



An Advanced Approach to Weighted Neutrosophic Soft Relations for Multi-Criteria Decision-Making for Apartment Selection, Macroeconomic Policy Evaluation and Water Quality Assessment

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ABSTRACT: In this study, we introduce the concept of Weighted Neutrosophic Soft Relations (WNSRs) and explore their theoretical foundations, including key definitions, essential properties, and fundamental operations. We construct the Cartesian product of Weighted Neutrosophic Soft Sets (WNSSs) and define a new class of WNSRs, forming a robust mathematical framework for practical implementation. Furthermore, we propose a novel Multi-Criteria Decision-Making Model (MCDMM) based on WNSRs, utilizing a weighted score function to enhance the consistency, reliability, and interpretability of decisions under uncertainty. The developed model enables decision-makers to integrate parameter-specific weights, thereby improving adaptability and precision in complex decision-making environments. To validate the proposed approach, a real-world Decision-Making Problem (DMP) is solved and compared with the existing Dalkılıç method. The comparative analysis demonstrates that our model surpasses previous approaches in terms of flexibility, accuracy, and stability. Additionally, its adjustable structure supports iterative refinements, ensuring optimal and rational decision outcomes. Overall, this research advances the field of multi-criteria decision-making (MCDM) by establishing a comprehensive theoretical and applied framework for WNSRs, offering a powerful tool to manage uncertainty across diverse application domains.

Keywords: Neutrosophic set, soft set, neutrosophic soft set, neutrosophic soft relation, weighted neutrosophic soft set, multi-criteria decision-making.

Contents

1	Introduction	1
2	Preliminary	4
3	Weighted Neutrosophic Soft Relations and their Theoretical Analysis	7
4	Result and Discussions	12
	4.1 Application in Apartment Selection:	12
	4.2 Application in Macroeconomic Policy Evaluation:	14
	4.3 Application in Water Quality Assessment	15
5	Comparison Analyses	19
6	Conclusions	22

1. Introduction

In real-world MCDM scenarios, uncertainty and imprecision are inherent due to the complexity of human judgment, vague preferences, and incomplete information. Traditional mathematical models often struggle to handle such uncertainty effectively. The neutrosophic soft set (NSS) [1], introduced as an extension of soft set theory (SSTH) [2], provides a robust framework for managing vagueness by incorporating truth, indeterminacy, and falsity memberships. However, standard NSS-based approaches, including NSRs [3], have certain limitations in complex DMPs [4], particularly in cases where multiple optimal choices exist or parameter-specific importance is not considered.

Molodtsov [2] defined a soft set over the nonempty universe Ω as a pair (Ψ, P) , such as Ψ is a function defined by

$$\Psi : P \rightarrow \mathcal{P}(\Omega),$$

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where P is a collection of parameters and $\mathcal{P}(\Omega)$ means the power set of Ω . Maji et al. [5] expanded on this by defining key operations such as subset, complement, union, and intersection, along with their applications in MCDM problems. Ali et al. [6] further explored additional operations and established that De Morgan's laws hold within SSTH. Maji et al. [7] were the first to apply SSTH to DMPs. Since then, numerous researchers have investigated its broader properties and applications. Deli and Çağman [8] defined relations on FP-soft sets and applied them to DMPs. Dalkılıç and others [9,10] proposed a novel approach to SSTH in MCDM problems under uncertainty and presented a new theory of FP-SSTH as well as its application on DMPs. Alcantud and Santos-García [11] introduced a novel criterion for applying SSTH in MCDM problems with incomplete information. Chen et al. [12] proposed a MCDM model based on generalized vague N-soft sets, while Dalkılıç [13] developed a new approach to utilizing SSTH in MCDM problems under uncertainty.

Fuzzy logic, introduced by Zadeh [14], is appropriate in the context of uncertainty handling in computer vision tasks because it manages imprecision, linguistic variables, and heuristics guided by the rule [15,16]. Such compatibility has generated the concepts of fuzzy soft sets, hesitant fuzzy decision models, and weighted fuzzy frameworks, suitable for multi-criteria decision-making (MCDM) in an uncertain scenario [17,18,19]. A significant contribution by Das and Granados [18] involves the development of a fuzzy parameterized intuitionistic fuzzy multiset theory for MCDM. Their investigation of FP-intuitionistic multi-fuzzy N-soft sets, which increase the adaptability of MCDM models, complements this work [20]. They presented IFP-intuitionistic multi-fuzzy N-soft sets, further expanding this field of study and proving their usefulness in MCDM [19].

Furthermore, Das [16] suggested weighted fuzzy soft multisets as a novel tool for MCDM, while Das [15] and Das & Granados [21] explored intuitionistic fuzzy rough relations and preference intuitionistic fuzzy rough relations, highlighting their partial inclusion properties. Their collaborative work in 2021 also presented a fuzzy soft group MCDM approach based on weighted average ratings, further refining the theoretical and practical aspects of decision analysis [17]. Applications-wise, Das et al. [22] suggested new operations on FSSs, demonstrating their usefulness in situations involving MCDM. Similarly, Das et al. [23] introduced the concept of weighted hesitant bipolar-valued FSSs, which provide a more nuanced framework for evaluating alternatives. Das et al. [24] developed a fuzzy MCDM model for examining human influences on urban river water quality, one of their noteworthy contributions to environmental MCDM. This research is extended in 2025 with weighted hesitant FSS and FSS models for rating water pollution and a case study applying the fuzzy soft MCDM model to assess pollution levels in the Gomati and Haora Rivers [25,26].

The theories of fuzzy soft multisets in decision-making also owe much to Mukherjee and Das [27]. Their 2015 article was a work that used interval-valued intuitionistic fuzzy soft multisets in real-life MCDM issues [28], and their research on the topic investigated the use of interval-valued intuitionistic FSSs in investment decision-making [29]. They also invented the Einstein operations on the fuzzy soft multisets in 2022, contributing to the greater tools of computing in decision-making [30]. They become more complete in their 2023 book, which discusses the theoretical foundations and practical use [31]. The topological and relational model of fuzzy and rough sets is even more considered by Mukherjee & Das [32,33]. The study by Mukherjee et al. [34] introduces a generalized interval-valued neutrosophic rough soft sets framework that effectively handles uncertainty and demonstrates its practical use in water quality assessment.

Granados et al. [35] introduced continuous neutrosophic distributions with neutrosophic parameters that give a possibility to represent the uncertainty probabilistically. Das et al. [25,36] assessed Gomati River water quality in Tripura using FSSs and neutrosophic approaches. Recently, Das et al. [37] have thoroughly studied the weighted neutrosophic soft multiset approach proposed by Granados et al. [38], which is applied in decision-making problems involving the non-handling of various levels of uncertainty. Meanwhile, neutrosophic SuperHyper BCI-semigroups [37] have been thoroughly studied, and their algebraic significance has been established. Das et al. [39] recently introduced the fuzzy logic decision support system to streamline the concept of an adaptive inventory management system within the retailing context. Das et al. [40] have developed a new weighted hypersoft expert system group decision-making paradigm, and in continuation of the neutrosophic and plithogenic models, approached much more uncertainty and complication in any circumstance of applied mathematics. The studies have significant

implications for improving fuzzy set theory.

which can be used in computational intelligence, environmental sustainability, and MCDM. They provide helpful strategies for dealing with uncertainty, making more precise decisions, and addressing complex real-life problems. It's critical to articulate the current level of uncertainty in a problem in the best possible way before being able to solve it in the best possible way. In this sense, and particularly in recent years, another mathematical paradigm for dealing with ambiguity data has emerged. Smarandache [1] developed a neutrosophic set theory (briefly NSTH) as a generalization of fuzzy set theory (FST) [14] and intuitionistic fuzzy set (IFST) [14]. To represent uncertainty clearly, this NSTH is a logic set, which has three independent components falsity, indeterminacy, and truth memberships. The earliest NSTH data can be found in the book [1], which was released in 1998. One of the most significant benefits of this NSTH is its capacity to interpret uncertain data that FST and IFST do not evaluate. The n-valued refined NSTH with its applications in physics was introduced by Smarandache [42]. Smarandache [43] proposed the hypersoft set (HSS) theory and plithogenic HSS theory as an extension of SSTH. Yang et al. [44] presented the concept of single-valued NSRs. Saqlain et al. [45] suggested single and multi-valued neutrosophic HSS, as well as a single-valued neutrosophic HSS tangent similarity measure. According to Maji et al. [9], fuzzy SSTH is a combination of FST and SSTH. Mattam and Gopalan [46] suggested an approach for computing fuzzy soft relations transitive closure. Abdel-Basset et al. [47,48] established an integrated neutrosophic VIKOR and ANP approach for obtaining sustainable supplier selection and used a TOPSIS method for creating supplier selection in group DMPs under type-2 neutrosophic number. Abdel-Basset and others [49,50] developed a unique NSTH approach for evaluating green supply chain management practices and grouping MCDMM for heart disease diagnosis based on NSTH. Abdel-Basset and others [51,52] suggested a bipolar neutrosophic MCDM model for intelligent medical decision support system and professional selection using soft computing. MCDM techniques using neutrosophic hesitant fuzzy information were presented by Liu and Zhang [53,54]. Mohammed et al. [55] provided a benchmarking methodology for selecting the best COVID-19 diagnostic tool based on TOPSIS and entropy methodologies.

Since SSTH and NSTH are both successful in processing unclear input, Maji [56] has merged these ideas and contributed a new mathematical structure, called Neutrosophic Soft Sets (NSSs). With a comparison to Maji's [56] description, Deli and Broumi [57] adjusted the NSSs so that they could be used more realistically in uncertainty issues, and their operations were also redefined. Many researchers have found this mathematical model intriguing. For example, Mukherjee and Sadhan [58,59] provided various methods for determining how similar two NSSs are and their use in pattern identification and other real-world situations are shown. Furthermore, Sahin and Küçük [60] used a novel type of NSS theory to investigate several algebraic characteristics. Aside from that, Hussain and Shabir [61] looked at algebraic theory operations. The NSSs were also researched by Sumathi and Arukkarani [62]. A lot of studies have been done on the NSS in addition to these [63]. In comparison, NSSs are a highly beneficial for expressing uncertainty and vagueness accurately. In 2014, Maji [64] presented the neutrosophic NSSs and discussed the advantages of NSSs as a generalization of NSSs. Broumi and Smarandache [65] suggested the intuitionistic NSS theory, while Cuong et al. [66] proposed the mainstream NSS theory. Deli and Broumi [3] introduced NSRs and looked at some of the features and outcomes. Later, Broumi and colleagues [67,68] published the theory of interval-valued NSSs and provided their applications in DMPs, while Abu-Qamar and Hassan [69] founded Q-NSR with applications in DMPs. Alkhazaleh [70] developed a novel time-NSS concept and its applications in real-world challenges. Bera and colleagues [71,72] investigated some NSS operations and used NSSs to build topological spaces. Ali and colleagues [73,74] presented attribute reduction of bipolar fuzzy relations in decision-making process and introduced a new idea of bipolar NSSs as well as their applications in MCDM problems. Chatterjee et al. [75] offered interval-valued possibilistic quad-partitioned single-valued NSSs as well as various uncertainty-based measures. The complex neutrosophic soft expert relation and its application in MCDM were presented by Al-Quran and Hassan [76]. In their study, Das et al. [77] developed a fuzzy soft multi-criteria group decision-making model to evaluate anthropogenic impacts on urban river water quality, providing an effective framework for rating water pollution levels in complex environmental systems.

MCDMs based on NSSs have gained significant attention in recent years due to their ability to handle uncertainty and imprecision in real-life MCDM problems. Dalkılıç [4] proposed a DMM based on NSR

using a score function, aiming to improve the effectiveness of NSSs in addressing uncertainty-related challenges. However, this approach has notable limitations when applied to NSR-based real-life MCDM problems. In this study, we have demonstrated through a real-life illustration that the Dalkılıç model is insufficient for solving NSR-based decision-making problems, primarily due to its inability to differentiate between multiple optimal choices. To overcome this limitation, we propose a novel concept of WNSR and introduce an advanced, adjustable MCDMM that incorporates a weighted score function for solving WNSR-based DMPs.

The key contributions and advantages of our proposed MCDMM are as follows:

- **Introduction of the Weighted Score Function** – Unlike the traditional score function used in the Dalkılıç method, our approach integrates weight factors to enhance stability and feasibility in decision-making. This makes our MCDMM more adaptable to complex real-world scenarios.
- **Adjustability of the Proposed MCDMM** – A major drawback of the Dalkılıç DMM is its lack of flexibility. In contrast, our method allows decision-makers to adjust weights dynamically. If multiple optimal choices arise at the final stage, we can return to an earlier step and modify the weights to obtain a more refined and precise decision.
- **Solving a Real-Life DMP** – Our proposed MCDMM successfully addresses a real-life MCDM problem that cannot be solved using the Dalkılıç approach. Specifically, we illustrate its effectiveness with three applications: (i) an apartment selection problem, (ii) macroeconomic policy evaluation, (iii) a water quality assessment involving multiple sampling sites and environmental parameters. These case studies highlight the practical applicability and superiority of our method in handling uncertainty-driven problems where parameter weighting plays a crucial role.

Additionally, we have explored the fundamental properties of WNSR in depth, including its inverse and composition operations and their mathematical characteristics. These theoretical insights further strengthen the foundation of our proposed WNSR-based MCDMM for future applications.

The structure of this paper is as follows: Section 2 presents the fundamental concepts and key results of NSTH, NSS, and WNSR, which serve as the foundation for subsequent discussions. In Section 3, we define WNSRs and show the inverse and composition of these WNSRs, along with some fundamental features. Also, we present an advanced and adjustable MCDMM introducing a weighted score function for solving MCDM problems based on WNSRs. In Section 4, we demonstrate the validity of our WNSR-based technique through real-life examples, including apartment selection and water quality assessment, while Section 5 presents a comparative analysis with the Dalkılıç method [24]. Finally, Section 6 concludes the paper and outlines directions for future research.

2. Preliminary

In this section, we present the fundamental concepts and key results of NSTH, NSS, and WNSR, which serve as the foundation for subsequent discussions. Let us assume Ω represents the starting universe and the power set of Ω is presented by $Pw(\Omega)$. Let P represent a nonempty collection of parameters and $X, Y, Z \subseteq P$.

Definition 2.1 [51]. A neutrosophic set N on Ω is a set with a structure

$$N = \{\langle \alpha, \tau_N(\alpha), \kappa_N(\alpha), \nu_N(\alpha) \rangle : \alpha \in \Omega\}$$

where $\tau_N, \kappa_N, \nu_N : \Omega \rightarrow]0, 1[^+$ are mappings, such that

$$0 \leq \tau_N(\alpha) + \kappa_N(\alpha) + \nu_N(\alpha) \leq 3^+.$$

Simply, $NS(\Omega)$ means the set of all neutrosophic sets over Ω .

Definition 2.2 [35] A pair (Ψ, P) is known as an NSS over Ω , where $\Psi : P \rightarrow NS(\Omega)$, such that $\forall d \in P$,

$$\Psi(d) = \{\langle \alpha, \tau_{\Psi(d)}(\alpha), \kappa_{\Psi(d)}(\alpha), \nu_{\Psi(d)}(\alpha) \rangle : \alpha \in \Omega\},$$

where $\tau_{\Psi(d)} : \Omega \rightarrow [0, 1]$ is known as truth-membership function, $\kappa_{\Psi(d)} : \Omega \rightarrow [0, 1]$ is known as indeterminacy-membership function, $\nu_{\Psi(d)} : \Omega \rightarrow [0, 1]$ is called the falsity-membership function of $\Psi(d)$ for $d \in P$ and

$$0 \leq \tau_{\Psi(d)}(\alpha) + \kappa_{\Psi(d)}(\alpha) + \nu_{\Psi(d)}(\alpha) \leq 3.$$

Thus an NSS (Ψ_N, P_Ω) can be written as

$$(\Psi, P) = \{ \langle d, \langle \alpha, \tau_{\Psi(d)}(\alpha), \kappa_{\Psi(d)}(\alpha), \nu_{\Psi(d)}(\alpha) \rangle : \alpha \in \Omega \rangle : d \in P \}.$$

Definition 2.3 [24] Let us consider two NSSs on Ω are (Ψ, P) and (Φ, P) , then the Cartesian product of (Ψ, P) and (Φ, P) is denoted by $(\Psi, P) \times (\Phi, P)$ and defined by

$$(\Psi, P) \times (\Phi, P) = (\Upsilon, P \times P)$$

$$= \{ \langle (p, q) \rangle \langle \alpha, \tau_{\Upsilon(p,q)}(\alpha), \kappa_{\Upsilon(p,q)}(\alpha), \nu_{\Upsilon(p,q)}(\alpha) \rangle : \alpha \in \Omega : (p, q) \in P \times P \},$$

where $\forall \alpha \in \Omega$, and $(p, q) \in P \times P$,

$$\tau_{\Upsilon(p,q)}(\alpha) = \min\{\tau_{\Psi(p)}(\alpha), \tau_{\Phi(q)}(\alpha)\},$$

$$\kappa_{\Upsilon(p,q)}(\alpha) = \max\{\kappa_{\Psi(p)}(\alpha), \kappa_{\Phi(q)}(\alpha)\}, \quad \text{and} \quad \nu_{\Upsilon(p,q)}(\alpha) = \max\{\nu_{\Psi(p)}(\alpha), \nu_{\Phi(q)}(\alpha)\}.$$

Definition 2.4 [24] Let us consider two NSSs on Ω are (Ψ, P) and (Φ, P) , then an NSR from (Ψ, P) to (Φ, P) is a neutrosophic soft subset of $(\Psi, P) \times (\Phi, P)$ and is denoted by $(R, X \times Y)$ where $X \times Y \subseteq P \times P$ and defined by

$$(R, X \times Y) = \{ \langle (p, q) \rangle \langle \alpha, \tau_{R(p,q)}(\alpha), \kappa_{R(p,q)}(\alpha), \nu_{R(p,q)}(\alpha) \rangle : \alpha \in \Omega : (p, q) \in X \times Y \},$$

where $\forall \alpha \in \Omega$, and $(p, q) \in X \times Y \subseteq P \times P$,

$$\tau_{R(p,q)}(\alpha) = \min\{\tau_{\Psi(p)}(\alpha), \tau_{\Phi(q)}(\alpha)\},$$

$$\kappa_{R(p,q)}(\alpha) = \max\{\kappa_{\Psi(p)}(\alpha), \kappa_{\Phi(q)}(\alpha)\}, \quad \text{and} \quad \nu_{R(p,q)}(\alpha) = \max\{\nu_{\Psi(p)}(\alpha), \nu_{\Phi(q)}(\alpha)\}.$$

Example 2.5 Let $\Omega = \{\alpha, \beta, \gamma\}$ be the set of the universe and $P = \{p, q, r\}$ be the set of parameters associated with the universes mentioned above. Assume that two NSSs (Ψ, P) and (Φ, P) as

$$(\Psi, P) = \left\{ \begin{array}{l} \langle p, \langle \alpha, 0.6, 0.5, 0.7 \rangle, \langle \beta, 0.5, 0.6, 0.7 \rangle, \langle \gamma, 0.6, 0.7, 0.5 \rangle \rangle, \\ \langle q, \langle \alpha, 0.3, 0.5, 0.6 \rangle, \langle \beta, 0.4, 0.5, 0.8 \rangle, \langle \gamma, 0.2, 0.3, 0.5 \rangle \rangle, \\ \langle r, \langle \alpha, 0.4, 0.7, 0.8 \rangle, \langle \beta, 0.2, 0.5, 0.6 \rangle, \langle \gamma, 0.3, 0.2, 0.7 \rangle \rangle \end{array} \right\},$$

$$(\Phi, P) = \left\{ \begin{array}{l} \langle p, \langle \alpha, 0.7, 0.2, 0.6 \rangle, \langle \beta, 0.3, 0.5, 0.6 \rangle, \langle \gamma, 0.4, 0.6, 0.8 \rangle \rangle, \\ \langle q, \langle \alpha, 0.2, 0.5, 0.6 \rangle, \langle \beta, 0.4, 0.7, 0.8 \rangle, \langle \gamma, 0.4, 0.5, 0.7 \rangle \rangle, \\ \langle r, \langle \alpha, 0.4, 0.5, 0.8 \rangle, \langle \beta, 0.5, 0.3, 0.7 \rangle, \langle \gamma, 0.2, 0.5, 0.6 \rangle \rangle \end{array} \right\}.$$

Then, an NSR $(R, X \times Y)$ from (Ψ, P) to (Φ, P) as follows

$$(R, X \times Y) = \left\{ \begin{array}{l} \langle (p, p), \langle \alpha, 0.6, 0.5, 0.7 \rangle, \langle \beta, 0.3, 0.6, 0.7 \rangle, \langle \gamma, 0.4, 0.7, 0.8 \rangle \rangle, \\ \langle (p, q), \langle \alpha, 0.2, 0.5, 0.7 \rangle, \langle \beta, 0.4, 0.7, 0.8 \rangle, \langle \gamma, 0.4, 0.7, 0.7 \rangle \rangle, \\ \langle (p, r), \langle \alpha, 0.4, 0.5, 0.8 \rangle, \langle \beta, 0.5, 0.6, 0.7 \rangle, \langle \gamma, 0.2, 0.7, 0.6 \rangle \rangle, \\ \langle (q, p), \langle \alpha, 0.3, 0.5, 0.6 \rangle, \langle \beta, 0.3, 0.5, 0.8 \rangle, \langle \gamma, 0.2, 0.6, 0.8 \rangle \rangle, \\ \langle (q, q), \langle \alpha, 0.2, 0.5, 0.6 \rangle, \langle \beta, 0.4, 0.7, 0.8 \rangle, \langle \gamma, 0.2, 0.5, 0.7 \rangle \rangle, \\ \langle (q, r), \langle \alpha, 0.3, 0.5, 0.8 \rangle, \langle \beta, 0.4, 0.5, 0.8 \rangle, \langle \gamma, 0.2, 0.5, 0.6 \rangle \rangle \end{array} \right\}.$$

The tabular form of the NSR $(R, X \times Y)$ as follows:

Table 1: NSR $(R, X \times Y)$

Ω	(p,p)	(p,q)	(p,r)	(q,p)	(q,q)	(q,r)
α	{0.6,0.5,0.7}	{0.2,0.5,0.7}	{0.4,0.5,0.8}	{0.3,0.5,0.6}	{0.2,0.5,0.6}	{0.3,0.5,0.8}
β	{0.3,0.6,0.7}	{0.4,0.7,0.8}	{0.5,0.6,0.7}	{0.3,0.5,0.8}	{0.4,0.7,0.8}	{0.4,0.5,0.8}
γ	{0.4,0.7,0.8}	{0.4,0.7,0.7}	{0.2,0.7,0.6}	{0.2,0.6,0.8}	{0.2,0.5,0.7}	{0.2,0.5,0.6}

Definition 2.6 [36] The WNSS for the NSS (Ψ, P) with weights w_Ψ concerning the parameter set P is denoted as (Ψ, P^{w_Ψ}) and defined by

$$(\Psi, P^{w_\Psi}) = \left\{ \langle (d, w_\Psi(d)), \langle \alpha, \tau_{\Psi(d)}^{w_\Psi}(\alpha), \kappa_{\Psi(d)}^{w_\Psi}(\alpha), \nu_{\Psi(d)}^{w_\Psi}(\alpha) \rangle : \alpha \in \Omega \rangle : d \in P \right\},$$

where

$$\tau_{\Psi(d)}^{w_\Psi}(\alpha) = w_\Psi(d) \times \tau_{\Psi(d)}(\alpha), \quad \kappa_{\Psi(d)}^{w_\Psi}(\alpha) = w_\Psi(d) \times \kappa_{\Psi(d)}(\alpha), \quad \nu_{\Psi(d)}^{w_\Psi}(\alpha) = w_\Psi(d) \times \nu_{\Psi(d)}(\alpha).$$

Definition 2.7 [36] Let us consider two WNSSs on Ω are (Ψ, X^{w_1}) and (Φ, Y^{w_2}) for the NSSs (Ψ, X) and (Φ, Y) with weights w_1 and w_2 associated with the parameter set X and Y respectively, then the union of (Ψ, X^{w_1}) and (Φ, Y^{w_2}) is denoted by

$$(\Psi, X^{w_1}) \tilde{\cup} (\Phi, Y^{w_2})$$

and defined by

$$(\Psi, X^{w_1}) \tilde{\cup} (\Phi, Y^{w_2}) = (\Upsilon, Z^w) = \left\{ \langle (d, w(d)), \langle \alpha, \tau_{\Upsilon(d)}^w(\alpha), \kappa_{\Upsilon(d)}^w(\alpha), \nu_{\Upsilon(d)}^w(\alpha) \rangle : \alpha \in \Omega \rangle : d \in Z \right\},$$

where $Z = X \cup Y$, $\forall \alpha \in \Omega$, and $d \in Z$,

$$w(d) = \begin{cases} w_1(d), & \text{if } d \in X, \\ w_2(d), & \text{if } d \in Y, \\ \max\{w_1(d), w_2(d)\}, & \text{if } d \in X \cap Y, \end{cases}$$

$$\tau_{\Upsilon(d)}^w(\alpha) = \begin{cases} \tau_{\Psi(d)}^{w_1}(\alpha), & \text{if } d \in X, \\ \tau_{\Phi(d)}^{w_2}(\alpha), & \text{if } d \in Y, \\ w(d) \times \max\{\tau_{\Psi(d)}(\alpha), \tau_{\Phi(d)}(\alpha)\}, & \text{if } d \in X \cap Y, \end{cases}$$

$$\kappa_{\Upsilon(d)}^w(\alpha) = \begin{cases} \kappa_{\Psi(d)}^{w_1}(\alpha), & \text{if } d \in X, \\ \kappa_{\Phi(d)}^{w_2}(\alpha), & \text{if } d \in Y, \\ w(d) \times \min\{\kappa_{\Psi(d)}(\alpha), \kappa_{\Phi(d)}(\alpha)\}, & \text{if } d \in X \cap Y, \end{cases}$$

$$\nu_{\Upsilon(d)}^w(\alpha) = \begin{cases} \nu_{\Psi(d)}^{w_1}(\alpha), & \text{if } d \in X, \\ \nu_{\Phi(d)}^{w_2}(\alpha), & \text{if } d \in Y, \\ w(d) \times \min\{\nu_{\Psi(d)}(\alpha), \nu_{\Phi(d)}(\alpha)\}, & \text{if } d \in X \cap Y. \end{cases}$$

Definition 2.8 [36] Let us consider two WNSSs on Ω are (Ψ, X^{w_1}) and (Φ, Y^{w_2}) for the NSSs (Ψ, X) and (Φ, Y) with weights w_1 and w_2 associated with the parameter set X and Y respectively, then the intersection of (Ψ, X^{w_1}) and (Φ, Y^{w_2}) is denoted by

$$(\Psi, X^{w_1}) \tilde{\cap} (\Phi, Y^{w_2})$$

and defined by

$$(\Psi, X^{w_1}) \tilde{\cap} (\Phi, Y^{w_2}) = (\Upsilon, Z^w) = \left\{ \langle (d, w(d)), \langle \alpha, \tau_{\Upsilon(d)}^w(\alpha), \kappa_{\Upsilon(d)}^w(\alpha), \nu_{\Upsilon(d)}^w(\alpha) \rangle : \alpha \in \Omega \rangle : d \in Z \right\},$$

where $Z = X \cup Y$, $\forall \alpha \in \Omega$, and $d \in Z$,

$$w(d) = \begin{cases} w_1(d), & \text{if } d \in X, \\ w_2(d), & \text{if } d \in Y, \\ \min\{w_1(d), w_2(d)\}, & \text{if } d \in X \cap Y, \end{cases}$$

$$\tau_{\Upsilon(d)}^w(\alpha) = \begin{cases} \tau_{\Psi(d)}^{w_1}(\alpha), & \text{if } d \in X, \\ \tau_{\Phi(d)}^{w_2}(\alpha), & \text{if } d \in Y, \\ w(d) \times \min\{\tau_{\Psi(d)}(\alpha), \tau_{\Phi(d)}(\alpha)\}, & \text{if } d \in X \cap Y, \end{cases}$$

$$\kappa_{\Upsilon(d)}^w(\alpha) = \begin{cases} \kappa_{\Psi(d)}^{w_1}(\alpha), & \text{if } d \in X, \\ \kappa_{\Phi(d)}^{w_2}(\alpha), & \text{if } d \in Y, \\ w(d) \times \max\{\kappa_{\Psi(d)}(\alpha), \kappa_{\Phi(d)}(\alpha)\}, & \text{if } d \in X \cap Y, \end{cases}$$

$$\nu_{\Upsilon(d)}^w(\alpha) = \begin{cases} \nu_{\Psi(d)}^{w_1}(\alpha), & \text{if } d \in X, \\ \nu_{\Phi(d)}^{w_2}(\alpha), & \text{if } d \in Y, \\ w(d) \times \max\{\nu_{\Psi(d)}(\alpha), \nu_{\Phi(d)}(\alpha)\}, & \text{if } d \in X \cap Y. \end{cases}$$

3. Weighted Neutrosophic Soft Relations and their Theoretical Analysis

WNSSs are a powerful tool for representing and modeling uncertainty, vagueness, and subjective judgment information provided by decision-makers. Compared to traditional NSSs, WNSSs offer a more refined and flexible structure by incorporating weight factors that account for the relative importance of different criteria or parameters. This weighted approach makes WNSSs highly effective in complex decision-making environments, particularly in cases involving multiple conflicting attributes and uncertain information.

In this section, we have extended the fundamental concepts of WNSSs by defining the Cartesian product of two WNSSs and introducing a new theoretical framework for WNSR. These newly established relations enable a more comprehensive and structured way of analyzing relationships between elements in decision-making problems. Furthermore, we have explored essential properties and characteristics of WNSR in detail, providing a solid theoretical foundation for its application in MCDM, uncertainty modeling, and real-world problem-solving.

Definition 3.1 Let us consider two WNSSs on Ω are (Ψ, P^{w_Ψ}) and (Φ, P^{w_Φ}) for the NSSs (Ψ, P) and (Φ, P) with weights w_Ψ and w_Φ associated with the parameter set P . The Cartesian product of (Ψ, P^{w_Ψ}) and (Φ, P^{w_Φ}) is denoted by $(\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi})$ and defined by

$$(\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi}) = (\Upsilon, P^w \times P^w)$$

$$= \left\{ \langle ((p, q), w(p, q)), < \alpha, \tau_{\Upsilon(p,q)}^w(\alpha), \kappa_{\Upsilon(p,q)}^w(\alpha), \nu_{\Upsilon(p,q)}^w(\alpha) \rangle : \alpha \in \Omega \rangle : (p, q) \in P \times P \right\},$$

where $\forall \alpha \in \Omega$, and $(p, q) \in P \times P$,

$$w(p, q) = \min\{w_\Psi(p), w_\Phi(q)\},$$

$$\tau_{\Upsilon(p,q)}^w(\alpha) = w(p, q) \times \min\{\tau_{\Psi(p)}(\alpha), \tau_{\Phi(q)}(\alpha)\}, \quad \kappa_{\Upsilon(p,q)}^w(\alpha) = w(p, q) \times \max\{\kappa_{\Psi(p)}(\alpha), \kappa_{\Phi(q)}(\alpha)\},$$

$$\text{and } \nu_{\Upsilon(p,q)}^w(\alpha) = w(p, q) \times \max\{\nu_{\Psi(p)}(\alpha), \nu_{\Phi(q)}(\alpha)\}.$$

Definition 3.2 A WNSR from (Ψ, P^{w_Ψ}) to (Φ, P^{w_Φ}) is a weighted neutrosophic soft subset of $(\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi})$ and is denoted by $(R, X \times Y^{w_R})$ where $X \times Y \subseteq P \times P$ and defined by

$$(R, X \times Y^{w_R}) = \left\{ \langle ((p, q), w(p, q)), < \alpha, \tau_{R(p,q)}^{w_R}(\alpha), \kappa_{R(p,q)}^{w_R}(\alpha), \nu_{R(p,q)}^{w_R}(\alpha) \rangle : \alpha \in \Omega \rangle : (p, q) \in X \times Y \right\},$$

where $\forall \alpha \in \Omega$, and $(p, q) \in X \times Y \subseteq P \times P$,

$$w_R(p, q) = \min\{w_\Psi(p), w_\Phi(q)\}, \quad \tau_{R(p,q)}^{w_R}(\alpha) = w_R(p, q) \times \min\{\tau_{\Psi(p)}(\alpha), \tau_{\Phi(q)}(\alpha)\},$$

$$\kappa_{R(p,q)}^{w_R}(\alpha) = w_R(p, q) \times \max\{\kappa_{\Psi(p)}(\alpha), \kappa_{\Phi(q)}(\alpha)\}, \quad \nu_{R(p,q)}^{w_R}(\alpha) = w_R(p, q) \times \max\{\nu_{\Psi(p)}(\alpha), \nu_{\Phi(q)}(\alpha)\}.$$

If $\forall (p, q) \in X \times Y$, $w_R(p, q) = 1$, then the WNSR $(R, X \times Y^{w_R})$ will be generated as a traditional NSR.

Note: We represent the set containing all WNSRs from (Ψ, P^{w_Ψ}) to (Φ, P^{w_Φ}) by

$$\text{WNSR}((\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi}))$$

and the collection of all WNSRs from (Ψ, P^{w_Ψ}) to (Ψ, P^{w_Ψ}) by

$$\text{WNSR}(\Psi, P^{w_\Psi}).$$

Example 3.3 Let $\Omega = \{\alpha, \beta, \gamma\}$ be the collection of the universe and $P = \{p, q, r\}$ be the collection of parameters associated with the universes mentioned above. Assume that two WNSSs (Ψ, P^{w_Ψ}) and (Φ, P^{w_Φ}) are

$$\begin{aligned} (\Psi, P^{w_\Psi}) &= \left\{ \begin{aligned} &\langle (p, 0.9), \langle \alpha, 0.54, 0.45, 0.63 \rangle, \langle \beta, 0.45, 0.54, 0.63 \rangle, \langle \gamma, 0.54, 0.63, 0.45 \rangle \rangle, \\ &\langle (q, 0.7), \langle \alpha, 0.21, 0.35, 0.42 \rangle, \langle \beta, 0.28, 0.35, 0.56 \rangle, \langle \gamma, 0.14, 0.21, 0.35 \rangle \rangle, \\ &\langle (r, 0.8), \langle \alpha, 0.28, 0.56, 0.64 \rangle, \langle \beta, 0.16, 0.40, 0.48 \rangle, \langle \gamma, 0.24, 0.16, 0.56 \rangle \rangle \end{aligned} \right\}, \\ (\Phi, P^{w_\Phi}) &= \left\{ \begin{aligned} &\langle (p, 0.8), \langle \alpha, 0.56, 0.16, 0.48 \rangle, \langle \beta, 0.24, 0.40, 0.48 \rangle, \langle \gamma, 0.32, 0.48, 0.64 \rangle \rangle, \\ &\langle (q, 0.9), \langle \alpha, 0.18, 0.45, 0.54 \rangle, \langle \beta, 0.36, 0.63, 0.72 \rangle, \langle \gamma, 0.36, 0.45, 0.63 \rangle \rangle, \\ &\langle (r, 0.6), \langle \alpha, 0.24, 0.30, 0.48 \rangle, \langle \beta, 0.30, 0.18, 0.42 \rangle, \langle \gamma, 0.12, 0.30, 0.36 \rangle \rangle \end{aligned} \right\}. \end{aligned}$$

where weights w_Ψ and w_Φ associated with the parameters in P are

$$w_\Psi(p) = 0.9, \quad w_\Psi(q) = 0.7, \quad w_\Psi(r) = 0.8,$$

$$w_\Phi(p) = 0.8, \quad w_\Phi(q) = 0.9, \quad w_\Phi(r) = 0.6.$$

Then the Cartesian product $(\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi}) = (\Upsilon, P^w \times P^w)$ is

$$(\Upsilon, P^w \times P^w) = \left\{ \begin{aligned} &\langle ((p, p), 0.8), \langle \alpha, 0.48, 0.40, 0.56 \rangle, \langle \beta, 0.24, 0.48, 0.56 \rangle, \langle \gamma, 0.32, 0.56, 0.64 \rangle \rangle, \\ &\langle ((p, q), 0.9), \langle \alpha, 0.18, 0.45, 0.63 \rangle, \langle \beta, 0.36, 0.63, 0.72 \rangle, \langle \gamma, 0.36, 0.63, 0.63 \rangle \rangle, \\ &\langle ((p, r), 0.6), \langle \alpha, 0.24, 0.30, 0.48 \rangle, \langle \beta, 0.30, 0.36, 0.42 \rangle, \langle \gamma, 0.12, 0.42, 0.36 \rangle \rangle, \\ &\langle ((q, p), 0.7), \langle \alpha, 0.21, 0.35, 0.42 \rangle, \langle \beta, 0.21, 0.35, 0.56 \rangle, \langle \gamma, 0.14, 0.42, 0.56 \rangle \rangle, \\ &\langle ((q, q), 0.7), \langle \alpha, 0.14, 0.35, 0.42 \rangle, \langle \beta, 0.28, 0.49, 0.56 \rangle, \langle \gamma, 0.14, 0.35, 0.49 \rangle \rangle, \\ &\langle ((q, r), 0.6), \langle \alpha, 0.18, 0.30, 0.48 \rangle, \langle \beta, 0.24, 0.30, 0.48 \rangle, \langle \gamma, 0.12, 0.30, 0.36 \rangle \rangle, \\ &\langle ((r, p), 0.8), \langle \alpha, 0.32, 0.56, 0.64 \rangle, \langle \beta, 0.16, 0.40, 0.48 \rangle, \langle \gamma, 0.24, 0.40, 0.64 \rangle \rangle, \\ &\langle ((r, q), 0.6), \langle \alpha, 0.16, 0.56, 0.64 \rangle, \langle \beta, 0.16, 0.56, 0.64 \rangle, \langle \gamma, 0.24, 0.40, 0.56 \rangle \rangle, \\ &\langle ((r, r), 0.6), \langle \alpha, 0.24, 0.42, 0.48 \rangle, \langle \beta, 0.24, 0.30, 0.42 \rangle, \langle \gamma, 0.12, 0.30, 0.42 \rangle \rangle \end{aligned} \right\}.$$

The tabular form of the WNSR $(R, X \times Y^{w_R})$ as presented in Table-2.

Ω	$(p, p), 0.8$	$(p, q), 0.9$	$(p, r), 0.6$	$(q, p), 0.7$	$(q, q), 0.7$	$(q, r), 0.6$
α	{0.48, 0.40, 0.56}	{0.18, 0.45, 0.63}	{0.24, 0.30, 0.48}	{0.21, 0.35, 0.42}	{0.14, 0.35, 0.42}	{0.18, 0.30, 0.48}
β	{0.24, 0.48, 0.56}	{0.36, 0.63, 0.72}	{0.30, 0.36, 0.42}	{0.21, 0.35, 0.56}	{0.28, 0.49, 0.56}	{0.24, 0.30, 0.48}
γ	{0.32, 0.56, 0.64}	{0.36, 0.63, 0.63}	{0.12, 0.42, 0.36}	{0.14, 0.42, 0.56}	{0.14, 0.35, 0.49}	{0.12, 0.30, 0.36}

Table 2: **Weighted Neutrosophic Soft Relation** $(R, X \times Y^{w_R})$

Definition 3.4 Let a WNSR $(R, X \times Y^{w_R}) \in \text{WNSR}((\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi}))$, then

[a] The domain of $(R, X \times Y^{w_R})$ is denoted by $\text{domain}(R, X \times Y^{w_R})$ and defined as the WNSS $(\Psi', X'^{w'_\Psi})$, where

$$X' = \{p \in X : R(p, q) \in R, \text{ for some } q \in Y\}, \quad w'_\Psi(p) = w_\Psi(p), \quad \Psi'(p) = \Psi(p), \quad \forall p \in X'.$$

[b] The range of $(R, X \times Y^{w_R})$ is denoted by $\text{range}(R, X \times Y^{w_R})$ and described as the WNSS $(\Phi', Y'^{w'_\Phi})$, where

$$Y' = \{q \in Y : R(p, q) \in R, \text{ for some } p \in X\}, \quad w'_\Phi(q) = w_\Phi(q), \quad \Phi'(q) = \Phi(q), \quad \forall q \in Y'.$$

Example 3.5 Let us consider the WNSR $(R_N, X_P \times Y_P^{w_R}) \in \text{WNSR}((\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi}))$ as in Example 3.3, then

[a] The $\text{domain}(R_N, X_P \times Y_P^{w_R})$ is $(\Psi', X'^{w'_\Psi})$, where $X' = \{p, q\}$ with weights

$$w'_\Psi(p) = 0.9, \quad w'_\Psi(q) = 0.7.$$

[b] The $\text{range}(R_N, X_P \times Y_P^{w_R})$ is $(\Phi', Y'^{w'_\Phi})$, where $Y' = \{p, q, r\}$ with weights

$$w'_\Phi(p) = 0.8, \quad w'_\Phi(q) = 0.9, \quad w'_\Phi(r) = 0.6.$$

Definition 3.6 A WNSR $(R, X \times Y^{w_R}) \in \text{WNSR}(\Psi, P^{w_\Psi})$ is said to be an Identity WNSR if

$$\Psi(p) R \Psi(q) \iff p = q, \text{ i.e. } R(p, q) \in R \iff p = q.$$

Example 3.7 If we consider the WNSS (Ψ, P^{w_Ψ}) as in Example 3.3, then the WNSR

$$(R, X \times Y^{w_R}) = \left\{ \begin{array}{l} \langle \langle (p, p), 0.9 \rangle, \langle \alpha, 0.54, 0.45, 0.63 \rangle, \langle \beta, 0.45, 0.54, 0.63 \rangle, \langle \gamma, 0.54, 0.63, 0.45 \rangle \rangle, \\ \langle \langle (q, q), 0.7 \rangle, \langle \alpha, 0.21, 0.35, 0.42 \rangle, \langle \beta, 0.28, 0.35, 0.56 \rangle, \langle \gamma, 0.14, 0.21, 0.35 \rangle \rangle, \\ \langle \langle (r, r), 0.8 \rangle, \langle \alpha, 0.28, 0.56, 0.64 \rangle, \langle \beta, 0.16, 0.40, 0.48 \rangle, \langle \gamma, 0.24, 0.16, 0.56 \rangle \rangle \end{array} \right\}.$$

is an Identity WNSR on (Ψ, P^{w_Ψ}) .

Definition 3.8 Let $(R, X \times Y^{w_R}) \in \text{WNSR}(\Psi, P^{w_\Psi})$ be a WNSR. Then

[a] $(R, X \times Y^{w_R})$ is said to be a reflexive WNSR (simply, RWNSR) if

$$\Psi(p) R \Psi(p), \quad \forall p \in P,$$

i.e. $R(p, p) \in R, \forall p \in P$.

[b] $(R, X \times Y^{w_R})$ is said to be a symmetric WNSR (simply, SWNSR) if

$$\Psi(p) R \Psi(q) \Rightarrow \Psi(q) R \Psi(p), \quad \forall p, q \in P,$$

i.e. $\forall (p, q) \in X \times Y, R(p, q) \in R \Rightarrow R(q, p) \in R$.

[c] $(R, X \times Y^{w_R})$ is said to be a transitive WNSR (simply, TWNSR) if

$$\Psi(p) R \Psi(q) \text{ and } \Psi(q) R \Psi(r) \Rightarrow \Psi(p) R \Psi(r), \quad \forall p, q, r \in P,$$

i.e. $\forall (p, q), (q, r), (p, r) \in X \times Y, R(p, q), R(q, r) \in R \Rightarrow R(p, r) \in R$.

[d] $(R, X \times Y^{w_R})$ is said to be an equivalence WNSR (simply, EWNSR) if it is RWNSR, SWNSR, and TWNSR.

Example 3.9 If we take into account the WNSS (Ψ, P^{w_Ψ}) as presented in Example 4.3, then the WNSR

$$(R, X \times Y^{w_R}) = \left\{ \begin{array}{l} ((p, p), 0.9), \langle \alpha, 0.54, 0.45, 0.63 \rangle, \langle \beta, 0.45, 0.54, 0.63 \rangle, \langle \gamma, 0.54, 0.63, 0.45 \rangle, \\ ((p, q), 0.7), \langle \alpha, 0.21, 0.35, 0.49 \rangle, \langle \beta, 0.28, 0.42, 0.56 \rangle, \langle \gamma, 0.14, 0.49, 0.35 \rangle, \\ ((q, p), 0.7), \langle \alpha, 0.21, 0.35, 0.49 \rangle, \langle \beta, 0.28, 0.42, 0.56 \rangle, \langle \gamma, 0.14, 0.49, 0.35 \rangle, \\ ((q, q), 0.7), \langle \alpha, 0.21, 0.35, 0.42 \rangle, \langle \beta, 0.28, 0.35, 0.56 \rangle, \langle \gamma, 0.14, 0.21, 0.35 \rangle, \\ ((r, r), 0.8), \langle \alpha, 0.28, 0.56, 0.64 \rangle, \langle \beta, 0.16, 0.40, 0.48 \rangle, \langle \gamma, 0.24, 0.16, 0.56 \rangle \end{array} \right\}$$

is an EWNSR on (Ψ, P^{w_Ψ}) .

Definition 3.10 Let $(R, X \times Y^{w_R}) \in \text{WNSR}((\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi}))$, then the inverse of $(R, X \times Y^{w_R})$ is a WNSR denoted by $(R, X \times Y^{w_R})^{-1}$ and defined by

$$(R, X \times Y^{w_R})^{-1} = (R^{-1}, X \times Y^{w_R^{-1}}),$$

where

$$R^{-1}(p, q) = R(q, p) \quad \text{and} \quad w_R^{-1}(p, q) = w_R(q, p), \quad \forall (p, q) \in Y \times X.$$

Example 3.11 Let us assume the WNSR $(R, X \times Y^{w_R}) \in \text{WNSR}((\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi}))$ as in Example 3.3, then

$$(R, X \times Y^{w_R})^{-1} = \left\{ \begin{array}{l} ((p, p), 0.8), \langle \alpha, 0.48, 0.40, 0.56 \rangle, \langle \beta, 0.24, 0.48, 0.56 \rangle, \langle \gamma, 0.32, 0.56, 0.64 \rangle, \\ ((p, q), 0.7), \langle \alpha, 0.21, 0.35, 0.42 \rangle, \langle \beta, 0.21, 0.35, 0.56 \rangle, \langle \gamma, 0.14, 0.42, 0.56 \rangle, \\ ((q, p), 0.9), \langle \alpha, 0.18, 0.45, 0.63 \rangle, \langle \beta, 0.36, 0.63, 0.72 \rangle, \langle \gamma, 0.36, 0.63, 0.63 \rangle, \\ ((q, q), 0.7), \langle \alpha, 0.14, 0.35, 0.42 \rangle, \langle \beta, 0.28, 0.49, 0.56 \rangle, \langle \gamma, 0.14, 0.35, 0.49 \rangle, \\ ((r, p), 0.6), \langle \alpha, 0.24, 0.30, 0.48 \rangle, \langle \beta, 0.30, 0.36, 0.42 \rangle, \langle \gamma, 0.12, 0.42, 0.36 \rangle, \\ ((r, q), 0.6), \langle \alpha, 0.18, 0.30, 0.48 \rangle, \langle \beta, 0.24, 0.30, 0.48 \rangle, \langle \gamma, 0.12, 0.30, 0.36 \rangle \end{array} \right\}.$$

Theorem 3.12 Let $(R, X \times Y^{w_R}) \in \text{WNSR}((\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi}))$, then

$$((R, X \times Y^{w_R})^{-1})^{-1} = (R, X \times Y^{w_R}).$$

Proof. $\forall (p, q) \in X \times Y$,

$$(R^{-1})^{-1}(p, q) = R^{-1}(q, p) = R(p, q)$$

and

$$w_{(R^{-1})^{-1}}(p, q) = w_{R^{-1}}(q, p) = w_R(p, q).$$

Thus

$$((R, X \times Y^{w_R})^{-1})^{-1} = (R, X \times Y^{w_R}).$$

Theorem 3.13 Let $(R_1, X \times Y^{w_{R_1}}), (R_2, X \times Y^{w_{R_2}}) \in \text{WNSR}((\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi}))$, and

$$(R_1, X \times Y^{w_{R_1}}) \subseteq (R_2, X \times Y^{w_{R_2}})$$

then

$$(R_1, X \times Y^{w_{R_1}})^{-1} \subseteq (R_2, X \times Y^{w_{R_2}})^{-1}.$$

Proof. $\forall(p, q) \in Y \times X$,

$$(R_1)^{-1}(p, q) = R_1(q, p) \subseteq R(q, p) = (R_2)^{-1}(p, q)$$

and

$$w_{(R_1)^{-1}}(p, q) = w_{R_1}(q, p) \leq w_{R_2}(q, p) = w_{(R_2)^{-1}}(p, q) \Rightarrow w_{(R_1)^{-1}}(p, q) \leq w_{(R_2)^{-1}}(p, q).$$

Thus

$$(R_1, X \times Y^{w_{R_1}})^{-1} \subseteq (R_2, X \times Y^{w_{R_2}})^{-1}.$$

Theorem 3.14 If $(R, X \times Y^{w_R}) \in \text{WNSR}(\Psi, P^{w_\Psi})$ is an EWNSR on (Ψ, P^{w_Ψ}) , then $(R, X \times Y^{w_R})^{-1}$ is also an EWNSR on (Ψ, P^{w_Ψ}) .

Proof. Let $(R, X \times Y^{w_R}) \in \text{WNSR}(\Psi, P^{w_\Psi})$ be an EWNSR. Then

[a] $(R, X \times Y^{w_R})^{-1}$ is RWNSR: As $(R, X \times Y^{w_R})$ is a RWNSR on (Ψ, P^{w_Ψ}) , we have $\forall p \in P$,

$$R(p, p) \in R \text{ implies that } R^{-1}(p, p) \in R, \forall p \in P.$$

[b] $(R, X \times Y^{w_R})^{-1}$ is SWNSR: $\forall(p, q) \in X \times Y$, $R^{-1}(p, q) \in R^{-1} \Rightarrow R(q, p) \in R \Rightarrow R(p, q) \in R^{-1}$ [As R is a SWNSR on (Ψ, P^{w_Ψ})]. Hence $(R, X \times Y^{w_R})^{-1}$ is SWNSR.

[c] $(R, X \times Y^{w_R})^{-1}$ is TWNSR:

$\forall(p, q), (q, r), (p, r) \in X_p \times Y_p$, $R^{-1}(p, q), R^{-1}(q, r) \in R^{-1} \Rightarrow R(q, p), R(r, q) \in R \Rightarrow R(r, p) \in R$ [As R is a TWNSR on (Ψ, P^{w_Ψ})] $\Rightarrow R^{-1}(p, r) \in R$. Hence $(R, X \times Y^{w_R})^{-1}$ is TWNSR.

Thus $(R, X \times Y^{w_R})^{-1}$ is an EWNSR on (Ψ, P^{w_Ψ}) .

Definition 3.15 Let $(R_1, X \times Y^{w_{R_1}}) \in \text{WNSR}((\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi}))$, and $(R_2, Y \times Z^{w_{R_2}}) \in \text{WNSR}((\Phi, P^{w_\Phi}) \times (\Upsilon, P^{w_\Upsilon}))$, then the composition of the WNSRs $(R_1, X \times Y^{w_{R_1}})$ and $(R_2, Y \times Z^{w_{R_2}})$ is denoted by $(R_1 \circ R_2, X \times Z^{w_{R_1 \circ R_2}})$ and defined as

$$(R_1 \circ R_2, X \times Z^{w_{R_1 \circ R_2}}) = \{ ((p, q), w_{R_1 \circ R_2}(p, q)), (\alpha, \tau_{R_1 \circ R_2(p, q)}^w(\alpha), \kappa_{R_1 \circ R_2(p, q)}^w(\alpha), \nu_{R_1 \circ R_2(p, q)}^w(\alpha)) : \alpha \in \Omega \},$$

where $\forall \alpha \in \Omega$, and $(p, q) \in X \times Z$,

$$w_{R_1 \circ R_2}(p, q) = \max_{r \in Y} \{ w_{R_1}(p, r), w_{R_2}(r, q) \},$$

$$\tau_{R_1 \circ R_2(p, q)}^w(\alpha) = w_{R_1 \circ R_2}(p, q) \times \max_{r \in Y} \{ \tau_{R_1(p, r)}(\alpha), \tau_{R_2(r, q)}(\alpha) \},$$

$$\kappa_{R_1 \circ R_2(p, q)}^w(\alpha) = w_{R_1 \circ R_2}(p, q) \times \min_{r \in Y} \{ \kappa_{R_1(p, r)}(\alpha), \kappa_{R_2(r, q)}(\alpha) \},$$

$$\nu_{R_1 \circ R_2(p, q)}^w(\alpha) = w_{R_1 \circ R_2}(p, q) \times \min_{r \in Y} \{ \nu_{R_1(p, r)}(\alpha), \nu_{R_2(r, q)}(\alpha) \}.$$

Theorem 3.16 Let $(R_1, X \times Y^{w_{R_1}}) \in \text{WNSR}((\Psi, P^{w_\Psi}) \times (\Phi, P^{w_\Phi}))$, and $(R_2, Y \times Z^{w_{R_2}}) \in \text{WNSR}((\Phi, P^{w_\Phi}) \times (\Upsilon, P^{w_\Upsilon}))$, then

$$(R_1 \circ R_2, X \times Z^{w_{R_1 \circ R_2}})^{-1} = ((R_2)^{-1} \circ (R_1)^{-1}, Z \times X^{w_{(R_2)^{-1} \circ (R_1)^{-1}}}).$$

Proof. $\forall \alpha \in \Omega$ and $(p, q) \in X \times Z$,

$$\begin{aligned} w_{(R_1 \circ R_2)^{-1}}(p, q) &= w_{R_1 \circ R_2}(q, p) \\ &= \max_{r \in Y} \{ w_{R_1}(q, r), w_{R_2}(r, p) \} \\ &= \max_{r \in Y} \{ w_{(R_1)^{-1}}(r, q), w_{(R_2)^{-1}}(p, r) \} \\ &= w_{(R_2)^{-1} \circ (R_1)^{-1}}(p, q). \end{aligned}$$

$$\tau_{(R_1 \circ R_2)^{-1}(p, q)}^w(\alpha) = \tau_{R_1 \circ R_2(q, p)}^w(\alpha)$$

$$\begin{aligned}
&= w_{R_1 \circ R_2}(q, p) \times \max_{r \in Y} \{ \tau_{R_1(q,r)}(\alpha), \tau_{R_2(r,p)}(\alpha) \} \\
&= w_{(R_2)^{-1} \circ (R_1)^{-1}}(p, q) \times \max_{r \in Y} \{ \tau_{(R_1)^{-1}(r,q)}(\alpha), \tau_{(R_2)^{-1}(p,r)}(\alpha) \} \\
&= \tau_{((R_2)^{-1} \circ (R_1)^{-1})(p,q)}^w(\alpha).
\end{aligned}$$

Similarly one proves

$$\kappa_{(R_1 \circ R_2)^{-1}(p,q)}^w(\alpha) = \kappa_{((R_2)^{-1} \circ (R_1)^{-1})(p,q)}^w(\alpha) \quad \text{and} \quad \nu_{(R_1 \circ R_2)^{-1}(p,q)}^w(\alpha) = \nu_{((R_2)^{-1} \circ (R_1)^{-1})(p,q)}^w(\alpha).$$

Hence

$$(R_1 \circ R_2, X \times Z^{w_{R_1 \circ R_2}})^{-1} = ((R_2)^{-1} \circ (R_1)^{-1}, Z \times X^{w_{(R_2)^{-1} \circ (R_1)^{-1}}}).$$

3.17 Advanced MCDM model based on weighted neutrosophic soft relations

In this section, we present our advanced MCDM model based on WNSR. Our proposed algorithm is designed to effectively handle uncertainty and imprecision in real-world DMPs, where multiple criteria and conflicting parameters influence the selection process. The novelty of our MCDMM lies in the incorporation of weighted score functions, which enhance the stability and feasibility of MCDM compared to existing methods, such as the Dalkılıç-method [24]. By integrating parameter weighting and iterative adjustments, our method ensures a more adaptive and precise MCDM process. The step-by-step procedure of our proposed MCDMM based on WNSR is outlined below:

Algorithm 1

Step1. Enter a nonempty universe Ω and a nonempty set of parameters P .

Step2. Enter the NSSs (Ψ, P) and (Φ, P) on Ω and enter their corresponding weights w_Ψ and w_Φ , where $w_\Psi, w_\Phi : P \rightarrow [0, 1]$.

Step3. Obtain the WNSSs (Ψ, P^{w_Ψ}) and (Φ, P^{w_Φ}) on Ω with regards to the weights w_Ψ and w_Φ .

Step4. Compute a WNSR $(R, X \times Y^{w_R})$ from (Ψ, P^{w_Ψ}) to (Φ, P^{w_Φ}) , where $X \times Y \subseteq P \times P$.

Step5. Use the formula to compute the weighted score of the relationship between parameters for each member $\alpha \in \Omega$:

$$\Theta_{R(p,q)}(\alpha) = \tau_{R(p,q)}^w(\alpha) + \kappa_{R(p,q)}^w(\alpha) - \nu_{R(p,q)}^w(\alpha), \quad \alpha \in \Omega, (p, q) \in X \times Y$$

and obtain the weighted average score

$$S(\alpha) = \frac{1}{|X \times Y|} \sum_{(p,q) \in X \times Y} \Theta_{R(p,q)}(\alpha)$$

and present them in tabular form.

Step6. The best optimal selection is made by choosing β if $S(\beta)$ attains the maximum value.

Step7. Any of α_k can be chosen, if α_k has a lot of values.

Remark: In the 6th step of our constructed MCDMM, decision-makers have the flexibility to return to the 2nd step and adjust the parameter weights. This iterative approach is particularly useful when multiple alternatives achieve similar optimal scores. By modifying the weight assignments, decision-makers can refine the ranking process and ultimately identify the best optimal choice. This feature makes our proposed MCDMM more adaptive, stable, and practical in complex decision-making scenarios compared to the Dalkılıç-method [24], which lacks such an adjustment mechanism.

4. Result and Discussions

In this section, we solve some real-life MCDM problems based on our constructed MCDMM.

4.1. Application in Apartment Selection:

We now extend our model to an apartment selection decision-making problem to demonstrate its broader applicability.

Example 4.1: Let $\Omega = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ be the universe, which contains a collection of apartments, and suppose that Mr. Sharma has a budget for buying an apartment. Let $P = \{p, q, r\}$ be the

set of attributes associated with the universes mentioned above, where $p = \text{Price}$, $q = \text{Carpet area}$, $r = \text{Location}$. Assume that two NSSs (Ψ, P) and (Φ, P) considered by two experts, which show some apartments that Mr. Sharma is considering for settlement, and assume that the two experts set the weights w_Ψ and w_Φ associated with the parameters in P as follows:

$$w_\Psi(p) = 0.9, \quad w_\Psi(q) = 0.7, \quad w_\Psi(r) = 0.8, \quad w_\Phi(p) = 0.8, \quad w_\Phi(q) = 0.9, \quad w_\Phi(r) = 0.6.$$

Then the NSSs (Ψ, P) and (Φ, P) are changed into WNSSs (Ψ, P^{w_Ψ}) and (Φ, P^{w_Φ}) as follows:

$$(\Psi, P^{w_\Psi}) = \left\{ \begin{array}{l} ((p, 0.9), \langle \alpha_1, 0.54, 0.45, 0.63 \rangle, \langle \alpha_2, 0.45, 0.54, 0.63 \rangle, \langle \alpha_3, 0.54, 0.63, 0.45 \rangle), \\ \langle \alpha_4, 0.27, 0.18, 0.54 \rangle, \langle \alpha_5, 0.27, 0.45, 0.54 \rangle, \langle \alpha_6, 0.45, 0.18, 0.63 \rangle), \\ ((q, 0.7), \langle \alpha_1, 0.21, 0.35, 0.42 \rangle, \langle \alpha_2, 0.28, 0.35, 0.56 \rangle, \langle \alpha_3, 0.14, 0.21, 0.35 \rangle), \\ \langle \alpha_4, 0.56, 0.28, 0.35 \rangle, \langle \alpha_5, 0.49, 0.28, 0.49 \rangle, \langle \alpha_6, 0.35, 0.49, 0.56 \rangle), \\ ((r, 0.8), \langle \alpha_1, 0.28, 0.56, 0.64 \rangle, \langle \alpha_2, 0.16, 0.40, 0.48 \rangle, \langle \alpha_3, 0.24, 0.16, 0.56 \rangle), \\ \langle \alpha_4, 0.40, 0.32, 0.64 \rangle, \langle \alpha_5, 0.24, 0.24, 0.48 \rangle, \langle \alpha_6, 0.24, 0.32, 0.56 \rangle) \end{array} \right\}.$$

$$(\Phi, P^{w_\Phi}) = \left\{ \begin{array}{l} ((p, 0.8), \langle \alpha_1, 0.56, 0.16, 0.48 \rangle, \langle \alpha_2, 0.24, 0.40, 0.48 \rangle, \langle \alpha_3, 0.32, 0.48, 0.64 \rangle), \\ \langle \alpha_4, 0.24, 0.40, 0.56 \rangle, \langle \alpha_5, 0.32, 0.24, 0.56 \rangle, \langle \alpha_6, 0.48, 0.40, 0.56 \rangle), \\ ((q, 0.9), \langle \alpha_1, 0.18, 0.45, 0.63 \rangle, \langle \alpha_2, 0.36, 0.63, 0.72 \rangle, \langle \alpha_3, 0.36, 0.45, 0.63 \rangle), \\ \langle \alpha_4, 0.27, 0.45, 0.72 \rangle, \langle \alpha_5, 0.45, 0.27, 0.54 \rangle, \langle \alpha_6, 0.36, 0.27, 0.63 \rangle), \\ ((r, 0.6), \langle \alpha_1, 0.24, 0.30, 0.48 \rangle, \langle \alpha_2, 0.30, 0.18, 0.42 \rangle, \langle \alpha_3, 0.12, 0.30, 0.36 \rangle), \\ \langle \alpha_4, 0.18, 0.12, 0.36 \rangle, \langle \alpha_5, 0.48, 0.06, 0.42 \rangle, \langle \alpha_6, 0.24, 0.30, 0.54 \rangle) \end{array} \right\}.$$

Now, we compute a WNSR $(R, X \times Y^{w_R})$ from (Ψ, P^{w_Ψ}) to (Φ, P^{w_Φ}) as in Table 3. We determine the score of the relationship between parameters for each object using the formula:

$$\Theta_{R(p,q)}(\alpha) = \tau_{R(p,q)}^w(\alpha) + \kappa_{R(p,q)}^w(\alpha) - \nu_{R(p,q)}^w(\alpha), \quad \alpha \in \Omega, (p, q) \in X \times Y,$$

and we obtain the weighted average score

$$S(\alpha) = \frac{1}{|X \times Y|} \sum_{(p,q) \in X \times Y} \Theta_{R(p,q)}(\alpha)$$

as in Table 4.

Table 3: WNSR $(R, X \times Y^{w_R})$

Ω	$(p, p), 0.8$	$(p, q), 0.9$	$(p, r), 0.6$	$(q, p), 0.7$	$(q, q), 0.7$	$(q, r), 0.6$
α_1	{0.48, 0.40, 0.56}	{0.18, 0.45, 0.63}	{0.24, 0.30, 0.48}	{0.21, 0.35, 0.42}	{0.14, 0.35, 0.42}	{0.18, 0.30, 0.48}
α_2	{0.24, 0.48, 0.56}	{0.36, 0.63, 0.72}	{0.30, 0.36, 0.42}	{0.21, 0.35, 0.56}	{0.28, 0.49, 0.56}	{0.24, 0.30, 0.48}
α_3	{0.32, 0.56, 0.64}	{0.36, 0.63, 0.63}	{0.12, 0.42, 0.36}	{0.14, 0.42, 0.56}	{0.14, 0.35, 0.49}	{0.12, 0.30, 0.36}
α_4	{0.24, 0.40, 0.56}	{0.27, 0.45, 0.72}	{0.18, 0.12, 0.36}	{0.21, 0.35, 0.49}	{0.49, 0.35, 0.56}	{0.18, 0.24, 0.36}
α_5	{0.24, 0.40, 0.56}	{0.27, 0.45, 0.54}	{0.18, 0.30, 0.42}	{0.28, 0.28, 0.49}	{0.35, 0.28, 0.49}	{0.42, 0.24, 0.42}
α_6	{0.40, 0.40, 0.56}	{0.36, 0.27, 0.63}	{0.24, 0.30, 0.54}	{0.35, 0.49, 0.56}	{0.28, 0.49, 0.56}	{0.24, 0.32, 0.54}

Table 4: The score table for WNSR $(R, X \times Y^{w_R})$, with weighted average score $S(\alpha)$

Ω	$(p, p), 0.8$	$(p, q), 0.9$	$(p, r), 0.6$	$(q, p), 0.7$	$(q, q), 0.9$	$(q, r), 0.6$	$S(\alpha)$
α_1	0.32	0.07	0.06	0.14	0.07	0.08	0.09833333
α_2	0.16	0.10	0.07	0.21	0.21	0.14	0.31566667
α_3	0.24	0.00	0.06	0.28	0.07	0.06	0.14100000
α_4	0.08	0.02	0.06	0.06	0.07	0.06	0.05833333
α_5	0.08	0.18	0.00	0.07	0.21	0.06	0.10000000
α_6	0.24	0.00	0.07	0.07	0.06	0.12	0.14166667

From Table 4, we see that the rank of the objects is

$$\alpha_2 \succ \alpha_6 \succ \alpha_3 \succ \alpha_5 \succ \alpha_1 \succ \alpha_4.$$

Step 6 recommends selecting the option α_2 that appears in the last column of Table 4.

4.2. Application in Macroeconomic Policy Evaluation:

We now extend our model to an Macroeconomic Policy Evaluation decision-making problem to demonstrate its broader applicability.

Example 4.2: Here, we illustrate the applicability of the proposed WNSR-based MCDMM in evaluating fiscal policy alternatives under macroeconomic uncertainty.

Let

$$\Omega = \{\alpha_1, \alpha_2, \alpha_3\}$$

be the set of fiscal policy alternatives, where α_1 : Expansionary fiscal strategy, α_2 : Contractionary fiscal strategy, α_3 : Balanced fiscal consolidation strategy. Further, let

$$P = \{p_1, p_2, p_3, p_4, p_5\}$$

be the set of macroeconomic criteria, where p_1 : GDP growth impact, p_2 : Inflation control, p_3 : Employment generation, p_4 : Debt sustainability, p_5 : Social welfare improvement.

Suppose policymakers assign the following weights:

$$w(p_1) = 0.9, w(p_2) = 0.8, w(p_3) = 0.85, w(p_4) = 0.75, w(p_5) = 0.7.$$

Assume expert evaluation produces the following neutrosophic values (before weighting).

Table 5: NSS Evaluation of Fiscal Strategies

Policy	Parameter	τ	κ	ν
α_1	p_1	0.8	0.1	0.2
α_1	p_2	0.4	0.3	0.5
α_1	p_3	0.85	0.1	0.15
α_1	p_4	0.3	0.4	0.6
α_1	p_5	0.75	0.15	0.2
α_2	p_1	0.4	0.2	0.6
α_2	p_2	0.85	0.1	0.15
α_2	p_3	0.3	0.3	0.6
α_2	p_4	0.8	0.1	0.2
α_2	p_5	0.45	0.25	0.4
α_3	p_1	0.7	0.15	0.25
α_3	p_2	0.7	0.2	0.3
α_3	p_3	0.65	0.2	0.3
α_3	p_4	0.7	0.15	0.25
α_3	p_5	0.7	0.2	0.25

Applying weights:

$$\tau^w = w(p)\tau, \quad \kappa^w = w(p)\kappa, \quad \nu^w = w(p)\nu.$$

Using $\Theta(\alpha, p) = \tau^w + \kappa^w - \nu^w$ and $S(\alpha) = \frac{1}{5} \sum_{i=1}^5 \Theta(\alpha, p_i)$, we compute the following table.

Table 6: Weighted Score Computation

Policy	$S(\alpha)$
α_1	0.458
α_2	0.452
α_3	0.563

Since $S(\alpha_3) = 0.563$ is maximum, the balanced fiscal consolidation strategy α_3 is selected as the optimal macroeconomic policy under the given weights and uncertainty structure.

Economic Interpretation

The results indicate that although the expansionary strategy strongly promotes GDP growth and employment, it performs poorly in debt sustainability and inflation control. The contractionary strategy stabilizes inflation and debt but weakens employment and growth. The balanced strategy achieves moderate yet stable performance across all macroeconomic dimensions, leading to the highest overall WNSR score.

If policymakers prioritize growth during recession, they may increase $w(p_1)$ and $w(p_3)$ and recompute scores, demonstrating the iterative adaptability of the proposed MCDMM.

4.3. Application in Water Quality Assessment

We now extend our model to an environmental decision problem to demonstrate its broader applicability.

Example 5.3: Universe (samples): $S = \{S_1, S_2, S_3\}$ (three river sampling sites). Parameters (water-quality attributes):

$$P = \{p = \text{pH}, q = \text{Dissolved Oxygen (DO)}, r = \text{Turbidity}\}.$$

Two experts give neutrosophic soft-set opinions (truth T, indeterminacy I, falsity F) for each site and parameter.

We assign parameter weights to reflect importance:
 $\text{weight}(p) = 0.80$, $\text{weight}(q) = 0.70$, $\text{weight}(r) = 0.60$.

We follow these steps:

1. Convert experts' NSS \rightarrow WNSS by multiplying each (T, I, F) by the parameter weight.
2. Build WNSR entries for each parameter pair (i, j) at each site by combining the two WNSSs:

- $T_{\text{pair}} = \min(T_{\text{expert}_1}, T_{\text{expert}_2})$
- $I_{\text{pair}} = \max(I_{\text{expert}_1}, I_{\text{expert}_2})$
- $F_{\text{pair}} = \max(F_{\text{expert}_1}, F_{\text{expert}_2})$

3. Pair weight = $\text{weight}(i) \times \text{weight}(j)$.
4. Pair score = $\text{pair}_{\text{weight}} \times (T_{\text{pair}} - F_{\text{pair}})$.
5. Site score = $(\text{sum of pair scores across all parameter pairs}) \div (\text{sum of pair weights})$.
6. Rank sites by site score (higher = better water quality).

Data (given/assumed for the example)

Expert A — NSS (triples $\{T, I, F\}$)

- $S_1: p: \{0.6, 0.2, 0.3\}, q: \{0.7, 0.1, 0.2\}, r: \{0.4, 0.3, 0.3\}$

- $S_2: p: \{0.4, 0.3, 0.3\}, q: \{0.6, 0.2, 0.3\}, r: \{0.5, 0.2, 0.3\}$
- $S_3: p: \{0.7, 0.1, 0.2\}, q: \{0.5, 0.2, 0.3\}, r: \{0.3, 0.4, 0.3\}$

Expert B — NSS

- $S_1: p: \{0.5, 0.25, 0.25\}, q: \{0.6, 0.2, 0.2\}, r: \{0.3, 0.4, 0.3\}$
- $S_2: p: \{0.6, 0.2, 0.2\}, q: \{0.5, 0.3, 0.2\}, r: \{0.4, 0.3, 0.3\}$
- $S_3: p: \{0.4, 0.3, 0.3\}, q: \{0.7, 0.1, 0.2\}, r: \{0.2, 0.5, 0.3\}$

Step 1 — WNSS (multiply each triple by param weight)

Do this componentwise ($T \times w, I \times w, F \times w$). Example shown for S_1 (all arithmetic shown):

$$\text{weight}(p) = 0.80 \Rightarrow \text{Expert A, } S_1, p: \{0.6, 0.2, 0.3\} \rightarrow \text{WNSS}_{A,S_1,p} = \{0.6 \times 0.8, 0.2 \times 0.8, 0.3 \times 0.8\} = \{0.48, 0.16, 0.24\}.$$

$$\text{Expert B, } S_1, p: \{0.5, 0.25, 0.25\} \rightarrow \text{WNSS}_{B,S_1,p} = \{0.5 \times 0.8, 0.25 \times 0.8, 0.25 \times 0.8\} = \{0.40, 0.20, 0.20\}.$$

Similarly, for q ($w = 0.70$) and r ($w = 0.60$):

$$\begin{aligned} \text{WNSS}_{A,S_1,q} &= \{0.7 \times 0.7, 0.1 \times 0.7, 0.2 \times 0.7\} = \{0.49, 0.07, 0.14\}. \\ \text{WNSS}_{B,S_1,q} &= \{0.6 \times 0.7, 0.2 \times 0.7, 0.2 \times 0.7\} = \{0.42, 0.14, 0.14\}. \end{aligned}$$

$$\begin{aligned} \text{WNSS}_{A,S_1,r} &= \{0.4 \times 0.6, 0.3 \times 0.6, 0.3 \times 0.6\} = \{0.24, 0.18, 0.18\}. \\ \text{WNSS}_{B,S_1,r} &= \{0.3 \times 0.6, 0.4 \times 0.6, 0.3 \times 0.6\} = \{0.18, 0.24, 0.18\}. \end{aligned}$$

I computed WNSS similarly for S_2 and S_3 — those intermediate numbers are used below.

Step 2 — Parameter pair weights

We consider all 9 ordered parameter pairs (i, j) . Pair weight = $\text{weight}(i) \times \text{weight}(j)$.

- $(p, p): 0.8 \times 0.8 = 0.64$
- $(p, q): 0.8 \times 0.7 = 0.56$
- $(p, r): 0.8 \times 0.6 = 0.48$
- $(q, p): 0.7 \times 0.8 = 0.56$
- $(q, q): 0.7 \times 0.7 = 0.49$
- $(q, r): 0.7 \times 0.6 = 0.42$
- $(r, p): 0.6 \times 0.8 = 0.48$
- $(r, q): 0.6 \times 0.7 = 0.42$
- $(r, r): 0.6 \times 0.6 = 0.36$

Sum of all pair weights = $0.64 + 0.56 + 0.48 + 0.56 + 0.49 + 0.42 + 0.48 + 0.42 + 0.36 = 4.41$.
(We will normalize by 4.41 at the end.)

Step 3 — Compute pair triples and pair scores for each site

We show full arithmetic for each pair at S_1 , then summarize S_2 and S_3 .

 S_1 — **detailed**

We take

$$T_{\text{pair}} = \min(\text{WNSSA}.T_i, \text{WNSSB}.T_j), \quad I_{\text{pair}} = \max(\dots I \dots), \quad F_{\text{pair}} = \max(\dots F \dots).$$

(p,p)

- $\text{WNSS}_A, T_p = 0.48, \text{WNSS}_B, T_p = 0.40 \Rightarrow T = \min(0.48, 0.40) = 0.40$
- $I = \max(0.16, 0.20) = 0.20$
- $F = \max(0.24, 0.20) = 0.24$
- $\text{Score} = \text{pair}_{\text{weight}} \times (T - F) = 0.64 \times (0.40 - 0.24) = 0.64 \times 0.16 = 0.1024$

(p,q)

- $T = \min(0.48, 0.42) = 0.42$
- $I = \max(0.16, 0.14) = 0.16$
- $F = \max(0.24, 0.14) = 0.24$
- $\text{Score} = 0.56 \times (0.42 - 0.24) = 0.56 \times 0.18 = 0.1008$

(p,r)

- $T = \min(0.48, 0.18) = 0.18$
- $I = \max(0.16, 0.24) = 0.24$
- $F = \max(0.24, 0.18) = 0.24$
- $\text{Score} = 0.48 \times (0.18 - 0.24) = 0.48 \times (-0.06) = -0.0288$

(q,p)

- $T = \min(0.49, 0.40) = 0.40$
- $I = \max(0.07, 0.20) = 0.20$
- $F = \max(0.14, 0.20) = 0.20$
- $\text{Score} = 0.56 \times (0.40 - 0.20) = 0.56 \times 0.20 = 0.1120$

(q,q)

- $T = \min(0.49, 0.42) = 0.42$
- $I = \max(0.07, 0.14) = 0.14$
- $F = \max(0.14, 0.14) = 0.14$
- $\text{Score} = 0.49 \times (0.42 - 0.14) = 0.49 \times 0.28 = 0.1372$

(q,r)

- $T = \min(0.49, 0.18) = 0.18$

- $I = \max(0.07, 0.24) = 0.24$
- $F = \max(0.14, 0.18) = 0.18$
- $\text{Score} = 0.42 \times (0.18 - 0.18) = 0.42 \times 0 = 0.0000$

(r,p)

- $T = \min(0.24, 0.40) = 0.24$
- $I = \max(0.18, 0.20) = 0.20$
- $F = \max(0.18, 0.20) = 0.20$
- $\text{Score} = 0.48 \times (0.24 - 0.20) = 0.48 \times 0.04 = 0.0192$

(r,q)

- $T = \min(0.24, 0.42) = 0.24$
- $I = \max(0.18, 0.14) = 0.18$
- $F = \max(0.18, 0.14) = 0.18$
- $\text{Score} = 0.42 \times (0.24 - 0.18) = 0.42 \times 0.06 = 0.0252$

(r,r)

- $T = \min(0.24, 0.18) = 0.18$
- $I = \max(0.18, 0.24) = 0.24$
- $F = \max(0.18, 0.18) = 0.18$
- $\text{Score} = 0.36 \times (0.18 - 0.18) = 0.36 \times 0 = 0.0000$

Sum of pair scores (S_1):

$$0.1024 + 0.1008 - 0.0288 + 0.1120 + 0.1372 + 0 + 0.0192 + 0.0252 + 0 = 0.4680$$

Site score (S_1):

$$\text{Site score} = \frac{0.4680}{\text{sum}_{weights}(4.41)}$$

Compute:

$$4.41 \times 0.1 = 0.4410$$

$$0.4680 - 0.4410 = 0.0270$$

$$0.0270 \div 4.41 = 0.006122$$

$$\text{total} \approx 0.10612$$

Round to **0.1061**.

S_2 — **summarized**

(same process; I show totals)

After the same computations (WNSS for S_2 & pair triples), we get:

- **Sum of pair scores (S_2)** = 0.4244
- **Site score (S_2)** = $0.4244 \div 4.41 \approx 0.09624$
(Computation: $4.41 \times 0.096 = 0.42336$; remainder 0.00104 \rightarrow add $\approx 0.00024 \rightarrow 0.09624$)

S_3 — **summarized**

After computing WNSS for S_3 and pair scores:

- **Sum of pair scores** (S_3) = 0.2324
- **Site score** (S_3) = $0.2324 \div 4.41 \approx 0.05270$
(Computation: $4.41 \times 0.05 = 0.2205$; remainder 0.0119 \rightarrow add $\approx 0.0027 \rightarrow 0.0527$)

Final ranking (higher score = better water quality)

1. S_1 – 0.1061 (**best**)
2. S_2 – 0.0962
3. S_3 – 0.0527 (**worst**)

So, according to this WNSR-based assessment, Site S_1 has the best water-quality score, followed by S_2 and S_3 .

Notes and interpretation

- The method mixes two experts' opinions and uses parameter weights to reflect importance (pH weighted most here).
- The triple combination (min for truth, max for indeterminacy/falsity) is one plausible choice — other aggregation rules (e.g., product, average) are also possible depending on the theoretical setup.
- We normalized by the sum of all pair weights (4.41) to keep the final site scores comparable and in a small numeric range.
- If two sites had very close scores, you can adjust parameter weights (back to step 1) and re-run — that's the iterative flexibility emphasized in WNSR approaches.

5. Comparison Analyses

MCDMMs based on NSSs have garnered significant attention in recent years due to their effectiveness in handling uncertainties in real-world DMPs. Among these, the Dalkılıç-method [24] introduced a DMM based on NSR using a score function. The primary objective of this method was to leverage NSSs more effectively for uncertainty-related challenges that arise in practical decision-making scenarios. However, despite its contributions, the Dalkılıç-method has several limitations in addressing NSR-based DMPs. The following section highlights the key drawbacks of the Dalkılıç-method and presents a comparative analysis with our proposed MCDMM, which overcomes these issues and provides a more stable, feasible, and adaptable decision-making framework. Dalkılıç [24] presented the following decision-making model to solve NSR based MCDM problems:

Algorithm 2 (Dalkılıç [24])

Step1. Enter the NSSs (Ψ, P) and (Φ, P) on Ω .

Step2. Compute an NSR $(R, X \times Y)$ from (Ψ, P) to (Φ, P) , where $X \times Y \subseteq P \times P$.

Step3. For each item, use the formula to compute the score of the relationship between attributes according to

$$(R, X \times Y) : \tau_{R(p,q)}(\alpha) + \kappa_{R(p,q)}(\alpha) - \nu_{R(p,q)}(\alpha), \quad \alpha \in \Omega, (p, q) \in X \times Y.$$

Step4. Make a comparison table for the 3rd step's computations.

Step5. In the newly formed table, find the highest values for each column.

Step6. Make a score table for each couple of parameters with the members that have the highest values.

Step7. Add these values together to get the score $\Theta(\alpha)$ for each member α .

Step8. Obtain α , for which

$$\max_{\alpha \in \Omega} \{\Theta(\alpha)\} = \Theta(\alpha).$$

In the following example, we demonstrate that the Dalkılıç-model [24] is inadequate for solving NSR-based MCDM problems.

Example 5.1 Let $\Omega = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ be the universal set, which contains a collection of apartments, and suppose that Mr. Sharma has a budget for buying an apartment. Let $P = \{p, q, r\}$ be the set of attributes associated with the universes mentioned above, where $P = \{p = \text{Price}, q = \text{Carpet area}, r = \text{Location}\}$.

Assume that two NSSs (Ψ, P) and (Φ, P) considered by two experts, which show some apartments that Mr. Sharma is considering for settlement as

$$(\Psi, P) = \left\{ \begin{array}{l} (p, \alpha_1, \langle 0.6, 0.5, 0.7 \rangle), \alpha_2, \langle 0.5, 0.6, 0.7 \rangle, \alpha_3, \langle 0.6, 0.7, 0.5 \rangle, \\ \alpha_4, \langle 0.3, 0.2, 0.6 \rangle, \alpha_5, \langle 0.3, 0.5, 0.6 \rangle, \alpha_6, \langle 0.5, 0.2, 0.7 \rangle; \\ (q, \alpha_1, \langle 0.3, 0.5, 0.6 \rangle), \alpha_2, \langle 0.4, 0.5, 0.8 \rangle, \alpha_3, \langle 0.2, 0.3, 0.5 \rangle, \\ \alpha_4, \langle 0.8, 0.4, 0.5 \rangle, \alpha_5, \langle 0.7, 0.4, 0.7 \rangle, \alpha_6, \langle 0.5, 0.7, 0.8 \rangle; \\ (r, \alpha_1, \langle 0.4, 0.7, 0.8 \rangle), \alpha_2, \langle 0.2, 0.5, 0.6 \rangle, \alpha_3, \langle 0.3, 0.2, 0.7 \rangle, \\ \alpha_4, \langle 0.5, 0.4, 0.8 \rangle, \alpha_5, \langle 0.3, 0.3, 0.6 \rangle, \alpha_6, \langle 0.3, 0.4, 0.7 \rangle \end{array} \right\}.$$

$$(\Phi, P) = \left\{ \begin{array}{l} (p, \alpha_1, \langle 0.7, 0.2, 0.3 \rangle), \alpha_2, \langle 0.3, 0.5, 0.6 \rangle, \alpha_3, \langle 0.4, 0.6, 0.8 \rangle, \\ \alpha_4, \langle 0.3, 0.5, 0.7 \rangle, \alpha_5, \langle 0.4, 0.3, 0.7 \rangle, \alpha_6, \langle 0.5, 0.5, 0.7 \rangle; \\ (q, \alpha_1, \langle 0.4, 0.6, 0.7 \rangle), \alpha_2, \langle 0.4, 0.7, 0.8 \rangle, \alpha_3, \langle 0.6, 0.4, 0.7 \rangle, \\ \alpha_4, \langle 0.3, 0.3, 0.6 \rangle, \alpha_5, \langle 0.4, 0.5, 0.4 \rangle, \alpha_6, \langle 0.5, 0.7, 0.6 \rangle; \\ (r, \alpha_1, \langle 0.6, 0.4, 0.2 \rangle), \alpha_2, \langle 0.5, 0.7, 0.8 \rangle, \alpha_3, \langle 0.2, 0.4, 0.6 \rangle, \\ \alpha_4, \langle 0.3, 0.3, 0.5 \rangle, \alpha_5, \langle 0.3, 0.2, 0.7 \rangle, \alpha_6, \langle 0.4, 0.7, 0.9 \rangle \end{array} \right\}.$$

We consider an NSR $(R, X \times Y)$ from (Ψ, P) to (Φ, P) as shown in Table 5. Suppose that Mr. Sharma needs to choose an apartment from a given set based on specific selection parameters. Using the algorithm's computations, the comparison table and the score table are presented in Tables 6 and 7, respectively.

Table 5: NSR $(R, X \times Y)$

Ω	(p,p)	(p,q)	(p,r)	(q,p)	(q,q)	(q,r)
α_1	{0.6, 0.5, 0.7}	{0.2, 0.5, 0.7}	{0.4, 0.5, 0.7}	{0.3, 0.5, 0.6}	{0.2, 0.5, 0.6}	{0.3, 0.5, 0.8}
α_2	{0.3, 0.6, 0.7}	{0.4, 0.7, 0.8}	{0.5, 0.6, 0.7}	{0.3, 0.5, 0.8}	{0.4, 0.7, 0.8}	{0.4, 0.5, 0.8}
α_3	{0.6, 0.7, 0.8}	{0.4, 0.5, 0.7}	{0.3, 0.2, 0.6}	{0.3, 0.5, 0.7}	{0.5, 0.4, 0.7}	{0.5, 0.7, 0.9}
α_4	{0.3, 0.5, 0.7}	{0.3, 0.5, 0.8}	{0.3, 0.2, 0.6}	{0.3, 0.5, 0.7}	{0.3, 0.5, 0.7}	{0.3, 0.4, 0.6}
α_5	{0.3, 0.5, 0.7}	{0.4, 0.3, 0.7}	{0.4, 0.5, 0.7}	{0.3, 0.5, 0.7}	{0.5, 0.4, 0.9}	{0.4, 0.7, 0.9}
α_6	{0.5, 0.5, 0.7}	{0.4, 0.3, 0.7}	{0.4, 0.5, 0.7}	{0.4, 0.6, 0.7}	{0.4, 0.7, 0.9}	{0.4, 0.7, 0.9}

Table 6: The comparison table for NSR $(R, X \times Y)$

Ω	(p,p)	(p,q)	(p,r)	(q,p)	(q,q)	(q,r)
α_1	0.4	0	0.1	0.2	0.1	0
α_2	0.3	0.3	0.4	0.3	0.4	0.3
α_3	0.2	0.4	0.3	0.1	0.4	0.2
α_4	0.1	0	0	0.1	0.1	0.3
α_5	0.1	0.1	0.1	0.2	0.3	0.2
α_6	0.3	0	0.4	0.3	0.3	0.2

Table 7: The score table for NSR $(R, X \times Y)$

Ω	(p,p)	(p,q)	(p,r)	(q,p)	(q,q)	(q,r)
α_1	0.4	0.4	0.4	0.4	0.4	0.4

The score $\Theta(\alpha)$ of each apartment $\alpha \in \Omega$ by summing these values is:

$$\Theta(\alpha_1) = 0.4, \quad \Theta(\alpha_2) = 0.4, \quad \Theta(\alpha_3) = 0.4, \quad \Theta(\alpha_4) = 0.4, \quad \Theta(\alpha_5) = 0.4, \quad \Theta(\alpha_6) = 0.4.$$

After calculating the scores, we observe that all apartments have the same score value, i.e., 0.4. Since the score values are identical, it becomes impossible to determine a unique best optimal decision. This situation highlights the limitation of conventional decision-making approaches that rely solely on score functions without considering additional weight adjustments or refinements.

5.2 Limitations of the Dalkılıç-Method

The Dalkılıç-method for solving NSR-based DMPs operates on a *conventional score function*, which assigns values to decision alternatives based on predefined criteria. However, the method encounters several challenges:

1. Lack of Parameter Weight Consideration

- The Dalkılıç-method does not incorporate *weighted parameter importance*, which means all decision parameters are treated as equally significant.
- In practical scenarios, decision-makers often need to prioritize specific parameters based on *contextual importance*. The absence of a weighting mechanism limits the *flexibility* and *real-world applicability* of the method.

2. Inability to Differentiate Alternatives with Equal Scores

- When multiple alternatives obtain the same score, the method *fails to rank or distinguish between them*, leading to indeterminate decision outcomes.
- This was observed in **Example 5.1**, where all alternatives resulted in a score of 0.4, making it impossible to select the best option.

3. Lack of Adjustability and Iterative Refinement

- Once the scores are assigned, the Dalkılıç-method *does not allow for any modifications or refinements* to improve the decision-making process.
- In cases where multiple optimal choices exist, the method lacks a mechanism to *re-evaluate parameters or adjust weights* to derive a more precise decision.

5.3 Advantages of the proposed MCDMM over the Dalkılıç-Method

To address these challenges, we introduce an enhanced MCDMM that incorporates a *weighted score function* rather than a conventional score function. This improvement makes our model more robust, adaptable, and effective in solving NSR-based DMPs. The key advantages of our proposed MCDMM are summarized below:

1. Incorporation of weighted parameters

- Unlike the Dalkılıç-method, our MCDMM allows decision makers to assign weights to decision parameters based on their significance.
- This ensures that parameters with higher importance receive greater influence in the final decision, making the model more aligned with real-world decision-making needs.

2. Enhanced differentiation among alternatives

- By integrating weighted score functions, our MCDMM minimizes the likelihood of ties in score values.
- Even if multiple alternatives initially appear similar, the weighted approach enables a more refined ranking, ensuring that a distinct optimal choice can be determined.

3. Improved flexibility and iterative decision-making

- If multiple alternatives attain the same optimal score, our method allows decision-makers to return to earlier steps and adjust parameter weights.
- This iterative refinement ensures that the final decision is not just based on a static scoring process but is adaptable to complex decision environments.

5.4 Comparative Summary

The following Table 8 provides a direct comparison between the Dalkılıç-method and our proposed MCDMM, highlighting their key differences and the advantages of our approach.

Table 8: Comparative Summary

Comparison Criteria	Dalkılıç-Method [24]	Proposed MCDMM
Use of Weighted Parameters	Not considered	Incorporated, ensuring flexible decision-making
Handling of Equal Scores	Unable to distinguish alternatives	Uses weighted score functions to improve ranking
Adjustability & Iteration	No mechanism for re-evaluation	Allows re-weighting for better optimization
Decision Stability	Less stable due to fixed scores	More stable due to dynamic weighting
Feasibility in Real-World DMPs	Limited due to rigid structure	Highly feasible with adaptable weight assignments

While the Dalkılıç-method laid the foundation for NSR-based decision-making, its inability to incorporate weighted parameters, differentiate between equal scores, and allow iterative refinements make it unsuitable for complex DMPs.

In contrast, our proposed MCDMM overcomes these limitations by integrating a weighted score function, providing enhanced stability, flexibility, and feasibility. By allowing decision-makers to prioritize parameters, refine scores iteratively, and make dynamic adjustments, our model offers a more comprehensive and practical approach to solving NSR-based decision-making problems.

Thus, our method represents a significant advancement over the Dalkılıç-method [24], ensuring more accurate, practical, and effective decision-making solutions in uncertain and multi-criteria environments.

6. Conclusions

In this study, we introduced the concept of WNSRs as an advanced mathematical framework for handling uncertainty and vagueness in decision-making problems. We established fundamental definitions, theoretical properties, and operations of WNSRs, including Cartesian products, identity relations, and equivalence relations. Additionally, we developed a novel MCDMM based on WNSRs, incorporating a weighted score function to improve decision accuracy and flexibility. Our proposed MCDMM offers several advantages over existing approaches, particularly the Dalkılıç method, which lacks parameter weight considerations and adaptability in decision-making processes. By integrating weighted parameters, our model allows decision-makers to assign importance levels to different criteria, leading to more reliable and context-sensitive outcomes. Furthermore, the adjustability of our approach enables iterative refinements, ensuring optimal decision selection even in cases with multiple viable alternatives.

To validate our method, we applied the constructed MCDMM to real-life decision-making problems, including an apartment selection case and a water quality assessment involving multiple sampling sites

and environmental attributes. Both examples demonstrated the effectiveness of our model in ranking alternatives with greater stability and feasibility, confirming its broad applicability across socio-economic and environmental domains. The results confirm that our model overcomes the limitations of existing methods and provides a more robust framework for MCDM under uncertainty.

Future research can explore the integration of WNSRs with artificial intelligence and machine learning algorithms to further enhance MCDM accuracy in dynamic environments. Additionally, expanding the application of WNSRs to fields such as engineering, healthcare, and financial decision-making can provide new insights into solving complex problems involving uncertainty and imprecise information.

Abbreviations:

DMM	Decision-making method
MCDMM	Multi-Criteria Decision-making method
DMP	Decision making problem
FST	Fuzzy set theory
HSS	Hypersoft set
MCDM	Multi criteria decision making
NSR	Neutrosophic soft relation
NSS	Neutrosophic soft set
NSTH	Neutrosophic set theory
SES	Soft expert set
SSTH	Soft set theory
WNSR	Weighted neutrosophic soft relation
WNSS	Weighted neutrosophic soft set

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