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Well-Posedness of Mixed Fractional-Order Nonlocal Problems With Absorption Phenomena

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ABSTRACT: This paper proves the existence of entropy solutions for a generalized class of mixed fractional-order systems (M-FOS) incorporating nonlocal operators and perturbation effects under Dirichlet boundary conditions. We initiate our analysis under minimal regularity assumptions (treating $f \in L^1$ data) and employ the Minty-Browder theorem to address non-coercive settings. By imposing structural conditions on the perturbation term, we capture a broad spectrum of phenomena arising in physics, engineering, and biological systems. Using variational techniques in fractional Sobolev spaces, we establish rigorous existence results for the proposed model.

Key Words: Well-posedness solution, fractional p,q-Laplacian systems, nonlocal Sobolev spaces.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^N, (N \geq 2)$ be a smooth bounded domain. In this paper, we consider the following nonlocal mixed fractional (p, q)-Laplacian system, where we investigate the existence and uniqueness of entropy solutions :

$$\begin{cases} (-\Delta)^s_{p,q}(u - \mathcal{B}(u)) = f(x, u) & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p \leq q < \infty$ and $(-\Delta)^s_{p,q}$ is the fractional p, q-Laplacian operator with variable order, where modelise a mixed fractional-order Schrödinger-Kirchhoff equations (NLSKE) with perturbation phenomena such that refer to a class of nonlinear partial differential equations (PDEs) that combine fractional Laplacian operators, nonlocal Kirchhoff terms, and external perturbations " These equations arise in various contexts including quantum mechanics and Bose-Einstein condensates", defined as:

$$\begin{aligned} (-\Delta)^s_{p,q} u(x) \quad &= \quad P.V. \int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \\ &+ \quad a\Big(\frac{x + y}{2}\Big) \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x - y|^{N+sq}} dy. \qquad x \in \Omega. \end{aligned}$$

Assuming P.V. indicates the principal value (in the conventional sense), and $0 < s < 1$. In this problem $a : \overline{\Omega} \to \mathbb{R}$ is a Lipschitz continuous map, $a(x) \geq 0$ for all $x \in \overline{\Omega}$. The absorption function $\mathcal{B}$ is continuous, defined from $\mathbb{R}$ to $\mathbb{R}$ and satisfying specific conditions (see assumption (H_1) below), the given datum f, assumed to satisfy growth conditions (e.g., $f \in L^1$), for further details, see Hypothesis (H_2). Recently, fractional calculus " a generalization of classical differentiation and integration to non-integer (real or complex) orders" has finds extensive applications across diverse fields—including mathematics, physics,

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engineering, economics, and social sciences. This framework, pioneered by luminaries such as Abel, Caputo, Euler, Grünwald, Hadamard, Hardy, Heaviside, Jumarie, Laplace, Leibniz, Letnikov, Liouville, Riemann, Riesz, and Weyl, extends conventional operators to encompass arbitrary powers. A principal motivation for its study lies in its profound utility in addressing problems arising in fractal space-time physics, where fractional operators naturally model anomalous diffusion, memory effects, and non-local phenomena. Its capacity to describe complex systems with scaling invariance or long-range dependencies underscores its significance in both theoretical and applied realms. Closely related are fractional Sobolev spaces, which have long been a classical subject in functional and harmonic analysis. In recent years, these spaces—along with their associated nonlocal equations—have seen remarkable applications, further expanding the reach of fractional methods. Examples include: Physics and materials science: Anomalous diffusion [[1], [2], [3]], crystal dislocation [[4], [5], [6]], stratified materials [[7], [8], [9]], and soft thin films [10]. Fluid dynamics and wave phenomena: Quasigeostrophic flows [[11], [12], [13]], water waves [[14], [15], [16], [17] etc.], and flame propagation [18]. Optimization and geometry: Minimal surfaces [[19], [20]], gradient potential theory [21], and phase transitions [[22], [23], [24], [25], [26]]. Interdisciplinary applications: Finance [27], semipermeable membranes [18], and the ultra-relativistic limits of quantum mechanics [28]. The synergy between fractional calculus and fractional Sobolev spaces highlights their shared role in modeling non-local interactions and irregular domains, from fractal geometries to discontinuous processes. For further context, foundational insights can be found in [[29], [30]], while broader theoretical treatments appear in [[31], [32]]. Fractional calculus extends classical calculus by incorporating non-commutative derivatives, which demonstrate considerable reliability when applied within non-commutative geometry. This advancement naturally prompts a generalization of information theory to fractional orders. The foundational works of Oldham and Spanier [33], Srivastava and Owa [34], Oustaloup [35], Miller and Ross [36], Samko et al. [37], Kiryakova [38], Mainardi [39], Podlubny [40], Hilfer [41], Zaslavsky [42], Kilbas et al. [43], Magin [44], Sabatier et al. [45], Hilfer [46], Mainardi [47], Monje et al. [48], Klafter et al. [49], Tarasov [50], Baleanu et al. [51], Yang [52], Jumarie [[53], [54], [55], [56]], among others, have significantly contributed to various applied sciences. Despite these developments, certain unresolved mathematical challenges persist, complicating progress in the field. While many key issues have been identified, practical formulations tailored to specific applications remain scarce. Research in this area continues to evolve [[33]-–[55]]. A major advantage of employing fractional differential equations (both ordinary and partial) in scientific modeling lies in their nonlocal nature. Unlike classical derivatives, which are local and linear operators, fractional derivatives are inherently nonlocal and nonlinear. Consequently, the behavior of a system described by fractional calculus depends not only on its immediate state but also on its entire history, offering a more comprehensive framework for modeling complex phenomena. The core of our approach relies on the Minty-Browder surjectivity theorem. First, we construct a sequence of weak solutions to problem (**??**). Using a priori estimates and Fatou's Lemma, we then establish the existence of at least one entropy solution. Finally, the uniqueness of the solution to (**??**) is proven by applying a fundamental Lemma.

2. Background Materials

This section is devoted to the definition of the fractional Sobolev spaces, which generalize the classical Sobolev spaces to non-integer orders of differentiation. These spaces are essential in the analysis of partial differential equations (PDEs) with fractional derivatives, as well as in applications where solutions exhibit irregular or non-local behavior. To establish their functional framework, we first report a quick tour of some of the useful basic definitions and properties of fractional calculus, including the Riemann-Liouville and Caputo fractional derivatives, along with their key operational properties. A solid grasp of these concepts is crucial for understanding both the theoretical structure and practical applications of fractional Sobolev spaces.

Definition 2.1 *Let :*

(i) Ω *be a bounded open domain in* $\mathbb{R}^N, (N \geq 2)$.

(ii) $a : \overline{\Omega} \longrightarrow [0, \infty)$ *is Lipschitz continuous function.*

(iii) γ *be function defined as follow*

$$\begin{array}{rcl}\gamma:\Omega\times\Omega\times[0,\infty) & \longrightarrow & [0,\infty)\\ ((x,y),t) & \longmapsto & \gamma((x,y);t):=t^p+a\Big(\frac{x+y}{2}\Big)t^q.\end{array}$$

(iv) A fractional order $s\in(0,1)$*, and* $u\in\mathcal{M}(\Omega,\mathbb{R})$.

The Gagliardo-type modular for a measurable function u *is defined as*

$$\begin{aligned}\rho_{s,\gamma}(u) &:= \int_{\Omega\times\Omega}\gamma\Big((x,y);\frac{|u(x)-u(y)|}{|x-y|^{s+N/.}}\Big)dydx.\\ &= \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}+a\Big(\frac{x+y}{2}\Big)\frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}dydx.\end{aligned}$$

where

- $1<p<q.$
- $\frac{q}{p}<1+\frac{1}{N}.$

The fractional Musielak-Orlicz space $L^\gamma(\Omega)$ is defined by

$$L^{s,\gamma}(\Omega)=\{u\in\mathcal{M}(\Omega,\mathbb{R}):\rho_{s,\gamma}(u)<+\infty\}.$$

The corresponding seminorm is given by the Luxemburg norm

$$[u]_{s,\gamma}=\inf\Big\{\lambda>0:\ \rho_{s,\gamma}\left(\frac{u}{\lambda}\right)\le 1\Big\}.$$

The corresponding Fractional Sobolev Space $W^{s,\gamma}(\Omega)$ of $L^\gamma(\Omega)$ is defined by

$$W^{s,\gamma}(\Omega)=\big\{u\in L^\gamma(\Omega):\ \nabla u\in L^{s,\gamma}(\Omega)^N\big\},$$

equipped with the norm

$$\|u\|_{s,\gamma}=[u]_{s,\gamma}+[u]_{s,\gamma,N}\quad\text{for all } u\in L^{s,\gamma}(\Omega),$$

such that

$$[u]_{s,\gamma,N}=\inf\left\{\lambda>0:\ \sum_{i=1}^{N}\rho_{s,\gamma}\left(\frac{u_i}{\lambda}\right)\le 1\right\}.$$

Now, we define the space $W_0^{s,\gamma}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in space $W^{s,\gamma}(\Omega)$

$$i.e: W_0^{s,\gamma}(\Omega)=\overline{C_0^\infty(\Omega)}^{W^{s,\gamma}(\Omega)}.$$

Or equivalently:

$$W_0^{s,p}(\Omega):=\{u\in W^{s,p}(\Omega):u=0\text{ a.e. in }\partial\Omega\}.$$

with respect the norm $\|.\|_{s,\gamma}$.

Proposition 2.1 (see [1]) *The both spaces* $(W^{s,\gamma}(\Omega),\|.\|_{s,\gamma})$ *and* $(W_0^{s,\gamma}(\Omega),\|.\|_{s,\gamma})$ *are uniformly convex and so, reflexive Banach spaces.*

Proposition 2.2 (see [1]) *Let* p,q *two real numbers such that* $1<p<q<N$*, then we have:*

(i) $W_0^{s,\gamma}(\Omega)\hookrightarrow L^r(\Omega)$. *is compact embedding whenever* $r<p^*$*, with* p^* *given by*

$$p^*:=\frac{Np}{N-p}.$$

(ii) $L^{s,\gamma}(\Omega) \hookrightarrow L^1(\Omega)$.

(iii) $L^q(\Omega) \hookrightarrow L^{s,\gamma}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_a(\Omega)$.

(iv) $\min\left\{\|u\|^p_{s,\gamma}; \|u\|^q_{s,\gamma}\right\} \leq \|u\|^p_p + \|u\|^q_{q,a} \leq \max\left\{\|u\|^p_{s,\gamma}; \|u\|^q_{s,\gamma}\right\}$.

(v) $\min\left\{\|u\|^p_{s,\gamma}; \|u\|^q_{s,\gamma}\right\} \leq \|\nabla u\|^p_p + \|\nabla u\|^q_{q,a} \leq \max\left\{\|u\|^p_{s,\gamma}; \|u\|^q_{s,\gamma}\right\}$.

Definition 2.2 (see [7]) *Let Y be a reflexive Banach space, Y^* its dual space and denote by $\langle .; . \rangle$its duality pairing. Let $B : Y \longrightarrow Y^*$, then B is called monotone if*

$$\langle Bu - Bv; u - v \rangle \geq 0 \qquad \forall\, u, v \in Y.$$

Theorem 2.1 (see [7]) *Let Y be a real, reflexive Banach space, and let $B : Y \longrightarrow Y^*$ be a bounded, hemi-continuous, coercive, pseudomonotone operator. Given $b \in Y^*$, there exists a solution to the equation $Bu - b = 0$.*

Definition 2.3 *Given a constant $k > 0$, we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as*

$$T_k(s) = \begin{cases} s & \text{if } |s| < k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases}$$

For a function $u = u(x)$ defined on Ω, we define the truncated function $T_k(u)$ as follows, for every $x \in \Omega$, the value of $(T_k u)$ at x is just $T_k(u(x))$.
By Ph. Bénilan, L. Boccardo, T. Gallouet [8] we define also the space

$$\mathcal{T}^{s,\gamma}_0(\Omega) = \{u \in \mathcal{M}(\Omega, \mathbb{R}) : \ T_k(u) \in W^{s,\gamma}_0(\Omega) \text{ for all } \ k > 0\} \quad .$$

Proposition 2.3 (see [8]) *For every $u \in \mathcal{T}^{s,\gamma}_0(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{|u|<k}$, a.e. in Ω for every $k > 0$, where χ_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $W^{s,\gamma}_0(\Omega)$, then v coincides with the standard distributional gradient of u, and we will denote it by $v = \nabla u$.*

Lemma 2.1 *For $\xi, \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have:*

(i) $\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi.(\xi - \eta)$,

(ii) $a(x)\frac{1}{p}|\xi|^p - a(x)\frac{1}{p}|\eta|^p \leq a(x)|\xi|^{p-2}\xi.(\xi - \eta)$, *for all x in $\overline{\Omega}$.*

where a dot symbol denotes the Euclidean scalar product in $\mathbb{R}^N$.

Proof: To establish the first claim (i), for $\eta \in \mathbb{R}^N$, we define the function $f : \mathbb{R}^N \to \mathbb{R}$ as

$$f(\eta) = \frac{1}{p}|\eta|^p,$$

where $1 < p < \infty$.
This function is smooth on $\mathbb{R}^N$ for $1 < p < \infty$, and its gradient is given by:

$$\nabla f(\eta) = |\eta|^{p-2}\eta,$$

where

$$\Big|\eta\Big| = \left(\sum_{i=1}^{N} \eta_i^2\right)^{\frac{1}{2}}.$$

Using the Taylor expansion of $f(\eta)$ at ξ, we have:

$$f(\eta) = f(\xi) + \nabla f(\xi).(\eta - \xi) + R.$$

Since $f(\eta) = \frac{1}{p}|\eta|^p$ is convex for $1 < p < \infty$.
This implies that the remainder term satisfies

$$R \geq 0.$$

This gives the inequality:

$$f(\eta) \geq f(\xi) + \nabla f(\xi).(\eta - \xi).$$

In other words:

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi.(\xi - \eta).$$

For the second claim (ii), it is sufficient to substitute f with f_a, where

$$f_a(\eta) = a(x)f(\eta)$$

for all (x,η) in $\overline{\Omega} \times \mathbb{R}^N$. □

Lemma 2.2 *For $a > 0, b > 0$ and $1 \leq p < \infty$, and $n \in \mathbb{N}$, we have*

(i)

$$(a + b)^p \leq 2^{p-1}(a^p + b^p).$$

(ii)

$$a^p + b^p \leq (a + b)^p.$$

Theorem 2.2 (see [15]) *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded set, and p be a real number such that $1 \leq p < \infty$. Then, for all $u \in W_0^{s,\gamma}(\Omega)$, the following inequalities hold:*

$$\|u\|_p \leq C_0\|\nabla u\|_p$$

$$\|u\|_{q,a} \leq C_1\|\nabla u\|_{q,a}$$

are satisfied where the constant C_0 depends on the exponent $p, diam(\Omega)$, and the dimension N, (respectively C_1 on the exponent $q, a, diam(\Omega)$, and the dimension N).

3. Key assumptions and proof of the central claim

This section presents the definition of an entropy solution for problem (1.1) and proves its existence through Theorem (3.1). Key assumptions on the nonlinearity f and the absorption term $\mathcal{B}$ are introduced to support the analysis. The proof is structured in two steps: First, we regularize (1.1) and construct a sequence of weak solutions for the approximate problems (3.2). Second, we derive crucial a priori estimates, enabling us to pass to the limit and establish the existence of an entropy solution to the original problem (1.1).

(H_1) $\mathcal{B}$ is a function defined from $\mathbb{R}^N$ to $\mathbb{R}$ such that $\mathcal{B}(0) = 0$ and for all real numbers x, y we have

- $\|\mathcal{B}(x) - \mathcal{B}(y)\| < \lambda_0|x - y|$.
- λ_0 is a real constant such that

$$0 < \lambda < \min\Big\{\frac{(N-p)(\text{meas}(\Omega))^{-\frac{1}{N}}}{2(N-1)p}, \frac{(N-q)(\text{meas}(\Omega))^{-\frac{1}{N}}}{2(N-1)q}\Big\}.$$

(H_2) $f \in L^1(\Omega)$.

Remark 3.1 *To simplify and clarify the discussion in this article, we assume:*

$$\begin{aligned}
\psi^u_{\mathcal{B}}(x,y) &= u(x)-u(y)-\mathcal{B}(u(x))+\mathcal{B}(u(y)) \\
&= [u(x)-\mathcal{B}(u(x))]-[u(y)-\mathcal{B}(u(y))] \\
\Psi^u_{\mathcal{B},p}(x,y) &= \frac{|\psi^u_{\mathcal{B}}(x,y)|^{p-2}\psi^u_{\mathcal{B}}(x,y)}{|x-y|^{N+sp}} \\
\Psi^u_{\mathcal{B},q,a}(x,y) &= a\Big(\frac{x+y}{2}\Big)\frac{|\psi^u_{\mathcal{B}}(x,y)|^{p-2}\psi^u_{\mathcal{B}}(x,y)}{|x-y|^{N+sq}}
\end{aligned}$$

Definition 3.1 *A function $u \in \mathcal{T}^{s,\gamma}_0(\Omega)$ is an entropy solution of degenerate elliptic problem (**??**) if and only if*

$$\begin{aligned}
&\int_\Omega\int_\Omega\left[\frac{|\psi^u_{\mathcal{B}}(x,y)|^{p-2}\psi^u_{\mathcal{B}}(x,y)}{|x-y|^{N+sp}}+a\left(\frac{x+y}{2}\right)\frac{|\psi^u_{\mathcal{B}}(x,y)|^{q-2}\psi^u_{\mathcal{B}}(x,y)}{|x-y|^{N+sq}}\right] \\
&\times\big(\nabla T_k(u-\varphi)(x)-\nabla T_k(u-\varphi)(y)\big)\,dydx \le \int_\Omega f(x,u)\,T_k(u-\varphi)(x)dx.
\end{aligned} \tag{3.1}$$

for all $k>0$, $\varphi \in W^{s,\gamma}_0(\Omega)\cap L^\infty(\Omega)$.

Theorem 3.1 *Under assumptions $(H_1)-(H_2)$, the problem (1.1) admits an entropy solution (in the sense of Definition (3.1)). In other words, there exists an element u in $\mathcal{T}^{s,\gamma}_0(\Omega)$ such that:*

$$\begin{aligned}
&\int_\Omega\int_\Omega\left[\frac{|\psi^u_{\mathcal{B}}(x,y)|^{p-2}\big(\psi^u_{\mathcal{B}}(x,y)\big)}{|x-y|^{N+sp}}+a\left(\frac{x+y}{2}\right)\frac{|\psi^u_{\mathcal{B}}(x,y)|^{q-2}\big(\psi^u_{\mathcal{B}}(x,y)\big)}{|x-y|^{N+sq}}\right] \\
&\times\big(\nabla T_k(u-\varphi)(x)-\nabla T_k(u-\varphi)(y)\big)\,dydx \le \int_\Omega f(x,u)\,T_k(u-\varphi)(x)dx.
\end{aligned}$$

for all $k>0$, $\varphi \in W^{s,\gamma}_0(\Omega)\cap L^\infty(\Omega)$.

Remark 3.2 *Further down, C_i; $i \in \{1;2;...\}$ is a positive constant and $meas\{B\}$ denotes the measure of the measurable set $B \subset \mathbb{R}^N$.*

3-1-The approximate problem

We consider the following approximate problem:

$$\begin{cases} (-\Delta)^s_{p,q}(u-\mathcal{B}(u)) = f_n & in\ \ \Omega, \\ u_n = 0 & on\ \ \partial\Omega, \end{cases} \tag{3.2}$$

where $(f_n)_{n\in\mathbb{N}}$ is a sequence of smooth functions defined by:

$$f_n = T_n(f), \quad \forall n \in \mathbb{N}^*.$$

The sequence $(f_n)_{n\in\mathbb{N}}$ satisfies the following properties:

1. $f_n \to f$ in $L^1(\Omega)$ as $n\to\infty$.
2. $|f_n| \le |f| \quad \forall n \in \mathbb{N}^*$.
3. $\|f_n\|_\infty \le \frac{\|f\|_1}{meas(\Omega)} \quad \forall n \in \mathbb{N}^*$.

Let us define the operator B_n as follows:

$$\begin{aligned}
B_n : W^{s,\gamma}_0(\Omega) &\longrightarrow (W^{s,\gamma}_0(\Omega))' \\
u &\longmapsto B_n u := B^1 u - L_n
\end{aligned}$$

where $(W^{s,\gamma}_0(\Omega))'$ is the dual space of $W^{s,\gamma}_0(\Omega)$

Remark 3.3 *It is important for the remainder of our work to note, using hypothesis* (H_2)*, that*

$$L_n \in (W_0^{s,\gamma}(\Omega))'.$$

where for $u, v \in W_0^{s,\gamma}(\Omega)$

$$\begin{aligned}
\langle B^1 u, v\rangle &= \int_\Omega\int_\Omega \left[\frac{|\psi^u_{\mathcal{B}}(x,y)|^{p-2}\big(\psi^u_{\mathcal{B}}(x,y)\big)}{|x-y|^{N+sp}} + a\left(\frac{x+y}{2}\right)\frac{|\psi^u_{\mathcal{B}}(x,y)|^{q-2}\big(\psi^u_{\mathcal{B}}(x,y)\big)}{|x-y|^{N+sq}}\right] \\
&\quad \times\big(\nabla v(x)-\nabla v(y)\big)dydx. \\
\langle L_n, v\rangle &= \int_\Omega f_n v dx \\
&= \int_\Omega T_n(f) v dx.
\end{aligned}$$

such first, we must establish that B_n is bounded, coercive, and pseudomonotone operator.

Step 1: B_n *is bounded.*

We will prove that B is bounded. Now, using propositions (2.1) and (2.2), Lemma (2.1) and hypothesis (H_1), let $u_n \in W_0^{s,\gamma}(\Omega)$ and $\varphi \in W_0^{s,\gamma}(\Omega) \cap W_0^{1,q'}(\Omega, a)$, and set:

$$\begin{aligned}
\lambda_1 &:= \frac{1}{2}\Big(\|u_n\|_p^{p-1} + \|u_n\|_{q,a}^{q-1}\Big). \\
\lambda_2 &:= \|\varphi\|_{s,\gamma}. \\
a &:= \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}}\Big(a\Big(\frac{x+y}{2}\Big)\Big).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\left|\left\langle B^1\,\frac{u_n}{\lambda_1}, \frac{\varphi}{\lambda_2}\right\rangle\right| &\leq \int_\Omega\int_\Omega\left[\Psi^{u_n/\lambda_1}_{\mathcal{B},p}(x,y) + \Psi^{\varphi/\lambda_2}_{\mathcal{B},q,a}(x,y)\right]\Big(\nabla\varphi(x)-\nabla\varphi(y)\Big)dydx \qquad (3.3) \\
&\leq \int_\Omega\int_\Omega \Psi^{u_n/\lambda_1}_{\mathcal{B},p}(x,y)\Big(\nabla\varphi(x)-\nabla\varphi(y)\Big)dydx \\
&+ \int_\Omega\int_\Omega \Psi^{\varphi/\lambda_2}_{\mathcal{B},q,a}(x,y)\Big(\nabla\varphi(x)-\nabla\varphi(y)\Big)dydx \\
&\leq 2^{p-2}\int_\Omega\int_\Omega\Big[\frac{|u_n(x)-u_n(y)|^{p-1}}{|x-y|^{N+sp}} + \frac{|\mathcal{B}^1 u_n(x)-\mathcal{B}^1 u_n(y)|^{p-1}}{|x-y|^{N+sp}}\Big] \\
&\times\ |\nabla\varphi(x)-\nabla\varphi(y)|dydx
\end{aligned}$$

$$
\begin{aligned}
+ &\quad 2^{q-2}a\int_\Omega\int_\Omega\Big[\frac{|u_n(x)-u_n(y)|^{q-1}}{|x-y|^{N+sq}}+\frac{|\mathcal{B}^1u_n(x)-\mathcal{B}^1u_n(y)|^{q-1}}{|x-y|^{N+sq}}\Big]\\
\times &\quad |\nabla\varphi(x)-\nabla\varphi(y)|dydx\\
\leq &\quad 2^{p-2}\Big(\lambda_1+1\Big)\int_\Omega\int_\Omega\frac{|u_n(x)-u_n(y)|^{p-1}}{|x-y|^{N+sp}}|\nabla\varphi(x)-\nabla\varphi(y)|dydx\\
+ &\quad 2^{q-2}a\Big(\lambda_2+1\Big)\int_\Omega\int_\Omega\frac{|u_n(x)-u_n(y)|^{q-1}}{|x-y|^{N+sq}}|\nabla\varphi(x)-\nabla\varphi(y)|dydx\\
\leq &\quad C_2\Big[\int_\Omega\int_\Omega\frac{|u_n(x)-u_n(y)|^{p-1}}{|x-y|^{N+sp}}|\nabla\varphi(x)-\nabla\varphi(y)|dydx\\
+ &\quad \int_\Omega\int_\Omega\frac{|u_n(x)-u_n(y)|^{q-1}}{|x-y|^{N+sq}}|\nabla\varphi(x)-\nabla\varphi(y)|dydx\Big]\\
\leq &\quad C_2\Big[\Big(\int_\Omega\int_\Omega\frac{|u_n(x)-u_n(y)|^{p-1}}{|x-y|^{N+sp}}dydx\Big)^{\frac{p-1}{p}}\\
\times &\quad \Big(\int_\Omega\int_\Omega|\nabla\varphi(x)-\nabla\varphi(y)|dydx\Big)^{\frac{1}{p}}\\
+ &\quad \Big(\int_\Omega\int_\Omega\frac{|u_n(x)-u_n(y)|^{q-1}}{|x-y|^{N+sq}}dydx\Big)^{\frac{q-1}{q}}\\
\times &\quad \Big(\int_\Omega\int_\Omega|\nabla\varphi(x)-\nabla\varphi(y)|dydx\Big)^{\frac{1}{q}}\Big]\\
\leq &\quad C_2\Big\|\frac{u_n}{\lambda_1}\Big\|_{s,\gamma}^{\gamma}\Big\|\frac{\varphi}{\lambda_2}\Big\|_{s,\gamma}\\
\leq &\quad C_2.
\end{aligned}
$$

where

$$C_2=\max\left\{2^{p-1}\left(1+(\lambda_0C_0)^{p-1}\right);2^{q-1}a\left(1+(\lambda_0C_1)^{q-1}\right)\right\}.$$

Now, using proposition (**??**) and hypothesis (H_2), we obtain, for $u_n\in W_0^{s,\gamma}(\Omega)$:

$$
\begin{aligned}
\int_\Omega T_n(f)u_ndx &\leq \|f_n\|_\infty\|u_n\|_1 \\
&\leq \frac{\|f_1\|_1}{meas(\Omega)}\|u_n\|_1\\
&\leq C_3\|u_n\|_{s,\gamma},
\end{aligned}
\tag{3.4}
$$

where

$$C_3=\frac{M_0\|f\|_1}{meas(\Omega)},$$

such that M_0 is the embedding constant of $L^\gamma(\Omega)$ into $L^1(\Omega)$.
Finally, using inequality (3.3) - (3.4), we deduce that B_n is bounded.
Step 2: B_n si coercive .

Now, using inequality (3.4), Lemmas (2.1) and (2.2), hypothesis (H_2), and proposition (2.2), let

$u_n \in W_0^{s,\gamma}(\Omega)$ and $\varphi \in W_0^{-s,\gamma'}(\Omega) \cap W_0^{-1,q'}(\Omega, a)$. We obtain that

$$\begin{aligned}
\langle B_n u_n, u_n\rangle &= \int_\Omega\int_\Omega \Psi^{u_n}_{\mathcal{B},p}(x,y)dydx + \int_\Omega\int_\Omega \Psi^{u_n}_{\mathcal{B},q,a}(x,y)dydx - \int_\Omega T_n(f)u_n dx \qquad (3.5)\\
&\geq 1/p\int_\Omega\int_\Omega \Big[1/2^{p-1}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} - 2\frac{|\mathcal{B}u_n(x)-\mathcal{B}u_n(y)|^p}{|x-y|^{N+sp}}\Big]dydx\\
&+ 1/q\int_\Omega\int_\Omega a\Big(\frac{x+y}{2}\Big)\Big[1/2^{q-1}\frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}} - 2\frac{|\mathcal{B}u_n(x)-\mathcal{B}u_n(y)|^q}{|x-y|^{N+sp}}\Big]dydx\\
&- C_3\|u_n\|_{s,\gamma}\\
&\geq 1/p\int_\Omega\int_\Omega \Big[1/2^{p-1}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} - 2\lambda_0^p\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\Big]dydx\\
&+ 1/q\int_\Omega\int_\Omega a\Big(\frac{x+y}{2}\Big)\Big[1/2^{q-1}\frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}} - 2\lambda_0^q\frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}}\Big]dydx\\
&- C_3\|u_n\|_{s,\gamma}\\
&\geq 1/p\Big(1/2^{p-1}-2\lambda_0^p\Big)\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}dydx\\
&+ 1/q\Big(1/2^{q-1}-2\lambda_0^q\Big)\int_\Omega\int_\Omega a\Big(\frac{x+y}{2}\Big)\frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}}dydx - C_3\|u_n\|_{s,\gamma}\\
&\geq C_6\rho_{s,\gamma}(u) - C_3\|u_n\|_{s,\gamma}
\end{aligned}$$

where

$$\begin{aligned}
C_4 &= 1/p\left(\frac{1}{2^{p-1}} - 2(\lambda_0 C_0)^p\right),\\
C_5 &= 1/q\left(\frac{1}{2^{q-1}} - 2(\lambda_0 C_1)^q\right),\\
C_6 &= \min(C_4; C_5),\\
\|u\|^{p_1}_{s,\gamma} &= \min\{\|u\|^p_{s,\gamma}, \|u\|^q_{s,\gamma}\}.
\end{aligned}$$

This implies that

$$\frac{\langle Bu_n, u_n\rangle}{\|u_n\|_{s,\gamma}} \longrightarrow +\infty \ \text{ as } \ \|u_n\|_{s,\gamma} \longrightarrow +\infty$$

Then, we conclude that B_n is coercive. *Step 4: B_n is pseudomonotone.*

Let $\{u_n\} \subset W_0^{s,\gamma}(\Omega)$ be a sequence such that:

1. $u_n \rightharpoonup u$ weakly in $W_0^{s,\gamma}(\Omega)$.
2. $\limsup_{n\to\infty}\langle B^1 u_n; u_n - u\rangle \leq 0$.

Prove that for all $v \in W_0^{s,\gamma}(\Omega)$, we have

$$\liminf_{n\to\infty}\langle B^1 u_n; u_n - v\rangle \geq \langle B^1 u; u - v\rangle.$$

Using the proposition (2.2), more precisely The embedding $W_0^{s,\gamma}(\Omega) \hookrightarrow L^{s,\gamma}(\Omega)$ is compact, so $u_n \longrightarrow u$ strongly in $L^{s,\gamma}(\Omega)$.

Since

$$\begin{aligned}
\Psi^{u_n}_{\mathcal{B},p} &\quad \text{is bounded in} \quad L^{-s,\gamma'}(\Omega).\\
\Psi^{u_n}_{\mathcal{B},q,a} &\quad \text{is bounded in} \quad L^{-s,\gamma'}(\Omega).
\end{aligned}$$

Then, we can extract a subsequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ such that

$$\begin{array}{lcll}\Psi^{u_{\varphi(n)}}_{\mathcal{B},p} & \rightharpoonup & \Psi^{u}_{\mathcal{B},p} & \text{as} \quad n\to\infty.\\ \Psi^{u_{\varphi(n)}}_{\mathcal{B},q,a} & \rightharpoonup & \Psi^{u}_{\mathcal{B},q,a} & \text{as} \quad n\to\infty.\end{array}$$

Moreover, we have

$$\nabla u_n(x)-\nabla u_n(y)\rightharpoonup \nabla u(x)-\nabla u(y) \quad \text{in} \quad L^{\gamma}(\Omega\times\Omega).$$

Now, using the weak lower semicontinuity of the $W^{s,\gamma}(\Omega\times\Omega)-$norm, we obtain

$$\liminf_{n\to\infty}\langle B^1u_n;u_n\rangle\geq\langle B^1u;u\rangle.$$

Finally, Combining this with $\limsup\limits_{n\to\infty}\langle B^1u_n;u_n-u\rangle\leq 0$ yields:

$$\liminf_{n\to\infty}\langle B^1u_n;u_n-v\rangle\geq\langle B^1u;u-v\rangle. \tag{3.6}$$

Then, the operator B^1 is pseudomonotone. This implies that B_n is pseudomonotone. Finally, using inequalities (3.3) - (3.6) and proposition (2.1), and then by Theorem (2.1), we can ensure that there exists $u_n\in W^{s,\gamma}_0(\Omega)$ such that

$$\begin{aligned}&\int_\Omega\int_\Omega\left[\frac{|\psi^{u_n}_{\mathcal{B}}(x,y)|^{p-2}\psi^{u_n}_{\mathcal{B}}(x,y)}{|x-y|^{N+sp}}+a\left(\frac{x+y}{2}\right)\frac{|\psi^{u_n}_{\mathcal{B}}(x,y)|^{q-2}\psi^{u_n}_{\mathcal{B}}(x,y)}{|x-y|^{N+sq}}\right] \qquad (3.7)\\ &\times\big(v(x)-v(y)\big)dydx=\int_\Omega T_n(f(x,u))v(x)dx.\end{aligned}$$

for all $v\in W^{s,\gamma}_0(\Omega)$.

4- A priori estimates

We use throughout this work a priori estimates based on the proof established by Boccardo and Gallouet's see [8] and [9].

Remark 3.4 *For simplicity, we use the two Lebesgue measures defined as $d\nu_1$ and $d\nu_2$, as follows*

$$\begin{aligned}d\nu_1 &= \frac{dydx}{|x-y|^{N+sp}}.\\ d\nu_2 &= \frac{dydx}{|x-y|^{N+sq}}.\end{aligned}$$

Lemma 3.1 *Let hypotheses (H_1) and (H_2) be satisfied, then $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ is bounded in $(L^{\gamma}(\Omega))^N$.*

Proof: Let k be a positive real number and $(u_n)_{n\in\mathbb{N}}$ a sequence of $W^{s,\gamma}_0(\Omega)$ which satisfies the inequality (3.7), we pose

$$\begin{aligned}F_k(n) &= \{(x,y)\in\Omega\times\Omega \mid |u_n(x)-u_n(y)|\leq k\}.\\ \Omega_k(n) &= \{|u_n|\leq k\}.\end{aligned}$$

Now, we take the test function $v=T_k(u_n)$ in equality (3.1), then we obtain that

$$\begin{aligned}&\int_{F_k(n)}\Psi^{u}_{\mathcal{B},p}(x,y) \times \big(\nabla T_k(u_n)(x)-\nabla T_k(u_n)(y)\big)d\nu 1 \qquad (3.8)\\ &+\int_{F_k(n)}\Psi^{u}_{\mathcal{B},q,a}(x,y) \times \big(\nabla T_k(u_n)(x)-\nabla T_k(u_n)(y)\big)d\nu 2\\ &\leq \int_{\Omega_k(n)}T_n(f)T_k(u_n)dx\\ &\leq k\|f\|_1,\end{aligned}$$

So, using (3.8), Lemmas (2.1) and (2.1), and hypotheses (H_1) and (H_2), we get that

$$\begin{aligned} C_6\|T_k(u_n)\|_\gamma^p &\le \int_\Omega\int_\Omega\left[\frac{|\psi_{\mathcal{B}}^u(x,y)|^{p-2}\psi_{\mathcal{B}}^u(x,y)}{|x-y|^{N+sp}}+a\left(\frac{x+y}{2}\right)\frac{|\psi_{\mathcal{B}}^u(x,y)|^{q-2}\psi_{\mathcal{B}}^u(x,y)}{|x-y|^{N+sq}}\right] \\ &\times \big(\nabla T_k(u_n)(x)-\nabla T_k(u_n)(y)\big)dydx \\ &\le \int_\Omega T_n(f(x,u))T_k(u_n)(x)dx \\ &\le k\|f\|_1, \end{aligned} \tag{3.9}$$

then

$$\|T_k(u_n)\|_\gamma \le C_7^{1/p}, \tag{3.10}$$

where

$$C_7=\frac{k\|f\|_1}{C_6}.$$

The same manner, we can assure the existence of a positive constant C_8 such that

$$\|T_k(u_n)\|_{q,a} \le C_8^{1/q}, \tag{3.11}$$

This implies that, for any $k>0$, $(T_k(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $W_0^{s,\gamma}(\Omega)$. Then, up to a subsequence, we can assume that for any $k>0$, $T_k(u_n)\rightharpoonup v_k$ in $W^{s,\gamma}(\Omega)$. Furthermore, by compact embedding, we have $T_k(u_n)\to v_k$ in $L^\gamma(\Omega)$ and a.e. in Ω. □

Lemma 3.2 *Let hypotheses $(H_1)-(H_2)$ be satisfied, the sequence $(u_n)_{n\in\mathbb{N}}$ converges in measure to some measurable function u.*

Proof: To prove this, we can prove that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Let $k>0$ be large enough positive number.

Remark 3.5 *It is obvious to see that*

$$\{|u_n|>k\}\subseteq\{|u_n|\ge k\}=\{|T_k(u_n)|\ge k\},$$

Indeed, we have

$$\begin{aligned} \{|T_k(u_n)|\ge k\} &= \{|T_k(u_n)|\ge k;|u_n|<k\}\cup\{|T_k(u_n)|\ge k;|u_n|\ge k\} \\ &= \{|u_n|\ge k\}. \end{aligned}$$

Now, using remark (3.5) and we take for $v=T_k(u_n)$ as a test function in (**??**), we have

$$\begin{aligned} \frac{C_6}{C_0^p}k^p meas(|u_n|>k) &\le \frac{C_6}{C_0^p}k^p meas(|u_n|\ge k) \\ &\le \frac{C_6}{C_0^p}\int_\Omega |u_n|^p dx, \\ &\le C_6\int_\Omega |\nabla u_n|^p dx \\ &\le \int_\Omega \Psi_{\mathcal{B},p}^{u_n}(x,y)\big(\nabla T_k(u_n)(x)-\nabla T_k(u_n)(y)\big)d\nu_1 \\ &+ \int_\Omega \Psi_{\mathcal{B},q,a}^{u_n}(x,y)\big(\nabla T_k(u_n)(x)-\nabla T_k(u_n)(y)\big)d\nu_2 \\ &\le \int_\Omega T_n(f)T_k(u_n)dx \\ &\le k\|f\|_1 \end{aligned} \tag{3.12}$$

this implies that

$$meas(|u_n| > k) \quad \leq \quad \frac{C_9}{k^{p-1}},$$

where

$$C_9 = \frac{C_0^p \|f\|_1}{C_6}.$$

Therefore

$$meas(|u_n| > k) \to 0 \quad \text{as} \quad k \to +\infty, \quad \text{uniformly with respect to n.} \tag{3.13}$$

Now, for every fixed $t > 0$, every real positive k, for all $n, m \in \mathbb{N}$ and $\varepsilon > 0$, we can write

$$\begin{aligned} u_n &= T_k(u_n) + (u_n - T_k(u_n)). \\ u_m &= T_k(u_m) + (u_m - T_k(u_m)). \end{aligned}$$

Moreover, using the triangle inequality, we have

$$|u_n - u_m| \leq |T_k(u_n) - T_k(u_m)| + |u_n - T_k(u_n)| + |u_m - T_k(u_m)|.$$

So, if $|u_n - u_m| > t$, Then at least one of the following three terms must satisfy :

- $|T_k(u_n) - T_k(u_m)| > t/3$.
- $|u_n - T_k(u_n)| > t/3$.
- $|u_m - T_k(u_m)| > t/3$.

This implies that

$$\begin{aligned} \{|u_n - u_m| > t\} &\subseteq \{|u_n - T_k(u_n| > t/3\} \cup \{|u_m - T_k(u_m| > t/3\} \\ &\cup \{|T_k(u_n) - T_k(u_m)| > t/3\} \\ &\subseteq \{|u_n| > k + t/3\} \cup \{|u_m| > k + t/3\} \\ &\cup \{|T_k(u_n) - T_k(u_m)| > t/3\} \end{aligned} \tag{3.14}$$

Since $T_k(u_n)$ converges strongly in $L^\gamma(\Omega)$, then it is a Cauchy sequence in $L^\gamma(\Omega)$, this implies by Markov inequality that

$$\begin{aligned} meas(|T_k(u_n) - T_k(u_m)| > t/3) &\leq meas(|T_k(u_n) - T_k(u_m)|^p > t^p/3^p) \\ &\leq \int_\Omega \frac{|T_k(u_n) - T_k(u_m)|^p}{t^p/3^p} dx \\ &\leq \frac{3^p}{t^p} \int_\Omega |T_k(u_n) - T_k(u_m)|^p dx \\ &\leq \frac{\varepsilon}{3}. \end{aligned} \tag{3.15}$$

Then, using (3.13) - (3.16), we get that

$$\begin{aligned} meas\{|u_n - u_m| > t\} &\leq meas\{|u_n| > k + t/3\} + meas\{|u_m| > k + t/3\} \\ &+ meas\{|T_k(u_n) - T_k(u_m)| > t/3\} \\ &\leq \varepsilon, \end{aligned} \tag{3.16}$$

for all $n, m \geq n_0(t, \varepsilon)$.
This proves that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure and it converges almost everywhere to some

measurable function u.
Therefore

$$\begin{array}{rcll} T_k(u_n) & \rightharpoonup & T_k(u) & \text{in } W_0^{s,\gamma}(\Omega), \\ T_k(u_n) & \to & T_k(u) & \text{in } L^{s,\gamma}(\Omega), \text{ and a.e. in } \Omega, \end{array} \tag{3.17}$$

□

Lemma 3.3 *Under* $(H_1)-(H_2)$, $\nabla u_n \to \nabla u$ *in measure, where* ∇u *is the weak gradient.*

Proof: Let ε, t, k, μ are positive real numbers and let $n \in \mathbb{N}$, we have

$$\begin{array}{rcl} \{|\nabla u_n - \nabla u| > t\} & \subset & \{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\} \\ & \cup & \{|\nabla T_k(u)| > k\} \cup \{|u_n - u| > \mu\} \cup \mathcal{Q}. \end{array} \tag{3.18}$$

Indeed, we have

$$\begin{array}{rcl} \{|\nabla u_n - \nabla u| > t\} & = & \{|\nabla u_n - \nabla u| > t, |u_n| > k\} \cup \{|\nabla u_n - \nabla u| > t, |u_n| \leq k\} \\ & \subset & \{|u_n| > k\} \cup \{|\nabla u_n - \nabla u| > t, |u_n| \leq k\} \\ & \subset & \{|u_n| > k\} \cup \{|\nabla u_n - \nabla u| > t, |u_n| \leq k, |u| > k\} \\ & \cup & \{|\nabla u_n - \nabla u| > t, |u_n| \leq k, |u| \leq k\} \\ & \subset & \{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla u_n - \nabla u| > t, |u_n| \leq k, |u| \leq k\} \\ & \subset & \{|u_n| > k\} \cup \{|u| > k\} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \\ & \subset & \{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\} \cup \mathcal{G}_2 \\ & \subset & \{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\} \cup \mathcal{G}_3 \cup \mathcal{G}_4 \\ & \subset & \{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\} \cup \{|\nabla T_k(u)| > k\} \\ & \cup & \mathcal{G}_4 \\ & \subset & \{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\} \cup \{|\nabla T_k(u)| > k\} \\ & \cup & \mathcal{G}_5 \cup \mathcal{Q} \\ & \subset & \{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\} \cup \{|\nabla T_k(u)| > k\} \\ & \cup & \{|u_n - u| > \mu\} \cup \mathcal{Q}. \end{array}$$

where

$$\begin{array}{rcl} \mathcal{G}_1 & = & \{|\nabla u_n - \nabla u| > t, |u_n| \leq k, |u| \leq k, |\nabla T_k(u_n)| > k\}. \\ \mathcal{G}_2 & = & \{|\nabla u_n - \nabla u| > t, |u_n| \leq k, |u| \leq k, |\nabla T_k(u_n)| \leq k\}. \\ \mathcal{G}_3 & = & \{|\nabla u_n - \nabla u| > t, |u_n| \leq k, |u| \leq k, |\nabla T_k(u_n)| \leq k, |\nabla T_k(u)| > k\}. \\ \mathcal{G}_4 & = & \{|\nabla u_n - \nabla u| > t, |u_n| \leq k, |u| \leq k, |\nabla T_k(u_n)| \leq k, |\nabla T_k(u)| \leq k\}. \\ \mathcal{G}_5 & = & \{|\nabla u_n - \nabla u| > t, |u_n| \leq k, |u| \leq k, |\nabla T_k(u_n)| \leq k, |\nabla T_k(u)| \leq k, |u_n - u| > \mu\}. \\ \mathcal{Q} & = & \{|\nabla u_n - \nabla u| > t, |u_n| \leq k, |u| \leq k, |\nabla T_k(u_n)| \leq k, |\nabla T_k(u)| \leq k, |u_n - u| \leq \mu\}. \end{array}$$

Using (3.12) and lemma (3.2), there exists an $n_1 \in \mathbb{N}$ and for k sufficiently large we obtain that

$$meas(\{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\}) \leq \frac{\varepsilon}{4}, \tag{3.19}$$

$$meas(\{|\nabla T_k(u)| > k\} \leq \frac{\varepsilon}{4}, \tag{3.20}$$

and

$$meas(\{|u_n - u| > \mu\}) \leq \frac{\varepsilon}{4} \quad \text{for all } \ n \geq n_1. \tag{3.21}$$

Now, taking $v = T_\mu(T_{k+\mu}(u_n) - T_k(u))$ in (**??**), we get that

$$\begin{aligned}&\int_\Omega\int_\Omega \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y)(h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_1 \\ +&\int_\Omega\int_\Omega \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y)(h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_2 \\ &\leq \mu\|f\|_2.\end{aligned} \tag{3.22}$$

where

$$h_\mu^{k+\mu,k}(u_n,u)(.) := T_\mu(T_{k+\mu}(u_n) - T_k(u))(.)$$

Remark 3.6 *We can show that we have*

$$T_{k+\mu}(u_n) \rightharpoonup T_{k+\mu}(u) \qquad \textit{in } W_0^{s,\gamma}(\Omega), \tag{3.23}$$

then

$$\nabla T_\mu(T_{k+\mu}(u_n)) \rightharpoonup \nabla T_\mu(T_{k+\mu}(u)) \ \text{ in } (L^{s,\gamma}(\Omega))^N. \tag{3.24}$$

Moreover, by using (H_1), we have that

$$\mathcal{B}(T_{k+\mu}(u_n)) \to \mathcal{B}(T_{k+\mu}(u)) \ \text{ in } (L^{s,\gamma}(\Omega))^N, \tag{3.25}$$

Now, let $v \in W_0^{s,\gamma}(\Omega)$, and by using the monotonicity of the operator $\Psi_{\mathcal{B},p}(.,.)$ and $\Psi_{\mathcal{B},q,a}(.,.)$, we have

$$\begin{aligned}&\langle \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y) - \Psi_{\mathcal{B},p}^{v}(x,y); \nabla T_k(u_n) - v\rangle > 0. \\ &\langle \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y) - \Psi_{\mathcal{B},q,a}^{v}(x,y); \nabla T_k(u_n) - v\rangle > 0.\end{aligned} \tag{3.26}$$

So, using (3.23) - (3.26), we have

$$\begin{aligned}&\liminf_{n\to+\infty}\langle \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y); \nabla T_k(u_n) - v\rangle > \langle \Psi_{\mathcal{B},p}^{v}(x,y); \nabla T_k(u_n) - v\rangle. \\ &\liminf_{n\to+\infty}\langle \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y); \nabla T_k(u_n) - v\rangle > \langle \Psi_{\mathcal{B},q,a}^{v}(x,y); \nabla T_k(u_n) - v\rangle.\end{aligned}$$

Since v is arbitrary in $W_0^{s,\gamma}(\Omega)$, this implies that

$$\Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y) \rightharpoonup \Psi_{\mathcal{B},p}^{u}(x,y) \quad \text{in} \quad L^{s,\gamma'}(\Omega). \tag{3.27}$$

$$\Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y) \rightharpoonup \Psi_{\mathcal{B},q,a}^{u}(x,y) \quad \text{in} \quad L^{s,\gamma'}(\Omega). \tag{3.28}$$

We consider the following decomposition

$$\Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y).(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y)) = L_n^0 + L_n^1. \tag{3.29}$$

$$\Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y).(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y)) = K_n^0 + K_n^1. \tag{3.30}$$

Such that L_n^0 and L_n^1 are defined as follows

$$\begin{aligned}L_n^0 &= \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y).(\nabla h_\mu^{k+\mu,k}(u_n,u_n)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u_n)(y)). \\ L_n^1 &= \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y).(\nabla h_\mu^{k,k}(u_n,u)(x) - \nabla h_\mu^{k,k}(u_n,u)(y)),\end{aligned}$$

where again K_n^0 and K_n^1 are defined as follows

$$\begin{aligned}K_n^0 &= \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y).(\nabla h_\mu^{k+\mu,k}(u_n,u_n)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u_n)(y)). \\ K_n^1 &= \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y).(\nabla h_\mu^{k,k}(u_n,u)(x) - \nabla h_\mu^{k,k}(u_n,u)(y)),\end{aligned}$$

• *For L_n^0 and K_n^0.*
By the truncation property of T_k we can prove that $\nabla h_\mu^{k+\mu,k}(u_n,u)$ is bounded in $L^\gamma(\Omega)$.
Moreover, using (**??**), we have

$$\lim_{n\to+\infty} L_n^0 = \int_\Omega\int_\Omega \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u)}(x,y).(\nabla h_\mu^{k+\mu,k}(u,u)(x) - \nabla h_\mu^{k+\mu,k}(u,u)(y))d\nu_1. \tag{3.31}$$

$$\lim_{n\to+\infty} K_n^0 = \int_\Omega\int_\Omega \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u)}(x,y).(\nabla h_\mu^{k+\mu,k}(u,u)(x) - \nabla h_\mu^{k+\mu,k}(u,u)(y))d\nu_2. \tag{3.32}$$

• *For L_n^1 and K_n^1*
Similarly, as stated in the remark (**??**), it can be proven that

$$\nabla T_\mu(T_k(u_n) - T_k(u)) \to 0 \quad \text{in} \quad L^\gamma(\Omega).$$

Furthermore, using (**??**), we have

$$\lim_{n\to+\infty} L_n^1 = 0. \tag{3.33}$$

$$\lim_{n\to+\infty} K_n^1 = 0. \tag{3.34}$$

Finally, using (3.29) - (3.33), we get that

$$\begin{aligned}
&\lim_{n\to+\infty}\int_\Omega\int_\Omega \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y).(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_1 + \\
&\lim_{n\to+\infty}\int_\Omega\int_\Omega \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y).(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_2 = \\
&\int_\Omega\int_\Omega \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u)}(x,y).(\nabla h_\mu^{k+\mu,k}(u,u)(x) - \nabla h_\mu^{k+\mu,k}(u,u)(y))d\nu_1 + \\
&\int_\Omega\int_\Omega \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u)}(x,y).(\nabla h_\mu^{k+\mu,k}(u,u)(x) - \nabla h_\mu^{k+\mu,k}(u,u)(y))d\nu_2
\end{aligned} \tag{3.35}$$

Let $v = T_K(u)$ be a test function. Using the monotonicity of $\Psi_{\mathcal{B},p}(.,.)$ and $\Psi_{\mathcal{B},q,a}(.,.)$ again, we have

$$\langle \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y) - \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u)}(x,y); (\nabla h_\mu^{k+\mu,k}(u,u)(x) - \nabla h_\mu^{k+\mu,k}(u,u)(y)) > 0. \tag{3.36}$$

$$\langle \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y) - \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u)}(x,y); (\nabla h_\mu^{k+\mu,k}(u,u)(x) - \nabla h_\mu^{k+\mu,k}(u,u)(y)) > 0. \tag{3.37}$$

Now, we consider the following sets

$$\begin{aligned}
\Omega_\mu^0 &= \{(T_{k+\mu}(u_n) - T_k(u))(x) - (T_{k+\mu}(u_n) - T_k(u))(y) \le \mu \mid (x,y) \in \Omega\times\Omega\}. \\
\Omega_\mu^1 &= \{(T_{k+\mu}(u_n) - T_k(u))(x) - (T_{k+\mu}(u_n) - T_k(u))(y) > \mu \mid (x,y) \in \Omega\times\Omega\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_\Omega \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_1 + \\
&\int_\Omega \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_2 = \\
&\int_{\Omega_\mu^0} \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_1 + \\
&\int_{\Omega_\mu^0} \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_2 + \\
&\int_{\Omega_\mu^1} \Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_1 + \\
&\int_{\Omega_\mu^1} \Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_2.
\end{aligned} \tag{3.38}$$

•• *For the set* Ω_μ^0.
We can prove that

- $\nabla T_\mu(T_{k+\mu}(u_n) - T_k(u))$ is bounded.
- There exist a set F_μ such that $supp(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y)) \subseteq F_\mu$ where

$$\lim_{\mu\to 0} F_\mu = 0.$$

Then, by using (3.36), we have

$$\lim_{\mu\to 0}\int_{\Omega_\mu^0}\int_{\Omega_\mu^0}\Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_1 + \int_{\Omega_\mu^0}\int_{\Omega_\mu^0}\Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_2 = 0. \tag{3.39}$$

•• *For the set* Ω_μ^1.
We can remark that

$$\nabla T_\mu(T_{k+\mu}(u_n) - T_k(u)) = 0.$$

Therefore, by using (**??**), we have

$$\lim_{\mu\to 0}\int_{\Omega_\mu^1}\int_{\Omega_\mu^1}\Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_1 + \int_{\Omega_\mu^1}\int_{\Omega_\mu^1}\Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_2 = 0. \tag{3.40}$$

Finally, using (**??**) - (**??**), we conclude that

$$\lim_{\mu\to 0}\int_{\Omega}\int_{\Omega}\Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_1 + \int_{\Omega}\int_{\Omega}\Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_2 = 0. \tag{3.41}$$

Let δ be a strictly positive number such that $\delta < \frac{\varepsilon}{8}$, there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, we have

$$-\int_{\Omega_\mu^0}\int_{\Omega_\mu^0}\Psi_{\mathcal{B},p}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_1 - \int_{\Omega_\mu^0}\int_{\Omega_\mu^0}\Psi_{\mathcal{B},q,a}^{\nabla T_{k+\mu}(u_n)}(x,y)(\nabla h_\mu^{k+\mu,k}(u_n,u)(x) - \nabla h_\mu^{k+\mu,k}(u_n,u)(y))d\nu_2 \leq \frac{\delta}{2}. \tag{3.42}$$

Now, the maps $\Psi_{\mathcal{B},p}(.,.)$ and $\Psi_{\mathcal{B},q,a}(.,.)$

$$\mathcal{T}_1 : (s,\xi_1,\xi_2) \to (\Psi_{\mathcal{B},p}^{\xi_1}(x,y) - \Psi_{\mathcal{B},p}^{\xi_2}(x,y)).(\xi_1-\xi_2). \tag{3.43}$$

$$\mathcal{T}_2 : (s,\xi_1,\xi_2) \to (\Psi_{\mathcal{B},q,a}^{\xi_1}(x,y) - \Psi_{\mathcal{B},q,a}^{\xi_2}(x,y)).(\xi_1-\xi_2). \tag{3.44}$$

are continuous and the set

$$\mathcal{Q} := \{(s,\xi_1,\xi_2) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N : |s| \leq k, |\xi_1| \leq k, |\xi_2| \leq k, |\xi_1-\xi_2| > t\} \tag{3.45}$$

is a compact Moreover, we have

$$(\Psi_{\mathcal{B},p}^{\xi_1}(x,y) - \Psi_{\mathcal{B},p}^{\xi_2}(x,y)).(\xi_1-\xi_2) > 0 \qquad \text{for all} \quad \xi_1 \neq \xi_2 \tag{3.46}$$

$$(\Psi_{\mathcal{B},q,a}^{\xi_1}(x,y) - \Psi_{\mathcal{B},q,a}^{\xi_2}(x,y)).(\xi_1-\xi_2) > 0 \qquad \text{for all} \quad \xi_1 \neq \xi_2 \tag{3.47}$$

Then, the maps $\mathcal{T}_1$ and $\mathcal{T}_2$ attain their minima on $\mathcal{Q}$, we shall note it by α, we have easily that $\alpha > 0$. Then using (3.2), (3.43) - (3.46), we conclude that

$$\begin{aligned} meas(\mathcal{Q}) &= \frac{1}{\alpha}\int_{\mathcal{Q}} \alpha dx \\ &\leq \int_\Omega\int_\Omega \Psi^{u_n}_{\mathcal{B},p}(x,y)(\nabla h^{k+\mu,k}_\mu(u_n,u)(x) - \nabla h^{k+\mu,k}_\mu(u_n,u)(y))d\nu_1 \\ &+ \int_\Omega\int_\Omega \Psi^{u_n}_{\mathcal{B},q,a}(x,y)(\nabla h^{k+\mu,k}_\mu(u_n,u)(x) - \nabla h^{k+\mu,k}_\mu(u_n,u)(y))d\nu_2 \\ &- \int_\Omega\int_\Omega \Psi^{u}_{\mathcal{B},p}(x,y)(\nabla h^{k+\mu,k}_\mu(u,u)(x) - \nabla h^{k+\mu,k}_\mu(u,u)(y))d\nu_1 \\ &- \int_\Omega\int_\Omega \Psi^{u}_{\mathcal{B},q,a}(x,y)(\nabla h^{k+\mu,k}_\mu(u,u)(x) - \nabla h^{k+\mu,k}_\mu(u,u)(y))d\nu_2 \\ &\leq \mu\|f\|_1 + \frac{\delta}{2} \\ &\leq \mu C_{11} + \frac{\delta}{2} \\ &\leq \delta \\ &\leq \frac{\varepsilon}{4} \end{aligned} \tag{3.48}$$

where

$$\mu < \frac{\delta}{2C_{11}}.$$

Eventually, from (3.18) - (3.20) and (3.48), we get that

$$meas\{|\nabla u_n - \nabla u| > t\} \leq \varepsilon. \tag{3.49}$$

This implies that the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges in measure to ∇u.
Now by passing to the limit we show that limit function u is an entropy solution of our problem (1.1).
Let $\varphi \in W^{s,\gamma}_0(\Omega) \cap L^\infty(\Omega)$ and we take $v = T_k(u_n - \varphi)$ in equality (3.1), we get

$$\begin{aligned} &\int_\Omega\int_\Omega \Psi^{u_n}_{\mathcal{B},p}(x,y)(\nabla T_k(u_n-\varphi)(x) - \nabla T_k(u_n-\varphi)(y))d\nu_1 \quad + \\ &\int_\Omega\int_\Omega \Psi^{u_n}_{\mathcal{B},q,a}(x,y)(\nabla T_k(u_n-\varphi)(x) - \nabla T_k(u_n-\varphi)(y))d\nu_2 \quad = \quad \int_\Omega T_n(f)T_k(u_n-\varphi)dx. \end{aligned} \tag{3.50}$$

Let $\overline{k} = k + \|\varphi\|_\infty$ and $\Omega_{n,\overline{k}} = \{|(T_{\overline{k}}(u_n)-\varphi)(x) - T_{\overline{k}}(u_n)-\varphi)(y))| < k \mid (x,y) \in \Omega\times\Omega\}$, we obtain that

$$\begin{aligned} &\int_\Omega\int_\Omega \Psi^{u_n}_{\mathcal{B},p}(x,y)(\nabla T_k(u_n-\varphi)(x) - \nabla T_k(u_n-\varphi)(y))d\nu_1 \quad + \\ &\int_\Omega\int_\Omega \Psi^{u_n}_{\mathcal{B},q,a}(x,y)(\nabla T_k(u_n-\varphi)(x) - \nabla T_k(u_n-\varphi)(y))d\nu_2 \quad = \\ &\int_\Omega\int_\Omega \Psi^{\nabla T_{\overline{k}}(u_n)}_{\mathcal{B},p}(x,y)(\nabla h^{k,\overline{k}}(u_n,\varphi)(x) - \nabla h^{k,\overline{k}}(u_n,\varphi)(y))d\nu_1 \quad + \\ &\int_\Omega\int_\Omega \Psi^{\nabla T_{\overline{k}}(u_n)}_{\mathcal{B},q,a}(x,y)(\nabla h^{k,\overline{k}}(u_n,\varphi)(x) - \nabla h^{k,\overline{k}}(u_n,\varphi)(y))d\nu_2 \quad = \\ &\int_\Omega\int_\Omega \Psi^{\nabla T_{\overline{k}}(u_n)}_{\mathcal{B},p}(x,y)(\nabla T_{\overline{k}(u_n)}(x) - \nabla T_{\overline{k}(u_n)}(y))\chi_{n,\overline{k}}d\nu_1 \quad + \\ &\int_\Omega\int_\Omega \Psi^{\nabla T_{\overline{k}}(u_n)}_{\mathcal{B},q,a}(x,y)(\nabla T_{\overline{k}(u_n)}(x) - \nabla T_{\overline{k}(u_n)}(y))\chi_{n,\overline{k}}d\nu_2 \quad - \\ &\int_\Omega\int_\Omega \Psi^{\nabla T_{\overline{k}}(u_n)}_{\mathcal{B},p}(x,y)(\nabla\varphi(x) - \nabla\varphi(y))\chi_{n,\overline{k}})d\nu_1 \quad - \\ &\int_\Omega\int_\Omega \Psi^{\nabla T_{\overline{k}}(u_n)}_{\mathcal{B},q,a}(x,y)(\nabla\varphi(x) - \nabla\varphi(y))\chi_{n,\overline{k}})d\nu_2, \end{aligned} \tag{3.51}$$

where χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^N$.
As the function $T_{\overline{k}}(u_n)$ is bounded in $W_0^{s,\gamma}(\Omega)$, then by hypothesis (H_1), $\mathcal{B}(T_{\overline{k}}(u_n))$ is also bounded in $L^\gamma(\Omega)^N$, this implies that $\Psi_{\mathcal{B},p}(.,.)$ and $\Psi_{\mathcal{B},q,a}(.,.)$ are bounded and weakly converges in $L^\gamma(\Omega)$.
However, we have

$$\begin{aligned} u_n &\rightarrow u \text{ a.e in } \Omega, \\ \nabla u_n &\rightarrow \nabla u \text{ a.e in } \Omega. \end{aligned} \tag{3.52}$$

Hence,

$$\begin{aligned} \mathcal{B}(T_{\overline{k}}(u_n)) &\rightarrow \mathcal{B}(T_{\overline{k}}(u)) \text{ a.e in } \Omega, \\ \nabla T_{\overline{k}}(u_n) &\rightarrow \nabla T_{\overline{k}}(u) \quad \text{a.e in } \Omega. \end{aligned} \tag{3.53}$$

This implies that

$$\nabla T_{\overline{k}}(u_n) - \mathcal{B}(T_{\overline{k}}(u_n)) \rightarrow \nabla T_{\overline{k}}(u) - \mathcal{B}(T_{\overline{k}}(u)) \text{ a.e in } \Omega. \tag{3.54}$$

Remark 3.7 *Let $(x, \eta) \in \overline{\Omega} \times \mathbb{R}^N$, The following functions are continuous:*

$$\begin{aligned} g_1(x) &= |\eta|^{p-2}\eta. \\ g_{2,a}(x) &= a(x)|\eta|^{q-2}\eta \end{aligned}$$

Now, using remark (3.6), (3.52) - (3.54), then we conclude that

$$\Psi_{\mathcal{B},p}^{\nabla T_{\overline{k}}(u_n)}(.,.) \rightarrow \Psi_{\mathcal{B},p}^{\nabla T_{\overline{k}}(u)}(.,.) \qquad \text{as } n \rightarrow \infty \tag{3.55}$$

$$\Psi_{\mathcal{B},q,a}^{\nabla T_{\overline{k}}(u_n)}(.,.) \rightarrow \Psi_{\mathcal{B},q,a}^{\nabla T_{\overline{k}}(u)}(.,.) \qquad \text{as } n \rightarrow \infty \tag{3.56}$$

As,

$$\nabla\varphi\chi_{\Omega_{n,\overline{k}}} \text{ converges to } \nabla\varphi\chi_{\Omega_{\overline{k}}} \text{ in } L^{\gamma'}(\Omega). \tag{3.57}$$

Then, using (3.55) - (3.57), and Dominated Convergence Theorem, we deduce that:

$$\begin{aligned} \int_\Omega\int_\Omega \Psi_{\mathcal{B},p}^{\nabla T_{\overline{k}}(u_n)}(x,y)(\nabla T_{\overline{k}(u_n)}(x) - \nabla T_{\overline{k}(u_n)}(y))\chi_{n,\overline{k}}d\nu_1 &\rightarrow \int_\Omega\int_\Omega \overline{h}_{1,k}(x,y,u)\chi_{\overline{k}}d\nu_1. \\ \int_\Omega\int_\Omega \Psi_{\mathcal{B},q,a}^{\nabla T_{\overline{k}}(u_n)}(x,y)(\nabla T_{\overline{k}(u_n)}(x) - \nabla T_{\overline{k}(u_n)}(y)\chi_{n,\overline{k}}d\nu_2 &\rightarrow \int_\Omega\int_\Omega \overline{h}_{2,k}(x,y,u)\chi_{\overline{k}}d\nu_2. \end{aligned}$$

where

$$\overline{h}_{1,k}(x,y,u) = \Psi_{\mathcal{B},p}^{\nabla T_{\overline{k}}(u)}(x,y)(\nabla T_{\overline{k}}(u)(x) - \nabla T_{\overline{k}}(u)(y)). \tag{3.58}$$

$$\overline{h}_{2,k}(x,y,u) = \Psi_{\mathcal{B},q,a}^{\nabla T_{\overline{k}}(u)}(x,y)(\nabla T_{\overline{k}}(u)(x) - \nabla T_{\overline{k}}(u)(y)). \tag{3.59}$$

Therefore, using (3.55) - (3.58) and Fatou's Lemma, we have

$$\begin{aligned} &\int_\Omega\int_\Omega (\Psi_{\mathcal{B},p}^{\nabla T_{\overline{k}}(u)}(x,y) + \Psi_{\mathcal{B},q,a}^{\nabla T_{\overline{k}}(u)}(x,y))(\nabla T_{\overline{k}}(u)(x) - \nabla T_{\overline{k}}(u)(y))\chi_{\Omega_{\overline{k}}}dydx \leq \\ &\liminf_{n\rightarrow\infty}\int_\Omega\int_\Omega (\Psi_{\mathcal{B},p}^{\nabla T_{\overline{k}}(u_n)}(x,y) + \Psi_{\mathcal{B},q,a}^{\nabla T_{\overline{k}}(u_n)}(x,y))(\nabla T_{\overline{k}}(u_n)(x) - \nabla T_{\overline{k}}(u_n)(y))\chi_{\Omega_{n,\overline{k}}}dydx \end{aligned} \tag{3.60}$$

Finally, taking limits as n goes to infinity in (3.60), then we conclude

$$\begin{aligned} \int_\Omega\int_\Omega (\Psi_{\mathcal{B},p}^{\nabla T_{\overline{k}}(u)}(x,y) + \Psi_{\mathcal{B},q,a}^{\nabla T_{\overline{k}}(u)}(x,y))(\nabla T_k(u-\varphi)(x) - \nabla T_k(u-\varphi)(y))dydx& \\ \leq \int_\Omega f T_k(u-\varphi)dx.& \end{aligned} \tag{3.61}$$

for all $k > 0$, $\varphi \in W_0^{s,\gamma}(\Omega) \cap L^\infty(\Omega)$.
This implies that our problem (1.1) has at least one entropy solution, which completes the demonstration. $\square$

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4. Bibliography

The BSPM bibliography style uses numeric labels, abbreviations from Mathematical Reviews, titles in italics, mixes upper-lower cases for book titles, lower cases for article titles, and use commas for separate fields.

References

1. R. Metzler, J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. **339** (2000) 1–77.
2. J.L. Vazquez, *Nonlinear diffusion with fractional Laplacian operators*, Proc. Abel Symp., in press.
3. A. Mellet, S. Mischler, C. Mouhot, *Fractional diffusion limit for collisional kinetic equations*, Arch. Ration. Mech. Anal. **199** (2) (2011) 493–525.
4. J.F. Toland, *The Peierls-Nabarro and Benjamin-Ono equations*, J. Funct. Anal. **145** (1) (1997) 136–150.
5. M.D.M. Gonzalez, R. Monneau, *Slow motion of particle systems as a limit of a reaction-diffusion equation with half-Laplacian in dimension one*, Discrete Contin. Dyn. Syst. A **32** (4) (2012) 1255–1286.
6. P. Biler, G. Karch, R. Monneau, *Nonlinear diffusion of dislocation density and self-similar solutions*, Comm. Math. Phys. **294** (1) (2010) 145–168.
7. O. Savin, E. Valdinoci, *Elliptic PDEs with fibered nonlinearities*, J. Geom. Anal. **19** (2) (2009) 420–432.
8. M. Chermisi, E. Valdinoci, *Fibered nonlinearities for p(x)-Laplace equations*, Adv. Calc. Var. **2** (2) (2009) 185–205.
9. M. Chermisi, E. Valdinoci, *A symmetry result for a general class of divergence form PDEs in fibered media*, Nonlinear Anal. **73** (3) (2010) 695–703.
10. M. Kurzke, *A nonlocal singular perturbation problem with periodic well potential*, ESAIM Control Optim. Calc. Var. **12** (1) (2006) 52–63.
11. A.J. Majda, E.G. Tabak, *A two-dimensional model for quasigeostrophic flow: comparison with the two-dimensional Euler flow*, Phys. D **98** (2–4) (1996) 515–522.
12. D. Cordoba, *Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation*, Ann. of Math. **148** (3) (1998) 1135–1152.
13. L. Caffarelli, A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math. **171** (3) (2010) 1903–1930.
14. J.J. Stoker, *Water Waves: The Mathematical Theory with Applications*, Pure Appl. Math., vol. IV, Interscience Publishers, New York, 1957.
15. V.E. Zakharov, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, Zh. Prikl. Mekh. Tekh. Fiz. **9** (1968) 86–94.
16. G.B. Whitham, *Linear and Nonlinear Waves*, Pure Appl. Math., Wiley-Interscience, New York, 1974.
17. W. Craig, C. Sulem, P.-L. Sulem, *Nonlinear modulation of gravity waves: a rigorous approach*, Nonlinearity **5** (2) (1992) 497–522.
18. L. Caffarelli, A. Mellet, Y. Sire, *Traveling waves for a boundary reaction-diffusion equation*, Preprint (2011).
19. L. Caffarelli, J.-M. Roquejoffre, O. Savin, *Non-local minimal surfaces*, Comm. Pure Appl. Math. **63** (2010) 1111–1144.
20. L. Caffarelli, E. Valdinoci, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations **41** (1–2) (2011) 203–240.
21. G. Mingione, *Gradient potential estimates*, J. Eur. Math. Soc. **13** (2011) 459–486.
22. G. Alberti, G. Bouchitté, P. Seppecher, *Phase transition with the line-tension effect*, Arch. Ration. Mech. Anal. **144** (1) (1998) 1–46.
23. X. Cabré, E. Cinti, *Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian*, Discrete Contin. Dyn. Syst. **28** (2010) 1179–1206.

24. Y. Sire, E. Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. **256** (6) (2009) 1842–1864.

25. A. Farina, E. Valdinoci, *Rigidity results for elliptic PDEs with uniform limits: an abstract framework with applications*, Indiana Univ. Math. J., in press.

26. A. Garroni, G. Palatucci, *A singular perturbation result with a fractional norm*, in: *Variational Problems in Material Science*, Progr. Nonlinear Differential Equations Appl., vol. 68, Birkhäuser, Basel, 2006, pp. 111–126.

27. R. Cont, P. Tankov, *Financial Modelling with Jump Processes*, Chapman and Hall/CRC Financ. Math. Ser., Chapman and Hall/CRC, Boca Raton, FL, 2004.

28. C. Fefferman, R. de la Llave, *Relativistic stability of matter. I*, Rev. Mat. Iberoam. **2** (1–2) (1986) 119–213.

29. L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, PhD thesis, University of Texas at Austin, 2005.

30. L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (1) (2007) 67–112.

31. N.S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren Math. Wiss., vol. 180, Springer-Verlag, Berlin, 1972.

32. E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser., vol. 30, Princeton University Press, Princeton, NJ, 1970.

33. R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

34. H.M. Srivastava, S. Owa, *Univalent Functions, Fractional Calculus, and Their Applications*, Ellis Horwood, Chichester, 1989.

35. A. Oustaloup, *La commande CRONE: commande robuste d'ordre non entier*, Hermes, Paris, 1991.

36. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

37. S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

38. V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman, New York, 1994.

39. F. Mainardi, *Fractional calculus*, in: *Fractals and Fractional Calculus in Continuum Mechanics*, CISM Courses and Lectures, vol. 378, Springer, Vienna, 1997, pp. 291–348.

40. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

41. R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

42. G.M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, 2005.

43. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.

44. R.L. Magin, *Fractional Calculus in Bioengineering*, Begell House, Redding, 2006.

45. J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), *Advances in Fractional Calculus*, Springer, Dordrecht, 2007.

46. R. Hilfer, *Threefold introduction to fractional derivatives*, in: *Anomalous Transport: Foundations and Applications*, Wiley-VCH, Weinheim, 2008, pp. 17–73.

47. F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, World Scientific, Singapore, 2010.

48. C.A. Monje, Y. Chen, B.M. Vinagre, D. Xue, V. Feliu-Batlle, *Fractional-Order Systems and Controls: Fundamentals and Applications*, Springer, London, 2010.

49. J. Klafter, S.C. Lim, R. Metzler (Eds.), *Fractional Dynamics: Recent Advances*, World Scientific, Singapore, 2011.

50. V.E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Berlin, 2011.

51. D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, Series on Complexity, Nonlinearity and Chaos, vol. 3, World Scientific, Singapore, 2012.

52. X.-J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science, New York, 2012.

53. G. Jumarie, *Maximum Entropy, Information Without Probability and Complex Fractals: Classical and Quantum Approach*, Fundamental Theories of Physics, vol. 112, Springer, Dordrecht, 2013.

54. J.A.T. Machado, *Entropy analysis of integer and fractional dynamical systems*, J. Appl. Nonlinear Dyn. **1** (2012) 109–112.

55. J.A.T. Machado, *Entropy analysis of fractional derivatives and their approximation*, J. Appl. Nonlinear Dyn. **1** (2012) 109–112.

56. J.A.T. Machado, *Entropy analysis of systems exhibiting negative probabilities*, Commun. Nonlinear Sci. Numer. Simul. **36** (2016) 58–64.

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