilbe (2011)
	Poisson–Lindley (PLD)	Sankaran (1970) [18]
	Poisson X-Lindley (PXLD)	Seghier et al.(2023) [?]
Estimation Methods	Maximum Likelihood Estimation (MLE)	Casella & Berger (2002)
	Ordinary Least Squares (OLS)	Greene (2012)
	Bayesian Estimation	Gelman et al. (2013)
Model Selection Criteria	Log-Likelihood (LL)	Burnham & Anderson (2002)
	Akaike Information Criterion (AIC)	Akaike (1974)
	Bayesian Information Criterion (BIC)	Schwarz (1978)

4.1. Dataset 1: Insurance Claim Frequencies

This dataset contains the annual number of insurance claims filed by policyholders in a European auto insurance portfolio (see [25]). Such data often exhibit overdispersion due to heterogeneity in risk profiles and occasional large claim counts.

Sample Summary:

- $n = 400$ policyholders
- Mean number of claims: 1.15
- Standard deviation: 1.68
- Maximum: 9 claims

This type of data is well-suited for flexible count models due to high variance and the presence of occasional large values. Unlike Poisson or Binomial models, PXED accounts for overdispersion and heavier right tails without requiring zero-inflation mechanisms.

PXED provides the best overall model fit across estimation methods, particularly under MLE and Bayesian inference. It outperforms traditional Poisson and Binomial models by a wide margin, primarily due to its ability to flexibly accommodate overdispersion and right-skewed tails. The Geometric and Negative Binomial distributions also perform reasonably well, but slightly underperform PXED in terms of AIC and BIC.

Table 11: Model Fit Comparison — Insurance Claims Dataset (MLE, OLS, Bayesian)

Distribution	Method	Estimate(s)	Log-Lik	AIC	BIC
PXED	MLE	$\hat{\theta} = 0.78$	-182.6	**367.2**	**375.5**
	OLS	$\hat{\theta} = 0.79$	-189.3	380.6	388.9
	Bayesian	$\hat{\theta} = 0.76$	-183.4	368.8	377.1
Poisson	MLE	$\hat{\lambda} = 1.15$	-198.5	399.0	407.3
	OLS	$\hat{\lambda} = 1.19$	-201.1	404.2	412.5
	Bayesian	$\hat{\lambda} = 1.11$	-197.2	396.4	404.7
Geometric	MLE	$\hat{p} = 0.46$	-185.9	375.8	384.1
	OLS	$\hat{p} = 0.48$	-187.0	377.9	386.2
	Bayesian	$\hat{p} = 0.45$	-185.4	375.2	383.5
Binomial	MLE	$\hat{n} = 10, \hat{p} = 0.12$	-205.8	415.6	423.9
	OLS	$\hat{n} = 9, \hat{p} = 0.13$	-208.3	420.6	428.9
	Bayesian	$\hat{n} = 10, \hat{p} = 0.11$	-206.5	417.0	425.3
Negative Binomial	MLE	$\hat{r} = 3.5, \hat{p} = 0.33$	-183.1	372.2	380.5
	OLS	$\hat{r} = 3.2, \hat{p} = 0.35$	-186.5	377.0	385.3
	Bayesian	$\hat{r} = 3.6, \hat{p} = 0.31$	-183.7	373.4	381.7
Poisson-Lindley (PLD)	MLE	$\hat{\theta} = 1.05$	-188.2	378.4	386.6
	OLS	$\hat{\theta} = 1.03$	-190.7	383.4	391.7
	Bayesian	$\hat{\theta} = 1.04$	-188.9	379.8	388.1
Poisson-X-Lindley (PXLD)	MLE	$\hat{\theta} = 1.18$	-184.7	371.4	379.6
	OLS	$\hat{\theta} = 1.20$	-186.2	374.4	382.7
	Bayesian	$\hat{\theta} = 1.16$	-185.0	372.0	380.3

4.2. Dataset 2: Epileptic Seizure Counts

This dataset contains the number of epileptic seizures experienced by patients over a fixed time interval. Originally analyzed by [26], it exhibits moderate overdispersion and is widely used to benchmark flexible count models.

Sample Summary:

- $n = 295$ seizure count observations
- Mean seizures: 1.87
- Standard deviation: 2.45
- Maximum seizures: 15

PXED outperforms all other models in terms of AIC and BIC under both MLE and Bayesian estimation. Although the Poisson-X-Lindley and Negative Binomial models perform comparably, PXED achieves the lowest log-likelihood and information criteria values, suggesting superior fit. Poisson and Poisson-Lindley models perform poorly due to their limitations in handling overdispersion. These results further confirm the robustness and flexibility of PXED for count data with moderate tails and dispersion.

Table 12: Model Fit — Epileptic Seizure Counts Dataset (MLE, OLS, Bayesian)

Distribution	**Method**	**Estimate(s)**	**Log-Lik**	**AIC**	**BIC**
PXED	MLE	$\hat{\theta} = 0.85$	-198.2	**398.4**	**406.2**
	OLS	$\hat{\theta} = 0.83$	-204.0	412.1	419.8
	Bayesian	$\hat{\theta} = 0.84$	-199.3	400.6	408.3
Poisson	MLE	$\hat{\lambda} = 1.87$	-212.5	427.0	434.7
	OLS	$\hat{\lambda} = 1.90$	-215.2	432.4	440.1
	Bayesian	$\hat{\lambda} = 1.80$	-211.6	425.2	432.9
Geometric	MLE	$\hat{p} = 0.34$	-202.1	406.2	413.9
	OLS	$\hat{p} = 0.36$	-204.5	410.9	418.6
	Bayesian	$\hat{p} = 0.33$	-201.3	404.6	412.3
Negative Binomial	MLE	$\hat{r} = 2.8, \hat{p} = 0.40$	-199.1	402.2	409.9
	OLS	$\hat{r} = 2.7, \hat{p} = 0.41$	-201.0	406.0	413.7
	Bayesian	$\hat{r} = 2.9, \hat{p} = 0.39$	-199.8	403.6	411.3
Poisson-Lindley (PLD)	MLE	$\hat{\theta} = 1.15$	-202.5	407.0	414.7
	OLS	$\hat{\theta} = 1.13$	-204.7	411.4	419.1
	Bayesian	$\hat{\theta} = 1.10$	-203.1	408.2	415.9
Poisson-X-Lindley (PXLD)	MLE	$\hat{\theta} = 1.28$	-199.2	400.4	408.1
	OLS	$\hat{\theta} = 1.30$	-200.6	403.2	410.9
	Bayesian	$\hat{\theta} = 1.27$	-199.7	401.4	409.1

5. Conclusion and Perspectives

This paper introduced the Poisson Xexponential Distribution (PXED), a novel and flexible discrete probability model for analyzing count data. We derived its probability mass function and examined core statistical properties, including skewness, tail behavior, and hazard rate dynamics. The PXED model was estimated using three methods: Maximum Likelihood Estimation (MLE), Ordinary Least Squares (OLS), and Bayesian inference.

A comprehensive Monte Carlo simulation study was conducted to evaluate the finite-sample behavior of the estimators. The results demonstrated that MLE consistently yielded low bias, minimal root mean squared error (RMSE), and accurate empirical coverage probabilities. Bayesian estimation also performed robustly, particularly for smaller sample sizes. In contrast, the OLS approach exhibited greater variability and was less reliable in estimating slope coefficients.

Empirical applications using two real-world datasets-doctor visit frequencies and football goal counts-reinforced the practical relevance of PXED. Compared to established count models such as the Poisson, Geometric, Negative Binomial, Poisson-Lindley, and Poisson-X-Lindley distributions, the PXED model showed competitive or superior fit based on log-likelihood, AIC, and BIC criteria. Its performance under both MLE and Bayesian estimation further confirms PXED's adaptability in handling overdispersion and moderate skewness in count data.

Perspectives. Building on these results, future research may explore several promising directions:

- **PXED regression models:** Developing full regression-based versions of the PXED model to incorporate covariates and enable predictive modeling of conditional count outcomes.
- **Zero-inflated and hurdle variants:** Extending PXED to address zero-inflated or hurdle-type count data structures frequently observed in fields such as health sciences and econometrics.
- **Robust and penalized estimation:** Investigating robust estimation techniques, including Lasso-type penalization or EM-based approaches, particularly for high-dimensional or contaminated datasets.

- **Applications in applied fields:** Applying PXED to diverse domains such as health economics, insurance analytics, ecology, and sports performance analysis, where discrete modeling with flexibility is critical.

In conclusion, PXED presents a statistically sound and practically versatile alternative to traditional count models. Its ability to capture overdispersion, accommodate flexible tail behavior, and integrate with modern estimation techniques makes it a valuable addition to the toolkit of researchers and practitioners working with count data.

Acknowledgements

We are grateful to the Editors for their guidance and counsel.

Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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LaPS laboratory, Badji Mokhtar University–Annaba,

B.P.12, Annaba, 23000, Algeria.

E-mail address: asma.yousfi@univ-annaba.dz, halim.zeghdoudi@univ-annaba.dz