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Uniform Continuity in Fuzzy Quasi-Metric Spaces: Foundations, Extensions, and Structural Properties

John Pamba, Hope Sabao and Isaac D. Tembo

ABSTRACT: This paper explores uniform continuity in fuzzy quasi-metric spaces, which generalize classical metric spaces by allowing asymmetry and fuzziness. The authors distinguish between pointwise and uniform continuity in these spaces and extend classical results to the fuzzy setting. Key contributions include an analysis of how uniform continuity behaves under composition, inversion, and symmetrization, and how it is preserved under uniform equivalence of fuzzy quasi-metrics. A central result is an Extension Theorem, showing that any uniformly continuous function defined on a dense subspace of a complete fuzzy quasi-metric space can be uniquely extended while preserving uniform continuity. The paper also shows that uniformly continuous functions preserve fuzzy Cauchy sequences and fuzzy total boundedness, linking these to convergence and compactness in fuzzy settings. While uniform continuity is preserved under restriction, the space's asymmetry complicates inverse mappings. Examples and counterexamples are provided to illustrate the behavior of such functions across various fuzzy quasi-metric environments.

Key Words: Asymmetric topology, Extension theorem, Fuzzy sets, Fuzzy Cauchy sequences, Fuzzy quasi-metric spaces, Total boundedness, Uniform continuity.

Contents

1. Introduction

The concept of distance is central to the study of analysis and topology. Classical metric spaces, which assume symmetry, non-negativity, the triangle inequality, and the identity of indistinguishable points, provide a robust framework for investigating continuity, convergence, and completeness. However, in many real-world scenarios, the assumption of exact distances or symmetric behavior is either too restrictive or unrealistic. This has motivated the development of generalized metric structures, among which fuzzy metric and fuzzy quasi-metric spaces have received considerable attention due to their ability to model uncertainty and asymmetry.

Fuzzy logic, introduced by Zadeh [16], provides a useful alternative by allowing partial membership and linguistic variables to express uncertainty. Fuzzy metric spaces were first introduced by Kramosil and Michálek [6] as a natural extension of statistical metric spaces, replacing precise distances with a fuzzy notion of closeness represented by a membership function $M(x, y, t) \in [0, 1]$, which expresses the degree to which the distance between two points x and y is less than a given threshold $t > 0$. George and Veeramani [2] later refined the definition of fuzzy metric spaces using continuous t-norms, ensuring compatibility with topological structures and enriching the analytical framework. See [10] for more details on the concept of $t-$norms.

In these fuzzy settings, the notion of continuity takes on new forms, and classical properties must be revisited and generalized. Of particular interest is the study of *uniform continuity*, which, in classical

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analysis, is critical for ensuring that function behavior is controlled globally rather than just locally. In fuzzy metric spaces, uniform continuity ensures that for any allowable degree of closeness in the output, there exists a universal threshold in the input space that suffices across the entire domain [8].

Further generalization arises with the introduction of *fuzzy quasi-metric spaces*, which drop the symmetry requirement, allowing $M(x, y, t) \neq M(y, x, t)$. This asymmetry enables the modeling of directional or time-dependent behavior, which is especially relevant in theoretical computer science, decision-making, and domains where cause-effect or temporal orderings are significant [7,9]. The foundational properties of such spaces have been explored in various contexts, including domain theory and generalized topologies [5].

Despite the advancement of fuzzy and quasi-metric structures, the study of uniformly continuous functions between such spaces remains relatively underdeveloped compared to their classical counterparts. In particular, questions concerning preservation properties (such as of Cauchy sequences and boundedness), extension of functions, and composition remain open or only partially addressed. This work aims to systematically study uniform continuity in the context of fuzzy quasi-metric spaces.

One of the central contributions of this paper is the formulation and proof of an Extension Theorem for uniformly continuous functions defined on dense subspaces of complete fuzzy quasi-metric spaces. In classical topology, a well-known result states that a uniformly continuous function from a dense subset of a complete metric space to another complete metric space can be uniquely extended to the whole space, preserving uniform continuity [11]. We generalize this result to the setting of fuzzy quasi-metric spaces by accounting for the lack of symmetry and the presence of fuzziness in the underlying structure. Additionally, we explore the following key properties:

- Preservation of Cauchy sequences: Uniform continuity is shown to preserve fuzzy Cauchy sequences, aligning with analogous results in classical and fuzzy metric settings [1].
- Uniform continuity of restrictions and compositions: We demonstrate that restriction of a uniformly continuous function to a subset remains uniformly continuous, and that composition of uniformly continuous functions preserves uniform continuity, generalizing standard analytical results to fuzzy quasi-metric spaces.
- Preservation of fuzzy total boundedness: We prove that the image of a fuzzy totally bounded set under a uniformly continuous function remains fuzzy totally bounded, extending compactness-like preservation results in fuzzy analysis [8].
- Uniqueness of uniformly continuous extensions: By leveraging completeness and the density of subspaces, we show that uniformly continuous functions admit a unique extension that retains uniform continuity.

We also highlight differences arising due to asymmetry: while a function ψ may be uniformly continuous, its inverse ψ^{-1} may not be, even if bijective. This directional sensitivity is unique to quasi-metric and fuzzy quasi-metric frameworks and distinguishes them sharply from symmetric metric settings.

To aid intuition, the paper includes illustrative examples and counterexamples, including well-behaved functions like $\sin(x)$ on rational domains extended to real domains, and functions defined over asymmetric fuzzy metrics. These help demonstrate the boundaries and applications of our results.

2. Preliminaries

In this section, we recall essential definitions and fundamental concepts related to fuzzy quasi-metric spaces and uniform continuity.

Definition 2.1 *[15] Let X be a nonempty set and let $q : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the non-negative reals. Then q is called a quasi-metric on X if for all $x, y, z \in X$:*

(i) $q(x,y) = 0 = q(y,x)$ *implies* $x = y$,

(ii) $q(x,z) \leq q(x,y) + q(y,z)$ *whenever* $x, y, z \in X$.

We say that the pair (X, q) *is a quasi-metric space.*

Definition 2.2 *[16] Let* $\tilde{A}$ *be a fuzzy set defined on the set of real numbers* $\mathbb{R}$. *Then the membership function*

$$\mu_{\tilde{A}} \colon X \to [0,1]$$

has the following characteristics:

1. $\tilde{A}$ *is* ***convex****:*
$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$$
for all $x \in [x_1, x_2]$ *and* $\lambda \in [0,1]$.

2. $\tilde{A}$ *is* ***normal*** *if*
$$\max \mu_{\tilde{A}}(x) = 1.$$

3. $\tilde{A}$ *is* ***piecewise continuous****.*

We recall from [10] that a binary operation $\star : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if* $\star$ *satisfies the following conditions: (i)* $\star$ *is commutative and associative; (ii)* $\star$ *is continuous; (iii)* $x \star 1 = x$; *for every* $x \in [0,1]$; *(iv)* $x \star y \leq w \star z$ *whenever* $x \leq w$ *and* $y \leq z$, *with* $w, x, y, z \in [0,1]$.

Definition 2.3 *[2][3] A 3-tuple* $(X, M, \star)$ *is said to be a fuzzy metric space if* X *is an arbitrary set,* $\star$ *is a continuous* $t-$*norm and* M *is a fuzzy set on* $X^2 \times (0,\infty)$ *satisfying the following conditions;*

(i) $M(x,y,t) > O$,

(ii) $M(x,y,t) = 1$ *if and only if* $x = y$,

(iii) $M(x,y,t) = M(y,x,t)$,

(iv) $M(x,y,t) \star M(y,z,s) \leq M(x,z,t+s)$,

(v) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous,*

$x, y, z \in X$ *and* $t, s > 0$.

We now define a fuzzy quasi-metric by generalising the definition of a fuzzy metric as in [2].

Definition 2.4 *A GV-fuzzy quasi-metric on a non-empty set* X *is a pair* $(M, \star)$ *where* $\star$ *is a continuous* $t-norm$*, and* M *is a fuzzy set on* $X^2 \times (0,\infty)$ *satisfying the following conditions for all* $x, y, z \in X$ *and* $t, s > 0$*:*

(i) $M(x,y,t) > 0$;

(ii) $M(x,y,t) = M(y,x,t) = 1$ *if and only if* $x = y$;

(iii) $M(x,z,t+s) \geq M(y,z,s) \star M(y,z,t)$;

(iv) $M(x,y,-) : (0,\infty) \to [0,1]$ *is continuous.*

The triple $(X, M, \star)$ such that X is a non-empty set and $(M, \star)$ is a fuzzy quasi-metric on X is called a GV-fuzzy quasi-metric space.
In this paper, we shall refer to a GV-fuzzy quasi-metric space as just a fuzzy quasi-metric space. If $(M, \star)$ is a fuzzy quasi-metric on X,then $(M^{-1}, \star)$ is also a fuzzy quasi-metric on X, where M^{-1} is the fuzzy set in $X^2 \times (0,\infty)$ defined by $M^{-1}(x,y,t) = M(y,x,t)$. Define M^i by $M^i(x,y,t) = min\{M(x,y,t), M^{-1}(x,y,t)\}$. Then $(M^i, \star)$ is a fuzzy metric on X.

Example 2.5 *Let (X, q) be a quasi-metric space. Denote by $a \cdot b$, the usual multiplication for every $a, b \in [0, 1]$ and let M_q be the function defined on $X^2 \times (0, \infty)$ by $M_q(x, y, t) = \frac{t}{t+q(x,y)}$. Then $(X, M, \star)$ is a fuzzy quasi-metric space which is induced by q and it is called the standard fuzzy quasi-metric space of (X, q).*

Proof: *To prove this, it suffices to only show that $M(x, y, t) \neq M(y, x, t)$ if $x \neq y$. That is, $\frac{t}{t+q(x,y)} \neq \frac{t}{t+q(y,x)}$, which is trivial since by definition, $q(x, y) \neq q(y, x)$.* □

Example 2.6 *Let $X = \mathbb{R}$ and $\star$ be a t-norm such that $a \star b = a \cdot b$ for all $a, b \in [0, 1]$. Define $M(x, y, t) = e^{\frac{-2|x-y|}{t}}$, for all $t > 0$. Then $(X, M, \star)$ is a fuzzy quasi-metric space.*

Proof: *For all $x, y \in X$ and $t > 0$, (i), (ii) and (iv) are trivially true. We show that (iii) holds. Let $x, y, z \in X$ and $t, s > 0$. Then*
$M(x, y, t) \star M(y, z, s) = e^{\frac{-2|x-y|}{t}} \star e^{\frac{-2|x-y|}{s}} = e^{-2(e^{\frac{|x-y|}{t}} + e^{\frac{|x-y|}{t}})}$.
Now, from the definition of the triangle inequality for real numbers, we obtain $| x - z | \leq | x - y | + | y - z |$ for all $x, y, z \in X$. Since $t, s > 0$, it follows that

$$2\frac{|x-z|}{t+s} \leq 2\frac{(|x-y|+|y-z|)}{t+s} = 2\frac{|x-y|}{t+s} + 2\frac{|y-z|}{t+s} \leq 2\frac{|x-y|}{t} + 2\frac{|y-z|}{s} \text{ holds.}$$

i.e $\quad 2\frac{|x-z|}{t+s} \leq 2\frac{|x-y|}{t} + 2\frac{|y-z|}{s}$, *so that,* $-2\frac{|x-z|}{t+s} \geq -2\frac{|x-y|}{t} - 2\frac{|y-z|}{s}$.

Thus, $e^{-2\frac{|x-z|}{t+s}} \geq e^{-2\frac{|x-y|}{t} - 2\frac{|y-z|}{s}}$
$\implies e^{-2\frac{|x-z|}{t+s}} \geq e^{-2\frac{|x-y|}{t}} \cdot e^{-2\frac{|y-z|}{s}}$.
$\implies M(x, z, t + s) \geq M(x, y, t) \star M(y, z, s)$ *as required.* □

Note that if $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$, then $(X, M, \star)$ becomes a fuzzy metric space. Thus, this definition generalises the concept of a fuzzy metric by allowing asymmetry in the distance function. For further study on fuzzy quasi-metric space and their topological properties, see [4].

Definition 2.7 *[2] A sequence (x_n) in a fuzzy quasi-metric space $(X, M, *)$ is said to converge to a point $x \in X$ if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$M(x_n, x, \epsilon) > 1 - \lambda.$$

Definition 2.8 *[2] A sequence (x_n) in $(X, M, *)$ is called a Cauchy sequence if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$,*

$$M(x_n, x_m, \epsilon) > 1 - \lambda.$$

Definition 2.9 *[2] A fuzzy metric space and similarly a fuzzy quasi metric space is said to be complete if every Cauchy sequence is convergent.*

Definition 2.10 *Let $(X, M, *)$ be a fuzzy quasi-metric space, where*

$$M : X \times X \times (0, \infty) \to [0, 1]$$

and $*$ is a continuous t-norm. A subset $D \subseteq X$ is said to be dense *in X if for every $x \in X$, $t > 0$, and $\lambda \in (0, 1)$, there exists $d \in D$ such that*

$$M(d, x, t) > 1 - \lambda \quad \text{and} \quad M(x, d, t) > 1 - \lambda.$$

In other words, for every point $x \in X$, there exists a net (or sequence) in D that converges to x with respect to the fuzzy quasi-metric M. If $(X, M, *)$ is complete (i.e., every Cauchy sequence converges in X), and $D \subseteq X$ is dense, then D is called a dense subspace *of the complete fuzzy quasi-metric space $(X, M, *)$.*

We now present some preliminary concepts on continuity in fuzzy metric spaces.

Definition 2.11 *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is said to be uniformly continuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) < \varepsilon$ whenever $d_X(x_1, x_2) < \delta$.*

Remark 2.12 *The key difference from ordinary (pointwise) continuity is that δ depends only on ε, not on the specific points x_1, x_2.*

Theorem 2.13 (Heine–Cantor [13, Theorem 4.19]) *If X is a compact metric space and $f : X \to Y$ is continuous, then f is uniformly continuous.*

Definition 2.14 *[2] Let X be any nonempty set and $(Y, M, \star)$ be a fuzzy metric space. Then a sequence f_n, of functions from X to Y is said to converge uniformly to a function f from X to Y if given $r, t > 0$, $0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that $M(f_n, (x), f(x), t) > 1 - r$ for all $n \geq n_0$ and for all $x \in X$.*

Remark 2.15 *All uniformly continuous functions are continuous, but the converse is not true in general.*

Definition 2.16 *(Uniform Limit) Let $f_n : X \to Y$ be a sequence of continuous functions from a topological space X to a fuzzy metric space Y. If f_n converges uniformly to f then f is continuous. One can follow [2] for the proof.*

3. Main Results

In this section, we introduce and study the notions of uniform continuity for functions defined on fuzzy quasi-metric spaces as our main results. We establish fundamental properties and characterizations that generalize classical uniform continuity results to these fuzzy asymmetric frameworks. We also adapt some classical properties of uniformly continuous functions to the settings of fuzzy quasi-metric spaces. Illustrative examples are provided to amplify our results. Throughout this section, X will denote a fuzzy quasi metric space.

In fuzzy quasi-metric spaces (and in classical analysis too), uniform continuity and pointwise continuity are related but not the same. The distinction lies in how the closeness of points is controlled, and in particular, whether the control depends on the point. We give the definition of pointwise continuity and uniform continuity in the setting of fuzzy quasi metric spaces as follows:

Definition 3.1 *(Continuity at a Point) Let $\psi : X \to X$ be a function on a fuzzy quasi-metric space $(X, M, \star)$. Then ψ is continuous at a point $x_0 \in X$ if for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $\delta \in (0, 1)$ such that for all $x \in X$,*

$$M(\psi(x), \psi(x_0), t) > 1 - \varepsilon \quad \textit{whenever} \quad M(x, x_0, t) > 1 - \delta$$

In the above definition, δ depends on the point x_0 which gives local control around each individual point so that we may have different δ's for different x_0's.

Definition 3.2 *(Uniform Continuity) The function ψ is uniformly continuous on X if for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $\delta \in (0, 1)$ such that for all $x, y \in X$,*

$$M(\psi(x), \psi(y), t) > 1 - \varepsilon \quad \textit{whenever} \quad M(x, y, t) > 1 - \delta$$

Here, the δ is independent of the choice of points. One can choose one global δ that works everywhere on X, such that it gives uniform control over the function's behavior across the whole space.

Definition 3.3 *Let $(X_1, M_1, *_1)$ and $(X_2, M_2, *_2)$ be fuzzy quasi-metric spaces. A mapping $\psi : X_1 \to X_2$ is said to be* uniformly continuous *if for every $\epsilon \in (0, 1)$ and $t_\epsilon > 0$, there exist $\delta \in (0, 1)$ and $t_\delta > 0$ such that for all $x, y \in X_1$,*

$$M_2(\psi(x), \psi(y), t_\epsilon) > 1 - \epsilon \quad \textit{whenever} \quad M_1(x, y, t_\delta) > 1 - \delta.$$

Definition 3.4 *A mapping ψ from a fuzzy quasi-metric space $(X, M, \star)$ to a fuzzy quasi-metric space $(X, M, \star)$ is called a uniformly continuous self-map if for each $\epsilon \in (0,1)$ and $t > 0$, there exists $\delta \in (0,1)$ such that $M(\psi(x), \psi(y), t) > 1 - \epsilon$ whenever $M(x, y, t) > 1 - \delta$*

Example 3.5 *Let $X = [0,1]$. For $x, y \in X$ such that $x \geq y$ and for $t > 0$, define a fuzzy quasi-metric by $M(x, y, t) = \frac{t}{t+(x-y)}$. Then the function $\psi : X \to X$ defined by $\psi(x) = 2x + 1$ is a uniformly continuous map on X.*

Proof: Clearly, $M(x, y, t)$ is a fuzzy quasi-metric. Let $\epsilon \in (0,1)$ be given and $t > 0$. Suppose $M(\psi(x), \psi(y), t) > 1 - \epsilon$. Then, $\frac{t}{t+(\psi(x)-\psi(y))} > 1 - \epsilon$,

$$\implies \quad \psi(x) - \psi(y) < \tfrac{\epsilon t}{1-\epsilon}$$

$$\implies x - y < \tfrac{\epsilon t}{2(1-\epsilon)}$$

For $\epsilon \in (0,1)$, choose δ such that $\epsilon = \frac{2\delta}{1+\delta}$. Then $M(x, y, t) = \frac{t}{t+(x-y)} > \frac{t}{t+\frac{\epsilon t}{2(1-\epsilon)}} = 1 - \delta$.
i.e $M(\psi(x), \psi(y), t) > 1 - \epsilon$ whenever $M(x, y, t) > 1 - \delta$. □

Remark 3.6
Let $(X, M, \star)$ be a fuzzy quasi-metric space. Then for each $x, y \in X$ and $t > 0$, the function $M(x, y, t)$ is non-decreasing.

Lemma 3.7 *(Compare [[12] Lemma 1])*
Let $(X, M, \star)$ be a fuzzy quasi-metric space. Then

1. *$\psi : (X, M, \star) \to (X, M, \star)$ is uniformly continuous if and only if*
   $\psi : (X, M^{-1}, \star) \to (X, M^{-1}, \star)$ is uniformly continuous.
2. *If $\psi : (X, M, \star) \to (X, M, \star)$ is uniformly continuous, then*
   $\psi : (X, M^i, \star) \to (X, M^i, \star)$ is uniformly continuous. The converse does not hold in general.

Proof:

1. Let $\psi : (X, M, \star) \to (X, M, \star)$ be uniformly continuous. Let $\epsilon \in (0,1)$ be given and let $t > 0$ and $\delta \in (0,1)$.

   Suppose $M^{-1}(\psi(x), \psi(y), t) > 1 - \epsilon$.

   Then $M^{-1}(\psi(x), \psi(y), t) = M(\psi(y), \psi(x), t) > 1 - \epsilon$, so that by

   continuity of $\psi(x)$ on $(M, X, \star)$, it follows that

   $M^{-1}(x, y, t) = M(y, x, t) > 1 - \delta$.

   Thus, $M^{-1}(\psi(x), \psi(y), t) > 1 - \epsilon$ holds whenever $M^{-1}(x, y, t) > 1 - \delta$. Hence, $\psi : (X, M^{-1}, \star) \to (X, M^{-1}, \star)$ is uniformly continuous.

   Conversely, let $\psi : (X, M^{-1}, \star) \to (X, M^{-1}, \star)$ be uniformly continuous. Suppose $M(\psi(x), \psi(y), t) > 1 - \epsilon$.

   Then $M(\psi(x), \psi(y), t) = M^{-1}(\psi(y), \psi(x), t) > 1 - \epsilon$, so that by

   continuity of $\psi(x)$ on $(X, M^{-1}, \star)$, it follows that

   $M(x, y, t) = M^{-1}(y, x, t) > 1 - \delta$. Thus, $M(\psi(x), \psi(y), t) > 1 - \epsilon$ holds whenever $M(x, y, t) > 1 - \delta$. Hence, $\psi : (X, M, \star) \to (X, M, \star)$ is uniformly continuous.

2. Let $\psi(x)$ be uniformly continuous on $(X, M, \star)$.

Suppose $M^i(\psi(x), \psi(y), t) > 1-\epsilon$. Then it follows that, $M(\psi(x), \psi(y, t) > 1-\epsilon$. By uniform continuity of ψ on $(X, M, \star)$, this implies that $M(x, y, t) > 1-\delta$. Similarly, $M^{-1}(x, y, t) > 1-\delta$. This means that $min\{M(x, y, t), M^{-1}(x, y, t)\} > 1-\delta$. Hence, $M^i(x, y, t) > 1-\delta$.

Therefore, $M^i(\psi(x, \psi(y, t) > 1-\epsilon$ whenever $M^i(x, y, t) > 1-\delta$, proving that $\psi(x)$ is uniformly continuous on $(X, M^i, \star)$.

We use a counter example below to show that the converse is not true in general.

Consider a function $\psi(x) = \sqrt{x}$ for all $x \in X = (0, b)$ where $b \in \mathbb{R}^+$ and let

$$M(x,y,t) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{1}{1+\frac{(x-y)}{t}} & \text{if } x > y. \end{cases}$$

which is a fuzzy quasi-metric on X.

Then $M^i(x, y, t) = min\{M(x, y, t)\ ,\ M(y, x, t)\} = \frac{1}{1+\frac{|x-y|}{t}}$.

We show that $\psi(x)$ is uniformly continuous on $M^i(x, y, t)$.

Suppose $M^i(\psi(x), \psi(y), t) > 1-\epsilon \ \ \forall \ \ \epsilon \in (0, 1)$.

$$\text{Then } \frac{1}{1+\frac{|\sqrt{x}-\sqrt{y}|}{t}} > 1-\epsilon. \quad \Longrightarrow \quad \frac{1}{1+\frac{|\sqrt{x}-\sqrt{y}|}{t}} > 1-\epsilon$$
$$\Longrightarrow \quad (1-\epsilon)(t+ \mid \sqrt{x}-\sqrt{y} \mid) < t$$
$$\Longrightarrow \quad \mid \sqrt{x}-\sqrt{y} \mid < \frac{\epsilon t}{1-\epsilon}$$
$$\Longrightarrow \quad \mid x-y \mid < \frac{\epsilon t|\sqrt{x}+\sqrt{y}|}{1-\epsilon}.$$

Since $\sqrt{x} \leq \sqrt{b}$ and $\sqrt{y} \leq \sqrt{b}$, for all $x, y \in X$ and where b is the least upper bound of $X = (0, b)$, we have that $\mid \sqrt{x}+\sqrt{y} \mid \leq \mid \sqrt{b}+\sqrt{b} \mid = 2\sqrt{b}$.

Thus, $\mid x-y \mid < \frac{2\epsilon t\sqrt{b}}{1-\epsilon}$.

So that $\mid x-y \mid < \frac{\epsilon t M}{1-\epsilon}$.

$$\text{Now } M^i(x,y,t) = \frac{1}{1+\frac{(x-y)}{t}}$$
$$> \frac{1}{1+\frac{\frac{2\epsilon t\sqrt{b}}{1-\epsilon}}{t}}$$
$$= \frac{t(1-\epsilon)}{t(1-\epsilon)+2\epsilon t\sqrt{b}}$$

Fix $\delta \in (0, 1)$ such that $\delta = \frac{2\epsilon t\sqrt{b}}{t(1-\epsilon)+2\epsilon t\sqrt{b}}$.

Then $M^i(x, y, t) > \frac{t(1-\epsilon)}{t(1-\epsilon)+2\epsilon t\sqrt{b}} = 1 - \frac{2\epsilon t\sqrt{b}}{t(1-\epsilon)+2\epsilon t\sqrt{b}} = 1-\delta$.

i.e $M^i(\psi(x), \psi(y), t) > 1-\epsilon$ whenever $M^i(x, y, t) > 1-\delta$, so that $\psi(x)$ is uniformly continuous on $M^i(x, y, t)$. We now check uniform continuity of $\psi(x) = \sqrt{x}$ on $M(x, y, t)$.

For $n \in \mathbb{N}$, fix $y_n = n$ and set $x_n = n + \frac{1}{n}$ so that $x_n > y_n$. Fix $t = 1$.

Then $M(x_n, y_n, 1) = \frac{1}{1+(x_n-y_n)} = \frac{1}{1+\frac{1}{n}} \approx 1$ as $n \to \infty$. This implies that x_n and y_n are close in the fuzzy quasi sense as $n \to \infty$. Thus, $\exists\ \delta \in (0, 1)$ such that $M(x_n, y_n, t) > 1-\delta$.

$$\text{Now, } M(\psi(x_n), \psi(y_n), 1) = \frac{1}{1+(\sqrt{x_n}-\sqrt{y_n})}$$
$$= \frac{1}{1+\sqrt{n+\frac{1}{n}}-\sqrt{n}}$$
$$\approx \frac{1}{1+\frac{1}{2n\sqrt{n}}}.$$

But $\frac{1}{1+\frac{1}{2n\sqrt{n}}}$ is monotonically increasing and its minimum is $\frac{2}{3}$ which occurs when $n = 1$, and it slowly increases towards 1 but never reaches 1. i.e $\{\frac{1}{1+\frac{1}{2n\sqrt{n}}} : n \in \mathbb{N}\} \subset [\frac{2}{3}, 1)$. Fix $\epsilon = \frac{1}{5} \in (0,1)$. Then
$M(\psi(x_n), \psi(y_n), 1) \approx \frac{1}{1+\frac{1}{2n\sqrt{n}}} < 1 - \epsilon = 1 - \frac{1}{5} = \frac{4}{5}$. This means that

$M(x, y, t) > 1 - \delta$ while $M(\psi(x), \psi(y), t) < 1 - \epsilon$, which is a

contradiction. Hence, $\psi(x) = \sqrt{x}$ is not uniformly continuous on $M(x, y, t)$. This completes the proof.

□

Remark 3.8 *Let $X = (0, b)$, where $b \in \mathbb{R}^+$, and define the quasi metric by $q(x, y,) = (y - x)^+$ for $x, y \in X$ so that $q^s(x, y) =| y - x |$, for $x, y \in \mathbb{R}$. Then by example 3.3 in [12],*
$\psi(x) = -\sqrt{x}$ is uniformly continuous on (X, q^s) but not on (X, q). Define a fuzzy quasi- metric by $M(x, y, t) = \frac{t}{t+q(x,y)}$ and the corresponding symmetrised fuzzy metric by
$M^i(x, y, t) = \frac{t}{t+q^s(x,y)}$. We first show that $\psi : M^i(x, y, t) \to M^i(x, y, t)$ is uniformly continuous. Let $\epsilon \in (0, 1)$ and $t > 0$. Suppose, $M^i(\psi(x), \psi(y), t) > 1-\epsilon$. Then $\frac{t}{t+q(\psi(y),\psi(x)} > 1-\epsilon$, so that, $\frac{t}{t+|-\sqrt{y}+\sqrt{x}|} > 1 - \epsilon$.

$$\Longrightarrow \quad (1-\epsilon)(t+ | \sqrt{x} - \sqrt{y} |) < t$$
$$\Longrightarrow \quad t+ | \sqrt{x} - \sqrt{y} | -\epsilon t - \epsilon | \sqrt{x} - \sqrt{y} |< t$$
$$\Longrightarrow \quad | \sqrt{x} - \sqrt{y} |< \frac{\epsilon t}{1-\epsilon}$$

But $| \sqrt{x} - \sqrt{y} |= \frac{|y-x|}{|\sqrt{x}+\sqrt{y}|}$. Following the same argument as in Lemma 3.7 above, we have that $| \sqrt{y} + \sqrt{x} |< 2\sqrt{b}$, so that $| \sqrt{x} - \sqrt{y} |> \frac{|y-x|}{2\sqrt{b}}$. Thus, from $| \sqrt{y} - \sqrt{x} |< \frac{\epsilon t}{1-\epsilon}$ above, we have that $\frac{|y-x|}{2\sqrt{b}} < \frac{\epsilon t}{1-\epsilon}$, so that $| y - x |< \frac{2\epsilon t\sqrt{b}}{1-\epsilon}$.
Now, $M^i(x, y, t) = \frac{t}{t+q^s(x,y)} = \frac{t}{t+|y-x|} > \frac{t}{t+\frac{2\epsilon t\sqrt{b}}{1-\epsilon}} = \frac{t(1-\epsilon)}{t(1-\epsilon)+2\epsilon t\sqrt{b}}$
Choose $\delta = \frac{2\epsilon t\sqrt{b}}{t(1-\epsilon)+2\epsilon t\sqrt{b}}$, which clearly belongs to $(0, 1)$. Then $M^i(x, y, t) > 1 - \delta$.

Thus, $M^i(\psi(x), \psi(y), t) > 1 - \epsilon$ whenever $M^i(x, y, t) > 1 - \delta$.

Therefore, $\psi(x) = -\sqrt{x}$ is uniformly continuous on $(X, M^i, \star)$. We now investigate the

uniform continuity of $\psi(x) = -\sqrt{x}$ on $(X, M, \star)$ by considering two cases.

Case 1: *$x < y$*
For x and y very close, choose $\epsilon = \frac{\delta t}{1-\delta}$, such that $y - x < \epsilon \in (0, 1)$.
Then
$$M(\psi(x), \psi(y), t) = \frac{t}{t+q(\psi(x),\psi(y)}.$$
$$= \frac{t}{t+max\{\sqrt{x}-\sqrt{y}\ ,\ 0\}}.$$
$$= 1, \text{ since } \sqrt{x} < \sqrt{y}.$$
So that, for any ϵ as defined above, we automatically have that
$M(\psi(x), \psi(y), t) = 1 > 1 - \epsilon$.
Now,
$$M(x, y, t) = \frac{t}{t+max\{y-x\ ,\ 0\}}$$
$$= \frac{t}{t+(y-x)}.$$
$$> \frac{t}{t+\epsilon}.$$
$$= \frac{t}{t+\frac{\delta t}{1-\delta}}.$$
$$= 1 - \delta.$$
i.e $M(\psi(x), \psi(y), t) = 1 > 1 - \epsilon$ whenever $M(x, y, t) > 1 - \delta$.
Therefore, $\psi(x) = -\sqrt{x}$ is uniformly continuous on $(X, M, \star)$.

Case 2: $x \geq y$

Suppose $M(\psi(X), \psi(y), t) > 1 - \epsilon$.
Then $M(x, y, t) = \frac{t}{t+q(x,y}$.
$= \frac{t}{t+max\{y-x\ ,\ 0\}}$.
$= 1$, *since* $x \geq y$.
Thus, for all $\delta \in (0, 1)$, $M(x, y, t) = 1 > 1 - \delta$, *so that* $M(\psi(x), \psi(y), t) > 1 - \epsilon$ *whenever* $M(x, y, t) > 1 - \delta$. *Hence,* $\psi(x) = -\sqrt{x}$ *is uniformly continuous on* $(X, M, \star)$.

The function $\psi(x) = -\sqrt{x}$ *is NOT uniformly continuous on the quasi-metric space* (X, q), *on* $\mathbb{R}^+$. *However, it has been found that the function* $\psi(x) = -\sqrt{x}$ *is uniformly continuous on the fuzzy quasi-metric induced by the quasi-metric* $q(x, y)$, *defined by* $M(x, y, t) = \frac{t}{t+q(x,y)}$. *This is so because the fuzzy quasi-metric introduces additional flexibility, particularly for the small values of t, which might smooth out or hide some of the discontinuities that were present in the original quasi-metric space. There is therefore no general guarantee that a function that is non-uniformly continuous on* (X, q) *implies that it is non-uniformly continuous on* $(X, M, \star)$, *where the metric is defined by* $M(x, y, t) = \frac{t}{t+q(x,y}$. *This is a case in particular.*

Proposition 3.9 *The composition of two fuzzy quasi-uniformly continuous self-maps is also fuzzy quasi-uniformly continuous.*

Proof: Let $\psi_1(x)$ and $\psi_2(x)$ be two uniformly continuous self-maps on a fuzzy quasi-metric space $(X, M, \star)$. Then we must show that $= \psi_1 \circ\ \psi_2$ is uniformly continuous on $(X, M, \star)$.
Now, by uniform continuity of $\psi_1(x)$, for every $\epsilon \in (0, 1)$ and $t > 0$, there exists $\delta_1 \in (0, 1)$ such that for all $x, y \in X$,

$$M(\psi_1(x), \psi_1(y), t) > 1 - \epsilon \quad whenever \quad M(x, y, t) > 1 - \delta_1 \tag{3.1}$$

.
Similarly, by uniform continuity of $\psi_2(x)$, there exists $\delta \in (0, 1)$ such that for all $x, y \in X$,

$$M(\psi_2(x), \psi_2(y), t) > 1 - \delta_1 \quad whenever \quad M(x, y, t) > 1 - \delta \tag{3.2}$$

Then from (1) we have that $M(\psi_1(\psi_2(x)), \psi_1(\psi_2(y)), t) > 1 - \epsilon$ whenever $M(\psi_2(x), \psi_2(y), t) > 1 - \delta_1$, so that from (2) $M(\psi_2(x), \psi_2(y), t) > 1 - \delta_1$ whenever $M(x, y, t) > 1 - \delta$. Hence, $M(\psi_1(\psi_2(x)), \psi_1(\psi_2(y)), t) > 1 - \epsilon$ whenever $M(x, y, t) > 1 - \delta$.
Therefore, $\psi_1 \circ\ \psi_2$ is uniformly continuous on $(X, M, \star)$. □

Example 3.10 *Using the fuzzy quasi-metric on* $\mathbb{R}$, *let* $\psi(x) = 2x$ *and* $\varphi(x) = x^2$ *restricted to a bounded interval (e.g.,* $[-1, 1]$*). Both are uniformly continuous on this interval, hence* $\varphi \circ \psi$ *is uniformly continuous.*

Proposition 3.11 *Let* $q(x, y)$ *be a quasi-metric and define a fuzzy quasi-metric by* $M(x, y, t) = \frac{t}{t+q(x,y)}$. *If* $\psi : (X, q) \to (X, q)$ *is a uniformly continuous self-map, then* $\psi : (X, M, \star) \to (X, M, \star)$ *is also uniformly continuous.*

Proof: Suppose $\psi(x)$ is uniformly continuous on (X, q).Then for every $\epsilon_1 > 0)$, there exists $\delta_1 > 0$ such that for all $x, y \in X$, $q(\psi(x), \psi(y)) < \epsilon_1$ whenever $q(x, y) < \delta_1$. Now, $M(x, y, t) = \frac{t}{t+q(x,y)} \implies t = M(x, y, t)(t + q(x, y))$, so that $q(x, y) = \frac{t(1-M(x,y,t))}{M(x,y,t)}$. Thus, $\frac{t(1-M(x,y,t))}{M(x,y,t)} < \delta_1$. Rearranging this yields; $M(x, y, t)) > \frac{t}{t+\delta_1}$.
Similarly, from $q(\psi(x), \psi(y)) < \epsilon_1$ and $q(x, y) = \frac{t(1-M(x,y,t))}{M(x,y,t)}$, we have that

$$\begin{aligned} q(\psi(x), \psi(y)) &= \frac{t(1-M(\psi(x),\psi(y),t))}{M(\psi(x),\psi(y),t)} < \epsilon_1 \\ &\implies t(1 - M(\psi(x), \psi(y), t) < \epsilon_1 M(\psi(x), \psi(y), t) \\ &\implies M(\psi(x), \psi(y), t) > \frac{t}{t+\epsilon_1}. \end{aligned}$$

Thus, $M(\psi(x), \psi(y), t)) > \frac{t}{t+\epsilon_1}$ whenever $M(x, y, t)) > \frac{t}{t+\delta_1}$. Letting $\delta = \frac{\delta_1}{t+\delta_1} \in (0, 1)$ and $\epsilon = \frac{\epsilon_1}{t+\epsilon_1} \in$

$(0,1)$, we have that $M(\psi(x),\psi(y),t) > 1-\epsilon$ whenever $M(x,y,t) > 1-\delta$. Thus, $\psi : (X,M,\star) \to (X,M,\star)$ is also uniformly continuous. Hence the proof.

□

We now give the definition in a case where two fuzzy quasi-metrics induce the same uniform structure on a given space.

Definition 3.12 *Two fuzzy quasi-metrics M_1 and M_2 on a set X are uniformly equivalent if $id_X : (X,M_1,\star) \to (X,M_2,\star)$ and $id_X : (X,M_2,\star) \to (X,M_1,\star)$ are both uniformly continuous maps of fuzzy quasi-metric spaces.*

As a consequence of the definition above, we have the following proposition.

Proposition 3.13 *Let $(X,M_1,\star)$ and $(X,M_2,\star)$ be two uniformly equivalent fuzzy quasi-metric spaces. Then $\psi : (X,M_1,\star) \to (X,M_1,\star)$ is uniformly continuous if and only if $\psi : (X,M_2,\star) \to (X,M_2,\star)$ is uniformly continuous.*

Proof: Suppose $\psi : (X,M_1,\star) \to (X,M_1,\star)$ is uniformly continuous. Then for any $\epsilon_1 \in (0,1)$ and $t > 0$, $\exists\ \delta_1 \in (0,1)$ such that $M_1(\psi(x),\psi(y),t) > 1-\epsilon_1$ whenever $M_1(x,y,t) > 1-\delta_1$.
i.e

$$M_1(x,y,t) > 1-\delta_1 \implies M_1(\psi(x),\psi(y),t) > 1-\epsilon_1 \tag{3.3}$$

By uniform equivalence of $(X,M_1,\star)$ and $(X,M_2,\star)$, $\exists\ \delta_1 \in (0,1)$ such that

$$M_2(x,y,t) > 1-\delta \implies M_1(x,y,t) > 1-\delta_1 \tag{3.4}$$

and

$$M_1(x,y,t) > 1-\delta \implies M_2(x,y,t) > 1-\delta_1 \tag{3.5}$$

Using uniform equivalence again but on the image under ψ we have that

$$M_1(\psi(x),\psi(y),t) > 1-\epsilon_1 \implies M_2(\psi(x),\psi(y),t) > 1-\epsilon \tag{3.6}$$

and

$$M_2(\psi(x),\psi(y),t) > 1-\epsilon_1 \implies M_1(\psi(x),\psi(y),t) > 1-\epsilon \tag{3.7}$$

Thus, from equations (3), (4) and (6), we have that for all $x,y \in X$, if $M_2(x,y,t) > 1-\delta$, then it implies that $M_1(x,y,t) > 1-\delta_1$, which implies that $M_1(\psi(x),\psi(y),t) > 1-\epsilon_1$, and which further implies that $M_2(\psi(x),\psi(y),t) > 1-\epsilon$. i.e $M_2(x,y,t) > 1-\delta$
$\implies M_2(\psi(x),\psi(y),t) > 1-\epsilon$, so that $M_2(\psi(x),\psi(y),t) > 1-\epsilon$ whenever $M_2(x,y,t) > 1-\delta$. Hence, $\psi : (X,M_2,\star) \to (X,M_2,\star)$ is uniformly continuous.
For the converse, the proof is identical to the one above since we just swap M_1 and M_2. Therefore, $\psi : (X,M_1,\star) \to (X,M_1,\star)$ is uniformly continuous if and only if $\psi : (X,M_2,\star) \to (X,M_2,\star)$ is uniformly continuous. Hence, the proof. □

In the following proposition, we establish that every uniformly continuous function on a fuzzy quasi-metric space is necessarily continuous.

Proposition 3.14 *(Uniform continuity implies continuity) Let $(X,M,\star)$ and $(Y,M',\star')$ be fuzzy quasi-metric spaces. Suppose*

$$\psi : X \to Y$$

is uniformly continuous, i.e., for every $\varepsilon \in (0,1)$ and $t > 0$, there exists $\delta \in (0,1)$ such that for all $x,y \in X$,

$$M(x,y,t) > 1-\delta \implies M'(\psi(x),\psi(y),t) > 1-\varepsilon.$$

Then ψ is continuous at every point $x_0 \in X$.

Proof: Fix an arbitrary point $x_0 \in X$. We want to show that for every $\varepsilon \in (0,1)$ and $t > 0$, there exists $\delta_{x_0} \in (0,1)$ such that for all $x \in X$,

$$M(x, x_0, t) > 1 - \delta_{x_0} \implies M'(\psi(x), \psi(x_0), t) > 1 - \varepsilon.$$

Since ψ is uniformly continuous, for the given ε and t, there exists a $\delta \in (0,1)$ such that for all $x, y \in X$,

$$M(x, y, t) > 1 - \delta \implies M'(\psi(x), \psi(y), t) > 1 - \varepsilon.$$

Set $\delta_{x_0} := \delta$. Then for any $x \in X$ with

$$M(x, x_0, t) > 1 - \delta_{x_0},$$

we apply uniform continuity with $y = x_0$ to obtain

$$M'(\psi(x), \psi(x_0), t) > 1 - \varepsilon.$$

Therefore, ψ is continuous at x_0. Since x_0 was arbitrary, ψ is continuous on X. □

Remark 3.15 *The converse of the above proposition does* not *necessarily hold in fuzzy quasi-metric spaces. That is, continuity of a function*

$$\psi : X \to Y$$

at every point $x_0 \in X$ does not *imply that ψ is uniformly continuous. Uniform continuity requires a single $\delta > 0$ (depending only on ε and t) that works for all pairs $x, y \in X$, whereas continuity at each point x_0 only guarantees the existence of such a δ locally around x_0, which may vary with the point. Hence, while uniform continuity implies continuity, the converse implication fails in general.*

Example 3.16 *Consider the fuzzy quasi-metric space $(X, M, \star)$, where $X = (0,1)$ and M is defined by*

$$M(x, y, t) = \begin{cases} 1 & \text{if } |x - y| < t, \\ 0 & \text{otherwise.} \end{cases}$$

Define the function $\psi : (0,1) \to \mathbb{R}$ by

$$\psi(x) = \frac{1}{x}.$$

Claim: ψ is continuous at every point in $(0,1)$ with respect to the fuzzy quasi-metric M,
but ψ is not uniformly continuous on $(0,1)$.
For any fixed $x_0 \in (0,1)$, given $\varepsilon > 0$, choose $\delta = \varepsilon \cdot x_0^2$. Then whenever $M(x, x_0, \delta) = 1$ (i.e., $|x - x_0| < \delta$), we have

$$|\psi(x) - \psi(x_0)| = \left| \frac{1}{x} - \frac{1}{x_0} \right| = \frac{|x - x_0|}{|x x_0|} < \frac{\delta}{x_0^2} = \varepsilon.$$

Hence, ψ is continuous at each point x_0. However, ψ is not uniformly continuous on $(0,1)$ because as $x \to 0^+$, the function values become arbitrarily large. For any $\delta > 0$, choose $x = \delta/2$ and $y = \delta/4$; then $M(x, y, t) = 1$ for sufficiently large t, but

$$|\psi(x) - \psi(y)| = \left| \frac{1}{\delta/2} - \frac{1}{\delta/4} \right| = \left| \frac{2}{\delta} - \frac{4}{\delta} \right| = \frac{2}{\delta},$$

which can be made arbitrarily large. Hence no single δ works for all x, y, so ψ is not uniformly continuous. This example shows that continuity does not imply uniform continuity in fuzzy quasi-metric spaces.

Lemma 3.17 (Restriction of Uniformly Continuous Functions) *Let $(X, M, \star)$ and $(Y, M', \star')$ be fuzzy quasi-metric spaces. Suppose $\psi : X \to Y$ is uniformly continuous, and let $A \subseteq X$. Then the restriction $\psi|_A : A \to Y$ is uniformly continuous.*

Proof: Let $t > 0$ and $\varepsilon \in (0, 1)$ be arbitrary. Since ψ is uniformly continuous on X, there exists $\delta \in (0, 1)$ such that for all $x, y \in X$,

$$M(x, y, t) > 1 - \delta \implies M'(\psi(x), \psi(y), t) > 1 - \varepsilon.$$

Now consider any $x, y \in A \subseteq X$. Since $x, y \in X$, the same implication holds:

$$M(x, y, t) > 1 - \delta \implies M'(\psi(x), \psi(y), t) > 1 - \varepsilon.$$

But this is exactly the condition required for $\psi|_A : A \to Y$ to be uniformly continuous (with the same δ and for all $x, y \in A$). Therefore, $\psi|_A$ is uniformly continuous.

□

Remark 3.18 *Due to asymmetry of M, the behavior of ψ and ψ^{-1} may differ regarding uniform continuity, so bijections do not necessarily preserve uniform continuity in both directions.*

Theorem 3.19 (Characterization via Cauchy Sequences) *A function $\psi : X \to Y$ between fuzzy quasi-metric spaces is uniformly continuous (in the sense that for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $\delta \in (0, 1)$ such that*

$$M(\psi(x), \psi(y), t) > 1 - \varepsilon \quad \textit{whenever} \quad M(x, y, t) > 1 - \delta$$

for all $x, y \in X$) if and only if ψ preserves Cauchy sequences, i.e., whenever (x_n) is a Cauchy sequence in X, then $(\psi(x_n))$ is a Cauchy sequence in Y.

Proof: Suppose ψ is uniformly continuous. Let (x_n) be a Cauchy sequence in X. Take any arbitrary $\varepsilon \in (0, 1)$ and $t > 0$. By uniform continuity, there exists $\delta \in (0, 1)$ such that

$$M(\psi(x), \psi(y), t) > 1 - \varepsilon \quad \text{whenever} \quad M(x, y, t) > 1 - \delta$$

Since (x_n) is Cauchy, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$M(x_n, x_m, t) > 1 - \delta.$$

By the implication from uniform continuity, we have

$$M(\psi(x_n), \psi(x_m), t) > 1 - \varepsilon \quad \text{for all } n, m \geq N.$$

Hence, $(\psi(x_n))$ is a Cauchy sequence in Y.

Conversely, suppose ψ is not uniformly continuous. Then there exist $\varepsilon_0 \in (0, 1)$ and $t_0 > 0$ such that for every $\delta \in (0, 1)$, there exist $x_\delta, y_\delta \in X$ with

$$M(x_\delta, y_\delta, t_0) > 1 - \delta \quad \text{but} \quad M(\psi(x_\delta), \psi(y_\delta), t_0) \leq 1 - \varepsilon_0.$$

Choose a sequence $\delta_n \to 0$ as $n \to \infty$, and for each $n \in \mathbb{N}$, pick $x_n, y_n \in X$ such that

$$M(x_n, y_n, t_0) > 1 - \delta_n \quad \text{but} \quad M(\psi(x_n), \psi(y_n), t_0) \leq 1 - \varepsilon_0.$$

Now construct a sequence (z_n) in X by setting $z_{2n} = x_n$, $z_{2n+1} = y_n$. We claim that (z_n) is Cauchy in X. For any $\eta > 0$, choose N such that $\delta_n < \eta$ for all $n \geq N$. Then for $m, n \geq 2N$, z_n and z_m are some x_k, y_k with $k \geq N$, and so:

$$M(z_n, z_m, t_0) > 1 - \delta_k > 1 - \eta.$$

Hence (z_n) is Cauchy in X. However, the image sequence $(\psi(z_n))$ is not Cauchy in Y, because

$$M(\psi(x_n), \psi(y_n), t_0) \leq 1 - \varepsilon_0 \quad \text{for all } n.$$

So for infinitely many pairs n, m, the closeness in Y is bounded away from 1. This contradicts the assumption that ψ preserves Cauchy sequences. Therefore, ψ must be uniformly continuous. □

Example 3.20 *Let* $X = \mathbb{R}$ *with fuzzy quasi-metric*

$$M(x, y, t) = \frac{t}{t + |x - y|}, \quad * = \min,$$

and define $\psi : \mathbb{R} \to \mathbb{R}$ *by* $\psi(x) = 2x$. *Then* ψ *is uniformly continuous, and the image of any fuzzy Cauchy sequence* (x_n) *is also fuzzy Cauchy since the scaled distances behave similarly.*

Theorem 3.21 *(Uniform Continuity and Completeness) Let* $(X, M, \star)$ *and* $(Y, N, \star)$ *be fuzzy quasi-metric spaces, with* Y *complete and Suppose* $\psi : X \to Y$ *is uniformly continuous. Then the image under* ψ *of every Cauchy sequence in* X *is a convergent sequence in* Y.

Proof: Let(x_n) be a Cauchy sequence in X. We want to show that $(\psi(x_n))$ converges in Y.
Let $\varepsilon \in (0, 1)$ and $t > 0$ be arbitrary. By uniform continuity of ψ, there exists $\delta \in (0, 1)$ such that

$$M(x, y, t) > 1 - \delta \quad \Rightarrow \quad N(\psi(x), \psi(y), t) > 1 - \varepsilon, \quad \text{for all } x, y \in X.$$

Since (x_n) is Cauchy in X, there exists $N_0 \in \mathbb{N}$ such that for all $n, m \geq N_0$,

$$M(x_n, x_m, t) > 1 - \delta.$$

Then, by the implication above,

$$N(\psi(x_n), \psi(x_m), t) > 1 - \varepsilon \quad \text{for all } n, m \geq N_0.$$

Hence, $(\psi(x_n))$ is a Cauchy sequence in Y.
Since Y is complete and $(\psi(x_n))$ is Cauchy in Y, there exists $y \in Y$ such that

$$\psi(x_n) \to y \quad \text{in } (Y, N, \star).$$

Therefore, for every Cauchy sequence (x_n) in X, $(\psi(x_n))$ converges in Y. □

Definition 3.22 *A subset* A *of a fuzzy quasi-metric space* $(X, M, *)$ *is said to be fuzzy totally bounded if for every* $t > 0$ *and* $\delta \in (0, 1)$, *there exists a finite set* $\{x_1, \ldots, x_n\} \subseteq X$ *such that*

$$A \subseteq \bigcup_{i=1}^{n} B(x_i, t, \delta),$$

where

$$B(x, t, \delta) = \{y \in X : M(x, y, t) > 1 - \delta\}.$$

Proposition 3.23 *Let* $(X, M, \star)$ *and* $(Y, M', \star')$ *be fuzzy quasi-metric spaces. Suppose* $\psi : X \to Y$ *is uniformly continuous, and let* $A \subseteq X$ *be fuzzy totally bounded. Then* $\psi(A) \subseteq Y$ *is also fuzzy totally bounded.*

Proof: Let $t > 0$ and $\varepsilon \in (0, 1)$ be arbitrary. Since ψ is uniformly continuous, there exists $\delta \in (0, 1)$ such that for all $x, y \in X$,

$$M(x, y, t) > 1 - \delta \implies M'(\psi(x), \psi(y), t) > 1 - \varepsilon.$$

Since $A \subseteq X$ is fuzzy totally bounded, there exists a finite set $\{x_1, x_2, \ldots, x_n\} \subseteq X$ such that

$$A \subseteq \bigcup_{i=1}^{n} B(x_i, t, \delta),$$

where

$$B(x_i, t, \delta) = \{y \in X : M(x_i, y, t) > 1 - \delta\}.$$

Now consider the images $\psi(x_1), \ldots, \psi(x_n) \in Y$. We claim that these form a fuzzy (t, ε)-cover of $\psi(A)$ in Y. Let $y \in \psi(A)$. Then there exists $a \in A$ such that $y = \psi(a)$. Since $a \in A$, there exists some $i \in \{1, \ldots, n\}$ such that

$$M(x_i, a, t) > 1 - \delta.$$

By uniform continuity of ψ, we then have:

$$M'(\psi(x_i), \psi(a), t) = M'(\psi(x_i), y, t) > 1 - \varepsilon.$$

So $y \in B'(\psi(x_i), t, \varepsilon)$, where

$$B'(\psi(x_i), t, \varepsilon) := \{z \in Y : M'(\psi(x_i), z, t) > 1 - \varepsilon\}.$$

Hence,

$$\psi(A) \subseteq \bigcup_{i=1}^{n} B'(\psi(x_i), t, \varepsilon),$$

which shows that $\psi(A)$ is covered by finitely many fuzzy balls in Y of radius t and fuzziness parameter ε. Since $t > 0$ and $\varepsilon \in (0, 1)$ were arbitrary, this proves that $\psi(A)$ is fuzzy totally bounded. □

Example 3.24 *Let $X = [0, 1] \subseteq \mathbb{R}$, and define a fuzzy quasi-metric $M : X \times X \times (0, \infty) \to [0, 1]$ by*

$$M(x, y, t) = \begin{cases} 1, & \text{if } x \leq y, \\ \max\left\{1 - \dfrac{|x - y|}{t}, 0\right\}, & \text{if } x > y, \end{cases}$$

with the standard t-norm $a \star b = ab$. Then $(X, M, \star)$ is a fuzzy quasi-metric space. The set $X = [0, 1]$ is bounded and closed in $\mathbb{R}$, and it can be shown that it is fuzzy totally bounded under M. Now, consider the function $\psi(x) = \sin(\pi x)$. Since ψ is uniformly continuous on $[0, 1]$ (as it is continuous on a compact interval), it follows by the previous theorem that $\psi(X) \subseteq \mathbb{R}$ is also fuzzy totally bounded. In fact, $\psi(X) = [0, 1]$, and under the induced fuzzy quasi-metric on $\mathbb{R}$, this image set remains fuzzy totally bounded.

To prove this example, one must see that the fuzzy quasi-metric defined on $X = [0, 1] \subseteq \mathbb{R}$ induces a structure under which X is fuzzy totally bounded, since it is bounded and can be covered by finitely many fuzzy balls for any $t > 0$ and $\delta \in (0, 1)$. The function $\psi(x) = \sin(\pi x)$ is uniformly continuous on $[0, 1]$, a compact interval in $\mathbb{R}$. By the theorem 3.23 above, it follows that the image $\psi(X)$ is also fuzzy totally bounded.

We now develop the Extension Theorem as a fundamental result building upon the concepts of completeness and uniform continuity in fuzzy quasi-metric spaces. As presented above, uniform continuity ensures the preservation of fuzzy Cauchy sequences, and completeness guarantees the existence of limits for such sequences. Combining these properties, the Extension Theorem asserts that any uniformly continuous function defined on a dense subspace of a complete fuzzy quasi-metric space admits a unique uniformly continuous extension to the entire space. This result is crucial, as it allows one to extend functions originally defined on manageable dense subsets to the full space without losing uniform control over their behavior, thereby facilitating further analysis and applications within fuzzy metric frameworks.

Theorem 3.25 (Extension Theorem in Fuzzy Quasi-Metric Spaces) *Let $(X, M, *)$ be a dense subspace of a complete fuzzy quasi-metric space $(\bar{X}, \bar{M}, \bar{\star})$, and let $(Y, M', \star')$ be a complete fuzzy quasi-metric space. Suppose $\psi : X \to Y$ is uniformly continuous in the sense that for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $\delta \in (0, 1)$ such that*

$$M'(f(x), f(y), t) > 1 - \varepsilon \quad \textit{whenever} \quad M(x, y, t) > 1 - \delta \quad \textit{for all } x, y \in X.$$

Then ψ admits a unique uniformly continuous extension $\bar{\psi} : \bar{X} \to Y$.

Proof: Let $\bar{x} \in \bar{X}$. Since X is dense in $\bar{X}$, there exists a sequence $(x_n) \subset X$ such that $x_n \to \bar{x}$ in $(\bar{X}, \bar{M}, \bar{*})$, i.e.,

$$\forall t > 0, \forall \delta \in (0,1),\ \exists N \in \mathbb{N} \text{ such that } n, m \geq N \Rightarrow \bar{M}(x_n, x_m, t) > 1 - \delta,$$

and

$$\forall t > 0, \forall \delta \in (0,1),\ \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow \bar{M}(x_n, \bar{x}, t) > 1 - \delta.$$

Since ψ is uniformly continuous on X, it preserves Cauchy sequences (as previously proved). Thus, $(\psi(x_n)) \subset Y$ is a Cauchy sequence in $(Y, M', \star)$. Since Y is complete, there exists a unique $y \in Y$ such that $\psi(x_n) \to y$.
Define $\bar{\psi}(\bar{x}) := \lim \psi(x_n) \in Y$. Suppose $(x'_n) \subset X$ is another sequence such that $x'_n \to \bar{x}$. We must show that $\psi(x'_n) \to y$ as well, so the limit is independent of the approximating sequence. Let $\varepsilon \in (0,1)$ and $t > 0$. By uniform continuity of ψ, there exists $\delta \in (0,1)$ such that:

$$M(x, y, t) > 1 - \delta \quad \Rightarrow \quad M'(\psi(x), \psi(y), t) > 1 - \varepsilon.$$

Since $x_n \to \bar{x}$ and $x'_n \to \bar{x}$, we can find N such that for all $n \geq N$,

$$\bar{M}(x_n, \bar{x}, t/2) > 1 - \delta/2 \quad \text{and} \quad \bar{M}(x'_n, \bar{x}, t/2) > 1 - \delta/2.$$

Using the triangularity of the fuzzy quasi-metric, for $n \geq N$:

$$\bar{M}(x_n, x'_n, t) \geq \bar{M}(x_n, \bar{x}, t/2) * \bar{M}(\bar{x}, x'_n, t/2) > (1 - \delta/2) * (1 - \delta/2).$$

Since $\star$ is a continuous $t-$norm and $(1 - \delta/2)^2 > 1 - \delta$, we get:

$$\bar{M}(x_n, x'_n, t) > 1 - \delta.$$

Then by uniform continuity:

$$M'(\psi(x_n), \psi(x'_n), t) > 1 - \varepsilon.$$

So $\psi(x_n)$ and $\psi(x'_n)$ are arbitrarily close for large n, hence $\psi(x'_n) \to y$. Thus, $\bar{\psi}(\bar{x}) := y$ is independent of the approximating sequence. Let $\varepsilon \in (0,1)$, $t > 0$ be given. Since ψ is uniformly continuous, there exists $\delta \in (0,1)$ such that

$$M(x, y, t) > 1 - \delta \quad \Rightarrow \quad M'(\psi(x), \psi(y), t) > 1 - \varepsilon \quad \text{for all } x, y \in X.$$

Let $\bar{x}, \bar{y} \in \bar{X}$. Take sequences $(x_n), (y_n) \subset X$ such that $x_n \to \bar{x}$, $y_n \to \bar{y}$. By completeness, $\bar{\psi}(\bar{x}) = \lim \psi(x_n)$, $\bar{\psi}(\bar{y}) = \lim \psi(y_n)$. If $\bar{M}(\bar{x}, \bar{y}, t) > 1 - \delta$, then for large n,

$$\bar{M}(x_n, y_n, t) > 1 - \delta \Rightarrow M'(\psi(x_n), \psi(y_n), t) > 1 - \varepsilon.$$

Taking limits, we conclude:

$$M'(\bar{\psi}(\bar{x}), \bar{\psi}(\bar{y}), t) \geq \limsup M'(\psi(x_n), \psi(y_n), t) \geq 1 - \varepsilon.$$

Thus, $\bar{\psi}$ is uniformly continuous on $\bar{X}$. If another uniformly continuous extension $\varphi : \bar{X} \to Y$ agrees with ψ on X, then for any $\bar{x} \in \bar{X}$ and any sequence $x_n \to \bar{x}$, we have:

$$\varphi(\bar{x}) = \lim \psi(x_n) = \bar{\psi}(\bar{x}).$$

So $\varphi = \bar{\psi}$, proving uniqueness.

□

Example 3.26 *Let $X = \mathbb{Q}$ with the usual fuzzy metric induced by the absolute value and $Y = \mathbb{R}$ similarly defined. The function $\psi : \mathbb{Q} \to \mathbb{R}$ defined by $\psi(x) = \sin(x)$ is uniformly continuous and can be extended to $\bar{\psi} : \mathbb{R} \to \mathbb{R}$ uniquely and uniformly continuously.*

4. Conclusion

In this paper, we have rigorously developed the theory of uniform continuity within the framework of fuzzy quasi-metric spaces, a significant generalization of classical metric spaces that accommodate both asymmetry and fuzziness. Our work extends foundational concepts of continuity, convergence, and completeness to these generalized settings. By differentiating between pointwise and uniform continuity, we have established key characterizations and preserved classical results in this more complex context.

We also demonstrated that uniform continuity ensures the preservation of fuzzy Cauchy sequences and fuzzy total boundedness, reinforcing the parallel between fuzzy and classical compactness concepts. We highlighted the subtle differences introduced by asymmetry, especially in the behavior of inverse mappings, which do not necessarily retain uniform continuity.

One of the central achievements is the formulation and proof of an Extension Theorem that guarantees the unique uniformly continuous extension of functions defined on dense subspaces of complete fuzzy quasi-metric spaces. This theorem not only bridges gaps in existing fuzzy analysis literature but also equips researchers with a powerful tool to extend functions without losing uniform control, which is crucial for both theoretical investigations and practical applications.

Our examples and counterexamples provided valuable insights into these characteristic behaviors, illustrating both the possibilities and limitations within fuzzy quasi-metric frameworks

Overall, our findings deepen the understanding of functional behavior in fuzzy topologies and pave the way for further theoretical advances and applied studies in fuzzy functional analysis.

Future research can explore uniform continuity in other generalized fuzzy structures such as fuzzy partial metric and probabilistic metric spaces, potentially uncovering new theoretical insights. Investigating the impact of uniform continuity on fixed point theorems within fuzzy quasi-metric spaces could enhance solution techniques in nonlinear analysis. Further study of the asymmetry effects on inverse functions and operators may reveal unique properties specific to these spaces. Additionally, developing computational methods to verify uniform continuity and function extension could facilitate practical applications in decision-making and fuzzy modeling.

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Competing interests
The authors declare no conflict of interest.

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John Pamba,
Department of Mathematics, Statistics and Actuarial Sciences,
University of Zambia,
Zambia.
E-mail address: jpamba@zut.edu.zm, https://orcid.org/0009-0002-0931-6725

and

Hope Sabao,
Department of Mathematics, Statistics and Actuarial Sciences,
University of Zambia,
Zambia.
E-mail address: hope.sabao@unza.zm, https://orcid.org/0000-0002-8649-5635

and

Isaac D. Tembo,
Department of Mathematics, Statistics and Actuarial Sciences,
University of Zambia,
Zambia.
E-mail address: isaactembo@unza.zm, http://0rcid.org/0009-0001-4393-0772