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Simplified Extension Zimmermann's Model for Fuzzy Multi-Objective Linear Programming Problems

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ABSTRACT: An important technique for making decisions under uncertainty, especially when balancing multiple objectives, is fuzzy multi-objective linear programming (FMOLP). Zimmermann's max-min model, which maximizes the minimal, satisfied among all fuzzy objectives to ensure equitable results, is one of the most used methods. Considering the importance it holds, this approach frequently results in conservative results and more mathematical complexity, particularly as the number of objectives increases. To overcome this limitation, we first proposed a hybrid ranking function that combines centroid-based and weighted-average measures, providing a more reliable evaluation of fuzzy numbers. Building on this idea, we extended Zimmermann's formulation by introducing a tunable parameter that balances the minimum satisfaction level with the average satisfaction across objectives. This extension enables decision makers to smoothly adjust the trade-off between fairness and efficiency, while the hybrid ranking function ensures a more accurate ranking of fuzzy objectives. The results showed that the created model yields more flexible and interpretable answers than the original formulation when the suggested method applied to a well-known oil plant scenario. Zimmermann's method improved for real-world applications by this work's controlled and computationally simple technique.

Key Words: Fuzzy multi-objective optimization, Zimmermann model, ranking function, membership function, oil production planning.

Contents

1. Introduction

Zimmermann first developed fuzzy multi-objective linear programming in 1978 [1,2]. This method involved converting each objective function into a membership function and then maximizing the minimum amount to be satisfied for all objectives. Many academics have expanded and used this idea extensively. Tiwari et al. (1987) [3] presented an additive model for fuzzy goal programming, for example, while Sakawa (1993) presented interactive techniques to consider decision-maker preferences [4].
In 2007, [5] S. Mohammadreza and others proposed an algorithm for eliminating the difficulties of the Zimmerman method. [6]. Several authors in 2011 put up different strategies to address those problems. Eman H. et al. carried a new method for solving the housing project problem using pentagonal fuzzy

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number [7]. Feyzan Arikan (2013) [8] investigated a multi-objective linear programming problem. A fuzzy mathematical model and a novel solution strategy were put out to address the problem and satisfy the decision maker's objectives for fuzzy objectives.

D. Samir and others [9], designed the objectives are (i) to minimize weight of the structure and (ii) to minimize the vertical deflection at loading point. In 2022 [10] , Chandra Sen's solution was compared with N. Samsun and other proposed statistical methods for addressing fuzzy multi-objective linear programming problems. The usage of Type-2 Fuzzy Sets to solve the Multi-Objective De Novo Programming problem without going over budget proposed by U. Nurullah [11]. Eman H. et al.(2025) utilized heptagonal fuzzy numbers with ranking function for solving multi-objective functions in [12].

In this work, we proposed a simplified form of Zimmermann's model. By maximizing the minimal membership level, the classical formulation offers balanced satisfaction; nevertheless, when additional objectives or constraints are added, it frequently results in computational complexity. This complexity is decreased by the suggested adjustment. The basic idea of Zimmermann's model is maintained in this condensed version, which also makes it easier to handle real-world issues like the production-planning case examined in this study.

Thus, this paper's contribution is a methodological extension that streamlines Zimmermann's methodology while preserving its decision-theoretic fundamentals, rather than just applying the fuzzy multi-objective programming models that are presently in use.

This paper consists of six sections. In Section 2, the principles are presented with basic definitions of fuzzy set theory. Section 3 proposes the pentagonal membership function and its $(\alpha - cut)$ function. The hybrid ranking function proposed in Section 4. A simplified extension of Zimmermann, max-min for fuzzy multi-objective linear programming (FMOLP) problem with application for oil factory problem proposed in Section 5. Finally, Section 6 presents conclusions.

2. Principles with Basic Definitions

This section includes some basic definitions.

Definition 2.1 [13]
In a universe set X, a fuzzy set $\widetilde{\mathcal{A}}(x)$ is a collection of elements where each element's membership $\mathcal{M}_{\widetilde{\mathcal{A}}}(x)$ is not strictly binary $\mathcal{M}_{\widetilde{\mathcal{A}}}(x)$, Where $\mathcal{M}_{\widetilde{\mathcal{A}}}(x) : X \to [0,1]$, but can be partial, unlike crisp sets, where an element either belongs to a set or does not ($\mathcal{M}_{\widetilde{\mathcal{A}}}(x) = 0$ or 1). $\widetilde{\mathcal{A}}(x) = \{(x, \mathcal{M}_{\widetilde{\mathcal{A}}}(x)) \mid x \in X, 0 < \mathcal{M}_{\widetilde{\mathcal{A}}}(x) < 1\}$, to put it another way.

Definition 2.2 [14]
The crisp set ($\alpha-cut$) satisfies the formula:
$\widetilde{\mathcal{A}_{\alpha pent}}(\alpha) = \{x \in X, \mathcal{M}_{\widetilde{\mathcal{A}}}(x) \geq \alpha, \alpha \in [0,1]\}$. Since every ($\alpha-cut$) is by definition an interval, it is easy to define them using their endpoints: $\inf \widetilde{\mathcal{A}_{\alpha pent}}(\alpha) = \widetilde{\mathcal{A}_{\alpha pent}}^{l}(\alpha)$, $\widetilde{\mathcal{A}_{\alpha pent}}^{u}(\alpha) = \sup \widetilde{\mathcal{A}_{\alpha pent}}(\alpha)$

Definition 2.3 [15,16]
If $\widetilde{\mathcal{A}}$ satisfies at least three of the following conditions, we classify it as a fuzzy number in the set of real numbers: An $\widetilde{\mathcal{A}}$ typical fuzzy set is required ($\exists x_0 \in \mathbb{R}, \mathcal{M}_{\widetilde{\mathcal{A}}}(x_0) = 1$).Every fuzzy number's ($\alpha-cut$) must be a closed interval for all $\alpha \in [0,1]$ for the fuzzy number to be a convex set. There must be an upper limit on the support of $\widetilde{\mathcal{A}}$.

Definition 2.4 [4]
A fuzzy multi-objective linear programming FMOLP problem is:

$$\begin{aligned} Max\widetilde{Z}(x) = &\quad (\widetilde{f}_1(x), \widetilde{f}_2(x), \cdots, \widetilde{f}_n(x)) \\ s.to &\quad \widetilde{A}x \leq \widetilde{b} \\ &\quad x \geq 0 \end{aligned}$$

Where: $\widetilde{f}_j(x) = \widetilde{c}_j^T x, j = 1, \cdots, n$
$\widetilde{c}_j, \widetilde{\mathcal{A}}, \widetilde{b}$ *are fuzzy numbers.*

3. Pentagonal Membership Function

Let $\widetilde{\mathcal{A}_{pent}} = (a_1, a_2, a_3, a_4, a_5)$ be pentagonal fuzzy number (PFN); $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \in \mathbb{R}$, The symmetric pentagonal membership function $SPM_{\widetilde{\mathcal{A}_{pent}}}(x)$:

$$SPM_{\widetilde{\mathcal{A}_{pent}}}(x) = \begin{cases} 0 & a_1 > x \\ \frac{1}{2}\frac{(x-a_1)}{a_2-a_1} & a_1 \leq x < a_2 \\ \frac{1}{2}\left(1 + \left(\frac{x-a_2}{a_3-a_2}\right)\right) & a_2 \leq x < a_3 \\ 1 & x = a_3 \\ 1 - \frac{1}{2}\left(\frac{x-a_3}{a_4-a_3}\right) & a_3 \leq x < a_4 \\ \frac{1}{2}\left(1 - \left(\frac{x-a_4}{a_5-a_4}\right)\right) & a_4 \leq x < a_5 \\ 0 & a_5 < x \end{cases} \tag{3.1}$$

3.1. $\alpha-$cut Function

Suppose that $\widetilde{\mathcal{A}_{pent}} = (a_1, a_2, a_3, a_4, a_5)$ is a PFN, $\alpha \in [0, 1]$ The crisp $\alpha-$cut Function $\widetilde{\mathcal{A}_{\alpha pent}}(\alpha)$ can be defined as:

$$\widetilde{\mathcal{A}_{\alpha pent}}(\alpha) = \begin{cases} 2\alpha(a_2 - a_1) + a_1 & \alpha \in [0, 0.5] \\ (2\alpha - 1)(a_3 - a_2) + a_2 & \alpha \in [0.5, 1] \\ (2 - 2\alpha)(a_4 - a_3) + a_3 & \alpha \in [0.5, 1] \\ (1 - 2\alpha)(a_5 - a_4) + a_4 & \alpha \in [0, 0.5] \end{cases} \tag{3.2}$$

3.2. Proposed Hybrid Ranking Function for PFNs

In decision-making under uncertainty, data often comes in imprecise or fuzzy forms. PFNs effectively represent such data, capturing a weighted average with blending coefficient $0 < \gamma < 1$. The proposed Ranking Function is designed to rank PFNs. This hybrid approach ensures a balanced and flexible ranking, suitable for decision-making, risk analysis, and fuzzy data evaluation. Let $\widetilde{\mathcal{A}_{pent}} = (a_1, a_2, a_3, a_4, a_5)$ be a PFN. which are:

- Let $\mathfrak{S}(\widetilde{\mathcal{A}_{pent}})$ is the center of gravity (centroid) ranking function that computes the mean of the five points [17].

$$\mathfrak{S}(\widetilde{\mathcal{A}_{pent}}) = \frac{\int_{a_1}^{a_5} x.SPM_{\widetilde{\mathcal{A}_{pent}}(x)} dx}{\int_{a_1}^{a_5} SPM_{\widetilde{\mathcal{A}_{pent}}(x)} dx} = \frac{a_1, a_2, a_3, a_4, a_5}{5} \tag{3.3}$$

The weighted average method $\mathcal{W}(\widetilde{\mathcal{A}_{pent}})$ that assigns different impotence weights to the elements:

$$\mathcal{W}(\widetilde{\mathcal{A}_{pent}}) = \frac{a_1 + 2a_2 + 3a_3 + 2a_4 + a_5}{9} \tag{3.4}$$

- The Dispersion measure $\mathcal{D}(\widetilde{\mathcal{A}_{pent}})$,to evaluates the uncertainty or fuzziness.

$$\mathcal{D}(\widetilde{\mathcal{A}_{pent}}) = (a_5 - a_1) + (a_4 - a_2) \tag{3.5}$$

- Let $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \geq 0$ and $(\mathfrak{r} + \mathfrak{s} + \mathfrak{t} = 1)$ are parameters that control the importance of each component. Then, the proposed hybrid ranking function for the symmetric pentagonal membership function is:

$$\mathcal{R}_{hybrid}(\widetilde{\mathcal{A}_{pent}}) = \mathfrak{r}.\mathfrak{S}(\widetilde{\mathcal{A}_{pent}}) + \mathfrak{s}.\mathcal{W}(\widetilde{\mathcal{A}_{pent}}) - \mathfrak{t}.\mathcal{D}(\widetilde{\mathcal{A}_{pent}}) \tag{3.6}$$

where:
$\mathfrak{r}$: The weight of the center of the PFN.
$\mathfrak{s}$ The weight of the average mean of the PFN.
$\mathfrak{t}$: The weight of the depression of the PFN.
Since, in a paper we dealt with the symmetrical membership function, therefore the dispersion will suppose ($\mathfrak{t} = 0$) and ($\mathfrak{r} = \mathfrak{s} = 0.5$).

4. Fuzzy Multi-Objective Oil Factory Production Problem (OFPP)

Suppose that the decision variables x_1, x_2 such that:
x_1 : Units of Oil 1 to produce.
x_2 : Units of Oil 2 to produce.
The Fuzzy Objective functions:

$$\begin{aligned} P_1: \quad & Max\ Z_1 = \widetilde{p}_1 x_1 + \widetilde{p}_2 x_2 \quad && \text{(Maximize Profit)} \\ & Max\ Z_2 = \widetilde{q}_1 x_1 + \widetilde{q}_2 x_2 && \text{(Maximize Quality Index)} \\ & Min\ Z_3 = \widetilde{e}_1 x_1 + \widetilde{e}_2 x_2 && \text{(Minimize Environmental Impact)} \end{aligned}$$

Subject to the constraints:

$$\begin{aligned} \widetilde{a}_1 x_1 + \widetilde{a}_2 x_2 \le \widetilde{M} \quad && \text{(Raw Material Constraint)} \\ \widetilde{t}_1 x_1 + \widetilde{t}_2 x_2 \le \widetilde{T} && \text{(Processing Time Constraint)} \\ x_1 \le \widetilde{D}_1, x_2 \le \widetilde{D}_2 && \text{(Demand Constraints)} \\ x_1, x_2 \ge 0 && \text{(Non-negativity condition)} \end{aligned}$$

Where all coefficients are Pentagonal Fuzzy Numbers (PFNs): $\widetilde{c} = (a_1, a_2, a_3, a_4, a_5)$

4.1. Numerical Example for OFPP:

Suppose that P2 be a mathematical multi-objective programming model with all coefficients are pentagonal fuzzy numbers.

$$\begin{aligned} P_2: \quad & MaxZ_1 = (36, 38, 40, 42, 44)x_1 + (27, 29, 30, 31, 33)x_2 && \text{(Profit)} \\ & MaxZ_2 = (0.70, 0.82, 0.90, 0.98, 1.06)x_1 + (0.45, 0.55, 0.60, 0.65, 0.75)x_2 && \text{(Quality)} \\ & MinZ_3 = (2.4, 2.7, 3.0, 3.3, 3.6)x_1 + (0.8, 0.9, 1.0, 1.1, 1.2)x_2 && \text{(Emission)} \\ \text{sub.to} \quad & \\ & (4, 4.5, 5, 5.5, 6)x_1 + (2.6, 2.85, 3, 3.15, 3.4)x_2 \le (230, 235, 240, 245, 250) && \text{(Row material)} \\ & (1.7, 1.9, 2, 2.1, 2.3)x_1 + (0.85, 0.95, 1.0, 1.05, 1.15)x_2 \le (95, 98, 100, 102, 105) && \text{(Processing Time)} \\ & x_1 \le (45, 48, 50, 52, 55) && \text{(Demand constraints)} \\ & x_2 \le (75, 78, 80, 82, 85) && \\ & x_1, x_2 \ge 0 && \end{aligned}$$

We, present two standard and complementary approaches:

1. Proposed Hybrid ranking function for PFN to fast single crisp LP.
2. Zimmermann max-min fuzzy LP using $\alpha-$ cut.

4.1.1. Hybrid Ranking function for PFN approach. In this section, we solve the numerical example for OFPP using the proposed Hybrid ranking function.

Step 1: use proposed hybrid ranking function for PFN eq.(3.6)

$$\mathcal{R}_{hybrid}(\widetilde{\mathcal{A}_{pent}}) = \mathfrak{r}.\mathfrak{S}(\widetilde{\mathcal{A}_{pent}}) + \mathfrak{s}.\mathcal{W}(\widetilde{\mathcal{A}_{pent}}) - \mathfrak{t}.\mathcal{D}(\widetilde{\mathcal{A}_{pent}}),$$

$$\mathfrak{r} = \mathfrak{s} = 0.5, \mathfrak{t} = 0$$

The multi-objective problem become as:

$$\begin{aligned}
Max Z_1 = &\quad 40x_1 + 30x_2 \\
Max Z_2 = &\quad 0.89x_1 + 0.6x_2 \\
Min Z_3 = &\quad 3x_1 + x_2 \\
sub.to &\quad 5x_1 + 3x_2 \leq 240 \\
&\quad 2x_1 + x_2 \leq 100 \\
&\quad x_1 \leq 50 \\
&\quad x_2 \leq 80 \\
&\quad x_1, x_2 \geq 0
\end{aligned}$$

Step 2: convert a multi-objective to the single LP by proposed weighted sum as follows:
Let $w_1 = 0.5$, $w_2 = 0.3$, $w_3 = 0.2$, such that $w_1 + w_2 + w_3 = 1$.
Then, the crisp objective function become:

$$\begin{aligned}
Max\ w = &\quad w_1 * Max Z_1 + w_2 * Max Z_2 - w_3 * Min Z_3 \\
= &\quad 0.5 * (40x_1 + 30x_2) + 0.3 * (0.89x_1 + 0.6x_2) - 0.2 * (3x_1 + x_2) \\
\therefore Max\ w = &\quad 19.667x_1 + 14.98x_2 \\
sub.to &\quad 5x_1 + 3x_2 \leq 240 \\
&\quad 2x_1 + x_2 \leq 100 \\
&\quad 0 \leq x_1 \leq 50 \\
&\quad 0 \leq x_2 \leq 80 \\
&\quad x_1, x_2 \geq 0
\end{aligned}$$

Step 3: Use simplex method to obtain the optimal solution:
$Maxw = 1198.4, x_1 = 0, x_2 = 80$
Then the profit $Z_1 = 30 * 80 = 2400$.
The quality $Z_2 = 0.6 * 80 = 48$.
The emission $Z_3 = 80$.
With these weights, the best crisp compromise produces only product ($x_2 =$ units of oil B) up to market cap.

4.1.2. Zimmermann max-min fuzzy LP approach. The crisp set $\alpha-$ cut eq.(3.2), $\alpha \in [0, 1]$. For any PFN $\utilde{\mathcal{A}}_{pent} = (a_1, a_2, a_3, a_4, a_5)$ define:
$\utilde{\mathcal{A}}_{\alpha pent}(\alpha) = [\utilde{\mathcal{A}}_{\alpha pent}(\alpha)^l, \utilde{\mathcal{A}}_{\alpha pent}(\alpha)^u] = [2\alpha(a_2 - a_1) + a_1, (1 - 2\alpha)(a_5 - a_4) + a_4]$
Zimmermann max-min for Multi-objective linear programming approach [1]: The fuzzy objectives from problem P_2, the (Profit, maximize) Z_1 and (Quality, maximize) Z_2

$$\frac{Z_i(x) - Z_i^{min}(\alpha)}{Z_1^{max}(\alpha) - Z_i^{min}(\alpha)} \geq \theta \;\; i = 1, 2,\; 0 < \theta < 1 \tag{4.1}$$

$$\therefore Z_i(x) - \theta(Z_1^{max}(\alpha) - Z_i^{min}(\alpha)) \geq Z_i^{min}(\alpha),\; i = 1, 2$$

$$Z_1(x) - \theta(Z_1^{max}(\alpha) - Z_1^{min}(\alpha)) \geq Z_1^{min}(\alpha) \tag{4.2}$$

$$Z_2(x) - \theta(Z_2^{max}(\alpha) - Z_2^{min}(\alpha)) \geq Z_2^{min}(\alpha) \tag{4.3}$$

For $Z_3 =$ (emission, minimization)

$$\frac{Z_3^{max}(\alpha) - Z_3(x)}{Z_3^{max}(\alpha) - Z_3^{min}(\alpha)} \geq \theta$$

$$\therefore Z_3(x) + \theta(Z_3^{max}(\alpha) - Z_3^{min}(\alpha)) \leq Z_3^{max}(\alpha) \tag{4.4}$$

The procedure Zimmermann max-min for FMOLP problem P_2 [4]
Step 1: Calculate $\widetilde{\mathcal{A}_{\alpha pent}}(\alpha)^l, \widetilde{\mathcal{A}_{\alpha pent}}(\alpha)^u$ at each fuzzy coefficient in the problem P_2 to convert to the crisp multi-objective LP.
Suppose that $\alpha = 0.5$
$\widetilde{\mathcal{A}_{\alpha pent}}(a_1, a_2, a_3, a_4, a_5)^l = 2\alpha(a_2 - a_1) + a_1,$
$\widetilde{\mathcal{A}_{\alpha pent}}(a_1, a_2, a_3, a_4, a_5)^u = (1 - 2\alpha)(a_5 - a_4) + a_4,$
$\therefore \widetilde{\mathcal{A}_{\alpha pent}}(36, 38, 40, 42, 44)^l = 38 \text{and} \widetilde{\mathcal{A}_{\alpha pent}}(36, 38, 40, 42, 44)^u = 42$
$\widetilde{\mathcal{A}_{\alpha pent}}(27, 29, 30, 31, 33)^l = 29, \widetilde{\mathcal{A}_{\alpha pent}}(27, 29, 30, 31, 33)^u = 31,$
$\widetilde{\mathcal{A}_{\alpha pent}}(0.70, 0.82, 0.90, 0.98, 1.06)^l = 0.82, \widetilde{\mathcal{A}_{\alpha pent}}(0.70, 0.82, 0.90, 0.98, 1.06)^u = 0.98$
And so on for all coefficients of the problem P_2
Step 2: construct the constraints:

$$\begin{aligned}
a^u_{1,\alpha} x_1 + a^u_{2,\alpha} x_2 \leq \mathcal{M}^L_\alpha \qquad & \text{(Raw Material)} \\
t^u_{1,\alpha} x_1 + t^u_{2,\alpha} x_2 \leq T^L_\alpha \qquad & \text{(Processing Time)} \\
0 \leq x_1 \leq D^L_{1,\alpha}, 0 \leq x_2 \leq D^L_{2,\alpha} \qquad & \text{(Demand Constraints)} \\
\text{The multi-objective functions:} & \\
MaxZ_1 = p^m_{1,\alpha} x_1 + p^m_{2,\alpha} x_2 \qquad & \text{(Maximize Profit)} \\
MaxZ_2 = q^m_{1,\alpha} x_1 + q^m_{2,\alpha} x_2 \qquad & \text{(Maximize Quality Index)} \\
MinZ_3 = e^m_{1,\alpha} x_1 + e^m_{2,\alpha} x_2 \qquad & \text{(Minimize Environmental Impact)}
\end{aligned}$$

Where $p^m_{1,\alpha} = \frac{38+42}{2} = 40$, $p^m_{2,\alpha} = \frac{0.82+0.98}{2} = 0.9$, and so on for all coefficients of the objective functions. Then the problem become:

$$\begin{aligned}
MaxZ_1 = & \quad 40x_1 + 30x_2 \\
MaxZ_2 = & \quad 0.9x_1 + 0.6x_2 \\
MinZ_3 = & \quad 3x_1 + x_2
\end{aligned}$$

Subject to:

$$\begin{aligned}
5.5x_1 + 3.15x_2 & \leq 235, \\
2.1x_1 + 1.05x_2 & \leq 98, \\
0 \leq x_1 & \leq 48, \\
0 \leq x_2 & \leq 78
\end{aligned}$$

By simplex method:

$$Z_1^{\min} = Z_2^{\min} = Z_3^{\min} = 0 \tag{4.5}$$

$$Z_1^{\max} = 2238.09, Z_2^{\max} = 44.7619, Z_3^{\max} = 222 \tag{4.6}$$

Step 3: Substitution $Z_i^{\min}, Z_i^{\max}$ in Eqs. (4.2), (4.3), and (4.4), the Zimmermann mathematical model at $\alpha = 0.5$

$$\begin{aligned}
P_3: \quad 40x_1 + 30x_2 - Z_1^{\min} & \geq \theta \left(Z_1^{\max} - Z_1^{\min}\right) \\
0.9x_1 + 0.6x_2 - Z_2^{\min} & \geq \theta \left(Z_2^{\max} - Z_2^{\min}\right) \\
Z_3^{\max} - (3x_1 + x_2) & \geq \theta \left(Z_3^{\max} - Z_3^{\min}\right)
\end{aligned}$$

sub.to

$$\begin{aligned} 5.5x_1 + 3.15x_2 &\leq 235, \\ 2.1x_1 + 1.05x_2 &\leq 98, \\ 0 \leq x_1 &\leq 48, \\ 0 \leq x_2 &\leq 78, \\ 0 \leq \theta &\leq 1. \end{aligned}$$

$$\begin{aligned} P_4: \quad 40x_1 + 30x_2 &\geq 2238.09\theta \\ 0.9x_1 + 0.6x_2 &\geq 44.7619\theta \\ 222 - (3x_1 + x_2)&222\theta \end{aligned}$$

sub.to

$$\begin{aligned} 5.5x_1 + 3.15x_2 &\leq 235, \\ 2.1x_1 + 1.05x_2 &\leq 98, \\ 0 \leq x_1 &\leq 48, \\ 0 \leq x_2 &\leq 78, \\ 0 \leq \theta &\leq 1. \end{aligned}$$

Then, solve the mathematical model by LP algorithm to find the optimal solution as follows:
$x_1 = 0, x_2 = 55.84, \theta = 0.748$
then $Z_1 = 1675.20, Z_2 = 33.504, Z_3 = 55.84$

5. Simplified Extension of Zimmermann Max-Min for Fuzzy Multi-Objective LP Problem

In this section, we depended on proposed ranking function to difuzzification
Step 1: define $f_i^{\min}$, $f_i^{\max}$ for each objective function
Step 2: define $\mu_i(x) = \frac{f_i(x) - f_i^{\min}}{f_i^{\max} - f_i^{\min}}$ for maximum objective function and for minimization used $\mu_i(x) = \frac{f_i^{\max} - f_i(x)}{f_i^{\max} - f_i^{\min}}$
Step 3: the new problem become:
$Max\ \theta\tau + (1-\tau)\,\bar{\mu}$
Sub. to

$$\begin{aligned} \mu_{f_i}(x) \ &\geq \ \theta, \quad j = 1, 2, \ldots, k \\ \mu_{g_i}(x) \ &\geq \ \theta, \quad j = 1, 2, \ldots, k \\ 0 \leq \theta \leq 1, \quad & x \geq 0, \quad \tau \in [0,1] \end{aligned}$$

θ : The minimum satisfaction level, as in the Zimmermann model.
$\bar{\mu} = \frac{1}{k}\sum_{j=1}^{k} \mu_{f_j}(x)$: The overall (average) satisfaction level across all objectives.
$\tau \in [0,1]$:is a balancing parameter that determines the relative importance given level versus the average satisfaction.
The procedure of simplified extension Zimmermann max-min (EZMM):
Step 1: Convert the fuzzy multi-objective problem by proposed hybrid ranking function for fuzzy pen-

tagonal membership function eq. (3.6), to obtain the following:

$$\begin{aligned} MaxZ_1 = &\quad 40x_1 + 30x_2 \\ MaxZ_2 = &\quad 0.89x_1 + 0.6x_2 \\ MinZ_3 = &\quad 3x_1 + x_2 \\ sub.to & \\ & 5x_1 + 3x_2 \leq 240 \\ & 2x_1 + x_2 \leq 100 \\ & x_1 \leq 50 \\ & x_2 \leq 80 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Step 2: Obtain $f_i^{\min}$ and $f_i^{\max}$ by splitting the multi-objective to the three linear programming problems and solve them to find:

$$\begin{aligned} \text{for} \quad MaxZ_1 = 40x_1 + 30x_2, &\quad f_1^{\max} = 2400, \quad f_1^{\min} = 0, \\ MaxZ_2 = 0.89x_1 + 0.6x_2, &\quad f_2^{\max} = 48, \quad f_2^{\min} = 0, \\ MinZ_3 = 3x_1 + x_2, &\quad f_3^{\max} = 144, \quad f_3^{\min} = 0. \end{aligned}$$

Step 3: The EZMM model become:
Max $\theta\tau + (1-\tau)\frac{1}{k}\sum_{j=1}^{k} \mu_{f_j}(x)$
Sub. to

$$\begin{aligned} \mu_i(x) = \frac{f_i(x) - f_i^{\min}}{f_i^{\max} - f_i^{\min}} \geq \theta, \quad i = 1, 2,\ j = 1, 2 \text{ (for max)} \\ \mu_i(x) = \frac{f_i^{\max} - f_i(x)}{f_i^{\max} - f_i^{\min}} \geq \theta, \quad j = 1 \text{ (for min)} \\ 0 \leq \theta \leq 1, \quad x \geq 0, \quad \tau \in [0, 1] \end{aligned}$$

and

$$Max\ \theta\tau + (1-\tau)\frac{1}{3}\sum_{j=1}^{3} \mu_{f_j}(x)$$

Sub. to:

$$\begin{aligned} \frac{Z_1(x) - 0}{2400} \geq \theta &\quad \Rightarrow \quad Z_1(x) \geq 2400\,\theta \\ \frac{Z_2(x) - 0}{48} \geq \theta &\quad \Rightarrow \quad Z_2(x) \geq 48\,\theta \\ \frac{144 - Z_3(x)}{144} \geq \theta &\quad \Rightarrow \quad Z_3(x) \leq 144\,(1-\theta) \\ 0 \leq \theta \leq 1 & \end{aligned}$$

So, the final EZMM mathematical model is:

$$\text{Max } \theta\tau + (1-\tau)\frac{1}{3}\left(\frac{Z_1(x)}{2400} + \frac{Z_2(x)}{48} + \frac{Z_3(x)}{144}\right)$$

Sub. to

$$\begin{aligned} Z_1(x) &\quad \geq 2400\,\theta \\ Z_2(x) &\quad \geq 48\,\theta \\ Z_3(x) &\quad \leq 144\,(1-\theta) \\ & 0 \leq \theta \leq 1 \end{aligned}$$

Where $Z_1(x) = 40x_1 + 30x_2, Z_2(x) = 0.89x_1 + 0.6x_2, Z_3(x) = 3x_1 + x_2$
The optimal solution of the problem as below Table(1) for $\tau = (0, 0.25, 0.5, 0.75, 1)$

Table 1: The optimal solutions of the problem for EZMM.

τ	x_1	x_2	$Z_1(x)$	$Z_2(x)$	$Z_3(x)$	μ_{f_1}	μ_{f_2}	μ_{f_3}	θ	$\bar{\mu}$
0.00	0.00	80.000	2400.00	48.000	80.000	1.000	1.000	0.444	0.000	0.814
0.25	0.00	80.000	2400.00	48.000	80.000	1.000	1.000	0.444	0.000	0.814
0.50	0.00	51.428	1542.857	30.857	51.428	0.642	0.642	0.642	0.642	0.642
0.75	0.00	51.428	1542.857	30.857	51.428	0.642	0.642	0.642	0.642	0.642
1.00	0.00	51.428	1542.857	30.857	51.428	0.642	0.642	0.642	0.642	0.642

Table (1) shows that the trade-off between the minimal satisfaction (fairness across objectives) and the average satisfaction (efficiency across all objectives) is controlled by the parameter (τ).

- In the case ($\tau = 0$) (pure Zimmermann's max–min), the method demonstrates equity by optimizing the minimum level of satisfaction. The maximum profit ($Z_1 = 2400$) and quality ($Z_2 = 48$) are obtained when the solution pushes x_2 to its top bound (80). The emission aim ($Z_3 = 80$) is still comparatively subpar, nevertheless. The total average satisfaction is very high ($\bar{\mu} = 0.815$) in spite of this discrepancy.

- In the case where $\tau = 1$, (pure Zimmermann's max–min): The model maximizes the least level of satisfaction in order to show fairness. All three objectives are approximately balanced in the solution ($\mu_{f_1} = \mu_{f_2} = \mu_{f_3} \approx 0.643$). The total average satisfaction is lower than in the $\tau = 0$ case, but this gives a "compromise" solution where no objective is compromised.

- Moderate numbers (0.25, 0.5, 0.75), cause the solution to gradually go from concentrating on efficiency to making sure that everyone is treated fairly. Although there is a penalty for the reduced emission performance, the result still supports average satisfaction with high profit and quality at $\tau = 0.25$. The solution converges to Zimmermann's method's fair distribution at $\tau = 0.5$ and above.

6. Conclusion

By adding a hybrid ranking function for fuzzy numbers and expanding the traditional formulation with an adjustable parameter τ, this work suggested an improvement to Zimmermann's FMOLP technique. The extended model offered a flexible balance between minimum and average satisfaction levels, while the hybrid ranking function increased the accuracy of fuzzy evaluations. When τ is small, the solution maximizes efficiency by maximizing overall satisfaction, while larger values of τ indicate equal treatment by equalizing membership degrees, as the computational results on the oil factory example showed. This suggests that the proposed model offers more flexibility than the traditional Zimmermann method. Because of its variety, the model is both useful and understandable for making decisions in the real world. It also gives decision-makers a reasonable trade-off mechanism that strengthens the robustness of fuzzy optimization. So this method may be applied to other types of fuzzy numbers.

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