



A New Perspective on Ideal Localized Sequences in \mathcal{A} -Metric Spaces

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ABSTRACT: This study presents and examines the perspectives of \mathcal{I} -localized sequences and \mathcal{I} -Cauchy sequences in the setting of an \mathcal{A} -metric space. We examine the interplay between these notions under suitable conditions. The study further extends to \mathcal{I} -boundedness, where we establish a significant relationship with \mathcal{I} -localized sequences. We define and explore the constructs of \mathcal{I} -limit points and \mathcal{I} -cluster points of sequences, emphasizing their connections with the \mathcal{I} -locator in an \mathcal{A} -metric space. The foundations of an \mathcal{I} -barrier for a sequence is also explored, and we utilize it to draw important links between \mathcal{I} -Cauchy and \mathcal{I} -localized sequences. Furthermore, we establish connections based on the presence of \mathcal{I} -nonthin Cauchy subsequences and thick subsets in the underlying space. Finally, we investigate the relationship between \mathcal{I} -localized and \mathcal{I}^* -localized sequences, offering new insights in the context of \mathcal{A} -metric spaces.

Key Words: \mathcal{A} -metric space, $\mathcal{I}^{\mathcal{A}}$ -localized sequence, $\mathcal{I}^{\mathcal{A}}$ -bounded, \mathcal{I} -limit point, \mathcal{I} -cluster point.

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1. Introduction and preliminaries

The notion of metric spaces, introduced by Fréchet in 1906 [9], forms a cornerstone of mathematics and its applications in science and engineering. Motivated by the need to analyze more complex structures, several generalizations have been proposed. Gähler’s 2-metric spaces [11] encountered continuity issues [18], while Dhage’s D -metric spaces [7] were later shown not to generalize classical metrics [34]. To overcome these limitations, Mustafa and Sims [35] introduced G -metric spaces, which inspired further developments including S -metric spaces [42] and \mathcal{A} -metric spaces [1]. These generalized frameworks provide greater flexibility in defining distance, modeling complex phenomena, and extending the scope of fixed point and summability theories, thereby opening new avenues for research and applications.

Statistical convergence, first hinted at by Zygmund in 1935, was formally introduced by Steinhaus [44] and Fast [8], and later revitalized by Schoenberg [41]. It quickly became an important tool in summability theory, with significant contributions by Fridy [10], Šalát [38], and Connor [5], who further advanced its role in modern analysis. Although rooted in natural density, statistical convergence has only recently gained momentum as a vibrant research area. Its applications now extend across diverse domains, including measure theory [28], approximation theory [4,29,30], topology [27], probability theory [43], locally convex spaces [26], and neutrosophic 2-normed spaces [19].

We now recall some fundamental notions related to ideals. The concept of \mathcal{I} -convergence, introduced by Kostyrko et al. [24] as a generalization of statistical convergence, has since been widely studied [13,31,32,39,40].

A family \mathcal{I} of subsets of a non-empty set \mathcal{U} is called an ideal [24] if (i) $\emptyset \in \mathcal{I}$; (ii) $\mathcal{C}, \mathcal{B} \in \mathcal{I} \implies \mathcal{C} \cup \mathcal{B} \in \mathcal{I}$; (iii) $\mathcal{C} \in \mathcal{I}, \mathcal{B} \subseteq \mathcal{C} \implies \mathcal{B} \in \mathcal{I}$.

An ideal \mathcal{I} is non-trivial if $\mathcal{U} \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$, and admissible if $\{\{x\} : x \in \mathcal{U}\} \subseteq \mathcal{I}$. For instance, the family \mathcal{I}_f of all finite subsets of \mathbb{N} is admissible. Given a non-trivial ideal \mathcal{I} , the class $\mathcal{F}(\mathcal{I}) = \{\mathcal{U} \setminus \mathcal{C} : \mathcal{C} \in \mathcal{I}\}$ forms the associated filter. A sequence $\{\zeta_s\}$ in a metric space (\mathcal{U}, ρ) is

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\mathfrak{I} -convergent to $v \in \mathcal{U}$ if for every $\varpi > 0$, the set $\{s \in \mathbb{N} : \rho(\zeta_s, v) \geq \varpi\} \in \mathfrak{I}$. This concept has since been extended to diverse frameworks, including neutrosophic n -normed linear spaces [20].

Definition 1.1 [24] *An admissible ideal $\mathfrak{I} \subset 2^{\mathbb{N}}$ satisfies condition (AP) if for every countable family of mutually disjoint sets $\{\mathcal{C}_i\}$ in \mathfrak{I} , there exists a family $\{\mathcal{B}_i\}$ such that each $\mathcal{C}_i \Delta \mathcal{B}_i$ is finite and $\bigcup_{i=1}^{\infty} \mathcal{B}_i \in \mathfrak{I}$.*

Ideal convergence has been investigated in numerous generalized settings: partial metric spaces [16], quasi-metric spaces [47], cone metric spaces [50,3], generalized metric spaces [23], quaternion-valued generalized metric spaces [22], asymmetric metric/normed spaces [12,33], and S -metric spaces [2]. Further studies include its development in probabilistic metric spaces [6], intuitionistic fuzzy metric spaces [21] and θ -metric spaces [14].

The concept of an \mathcal{A} -metric space, introduced in [1], extends the classical metric and S -metric structures into an n -dimensional framework. Formally, a function $\mathcal{A} : \mathcal{U}^n \rightarrow [0, \infty)$ on a nonempty set \mathcal{U} is called an \mathcal{A} -metric if it satisfies positivity, identity of indiscernibles, and a generalized triangle type inequality, i.e., if, for all $\vartheta_1, \vartheta_2, \dots, \vartheta_n, \tau \in \mathcal{U}$ the following conditions are satisfied:

1. $\mathcal{A}(\vartheta_1, \vartheta_2, \dots, \vartheta_n) \geq 0$,
2. $\mathcal{A}(\vartheta_1, \vartheta_2, \dots, \vartheta_n) = 0$ if and only if $\vartheta_1 = \vartheta_2 = \dots = \vartheta_n$,
3. The following inequality holds: $\mathcal{A}(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, \vartheta_n) \leq \sum_{i=1}^n \underbrace{\mathcal{A}(\vartheta_i, \vartheta_i, \dots, \vartheta_i, \tau)}_{n-1 \text{ times}}$.

The pair $(\mathcal{U}, \mathcal{A})$ is then called an \mathcal{A} -metric space. Both the usual metric ($n = 2$) and the S -metric ($n = 3$) appear as special cases, placing \mathcal{A} -metrics within a broader hierarchy of generalized metric spaces.

Example 1.1 *Let $\mathcal{U} = \mathbb{R}$, set of all real numbers. Define a function $\mathcal{A} : \mathcal{U}^n \rightarrow [0, \infty)$ by*

$$\mathcal{A}(\vartheta_1, \vartheta_2, \dots, \vartheta_n) = \sum_{i=1}^n \sum_{i < j} |\vartheta_i - \vartheta_j|.$$

Then $(\mathcal{U}, \mathcal{A})$ is a usual \mathcal{A} -metric space.

Lemma 1.1 *Let $(\mathcal{U}, \mathcal{A})$ be an \mathcal{A} -metric space. Then, $\mathcal{A}(\underbrace{\vartheta, \vartheta, \dots, \vartheta}_{n-1 \text{ times}}, \tau) = \mathcal{A}(\tau, \underbrace{\tau, \tau, \dots, \tau}_{n-1 \text{ times}}, \vartheta)$ for all $\vartheta, \tau \in \mathcal{U}$.*

Lemma 1.2 *Let $(\mathcal{U}, \mathcal{A})$ be an \mathcal{A} -metric space. Then, for all $\vartheta, \tau \in \mathcal{U}$ we obtain: $\mathcal{A}(\vartheta, \vartheta, \dots, \vartheta, \sigma) \leq (n-1)\mathcal{A}(\vartheta, \vartheta, \dots, \vartheta, \tau) + \mathcal{A}(\sigma, \sigma, \dots, \sigma, \tau)$ and $\mathcal{A}(\vartheta, \vartheta, \dots, \vartheta, \sigma) \leq (n-1)\mathcal{A}(\vartheta, \vartheta, \dots, \vartheta, \tau) + \mathcal{A}(\tau, \tau, \dots, \tau, \sigma)$.*

Lemma 1.3 *Let $(\mathcal{U}, \mathcal{A})$ be an \mathcal{A} -metric space. Then for all $\vartheta, \tau, \sigma \in \mathcal{U}$*

$$|\mathcal{A}(\vartheta, \vartheta, \dots, \vartheta, \tau) - \mathcal{A}(\tau, \tau, \dots, \tau, \sigma)| \leq (n-1)\mathcal{A}(\vartheta, \vartheta, \dots, \vartheta, \sigma).$$

Key structural properties include symmetry-type identities (Lemmas 1.1–1.2), boundedness criteria, and natural notions of open and closed balls. Convergence and Cauchy conditions are defined analogously to classical settings, with sequences $\{\zeta_s\}$ converging to v when $\mathcal{A}(\zeta_s, \dots, \zeta_s, v) \rightarrow 0$ as $s \rightarrow \infty$.

Recent work has further enriched this setting by introducing statistical convergence and Cauchy conditions in \mathcal{A} -metric spaces [45,46], extending earlier studies in classical, cone, and g -metric frameworks. Within this context, we also define statistical localization: a sequence is statistically localized on $\mathcal{H} \subseteq \mathcal{U}$ if $\mathcal{A}(\zeta_s, \dots, \zeta_s, v)$ is statistically convergent for each $v \in \mathcal{H}$. The maximal such set forms the statistical locator, and sequences localized everywhere play a central role in the analysis that follows.

1.1. Motivation

\mathcal{A} -metric spaces, introduced by Abbas et al. [1], extend the classical notion of metric spaces and have been extensively investigated within fixed point theory. However, their potential in summability theory and generalized convergence remains largely unexplored.

The concept of localized sequences, proposed by Krivonosov [25] in 1974 as a natural generalization of Cauchy sequences, provides a powerful framework for studying convergence through localized sequences and locator properties. A sequence $\{\zeta_n\}$ in a metric space (\mathcal{U}, ρ) is said to be localized on a subset $\mathcal{H} \subseteq \mathcal{U}$ if the sequence $\{\rho(\zeta_n, \zeta)\}$ converges for every $\zeta \in \mathcal{H}$. The maximal such subset \mathcal{H} is called the locator of the sequence. In particular, every Cauchy sequence is localized everywhere.

Building upon this foundation, Nabiev et al. [36,37] introduced the notions of statistically localized sequences and their \mathfrak{I} -extensions. Subsequently, the theory was further enriched by its development in 2-normed and n -normed spaces, as well as in double sequence settings; see, for instance, [48,49,17,15]. More recently, these ideas have been extended to \mathcal{A} -metric spaces [46].

Motivated by these developments, the present work introduces and systematically studies the concept of \mathfrak{I} -localized sequences in \mathcal{A} -metric spaces. We establish their basic properties, define the notion of \mathfrak{I} -locators, and investigate their connections with \mathfrak{I} -Cauchy sequences.

1.2. Key Contributions

In this work, we introduce and systematically develop several new notions in the framework of \mathcal{A} -metric spaces, including \mathfrak{I} -convergence, \mathfrak{I} -Cauchy sequences, \mathfrak{I} -localized sequences, and the \mathfrak{I} -locator of a sequence. We investigate in detail the interrelationships among \mathfrak{I} -convergence, \mathfrak{I} -localized sequences, and \mathfrak{I} -Cauchy sequences, together with the associated concepts of \mathfrak{I} -limit points, \mathfrak{I} -cluster points, and \mathfrak{I} -thin and non-thin subsequences in the setting of \mathcal{A} -metric spaces.

Furthermore, we introduce the notion of \mathfrak{I} -boundedness in \mathcal{A} -metric spaces and establish a fundamental connection between \mathfrak{I} -boundedness and \mathfrak{I} -localized sequences. We also examine important set-theoretic properties of the \mathfrak{I} -locator associated with a sequence in an \mathcal{A} -metric space. Key results concerning \mathfrak{I} -limit points and \mathfrak{I} -cluster points of \mathfrak{I} -localized sequences are derived, highlighting their intrinsic relationship with the \mathfrak{I} -locator of the sequence.

In addition, we introduce the concept of an \mathfrak{I} -barrier for a sequence in an \mathcal{A} -metric space and, by means of this notion, establish a meaningful connection between \mathfrak{I} -Cauchy sequences and \mathfrak{I} -localized sequences. In particular, we show that an $\mathfrak{I}^{\mathcal{A}}$ -localized sequence reduces to an $\mathfrak{I}^{\mathcal{A}}$ -Cauchy sequence whenever its \mathfrak{I} -barrier vanishes.

2. Main results

This section is devoted to presenting our main results. Throughout, we denote the \mathcal{A} -metric space simply by \mathcal{U} , omitting the pair notation $(\mathcal{U}, \mathcal{A})$ for brevity. Unless otherwise specified, \mathfrak{I} will represent a nontrivial admissible ideal. We begin by introducing the foundational definitions necessary for our study. To date, research on \mathcal{A} -metric spaces remains in its infancy, particularly in the context of summability theory. Notions such as ideal convergence and ideal Cauchy sequences have yet to be formally defined within this framework. Therefore, we first establish these fundamental concepts and subsequently introduce our central focus: \mathfrak{I} -localized sequences in \mathcal{A} -metric spaces.

Definition 2.1 Consider $\{\zeta_s\}$ to be a sequence in an \mathcal{A} -metric space $(\mathcal{U}, \mathcal{A})$. Then, $\{\zeta_s\}$ is regarded as \mathfrak{I} -convergent (in short $\mathfrak{I}^{\mathcal{A}}$ -convergence) to $v \in \mathcal{U}$ if, for every $\varpi > 0$ the set $\mathcal{C}(\varpi) = \{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) \geq \varpi\} \in \mathfrak{I}$. In this scenario, we express $\mathfrak{I}^{\mathcal{A}} - \lim \zeta_s = v$ or $\zeta_s \xrightarrow{\mathfrak{I}^{\mathcal{A}}} v$, and v is called $\mathfrak{I}^{\mathcal{A}}$ -limit of $\{\zeta_s\}$.

Definition 2.2 Consider $\{\zeta_s\}$ to be a sequence in an \mathcal{A} -metric space $(\mathcal{U}, \mathcal{A})$. Then, $\{\zeta_s\}$ is defined as \mathfrak{I} -Cauchy sequence (in short $\mathfrak{I}^{\mathcal{A}}$ -Cauchy) if, for every $\varpi > 0$ there can be found $t = t(\varpi) \in \mathbb{N}$ such that the set $\{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \zeta_t) \geq \varpi\} \in \mathfrak{I}$.

Definition 2.3 Consider $\{\zeta_s\}$ to be a sequence in an \mathcal{A} -metric space $(\mathcal{U}, \mathcal{A})$. Then, $\{\zeta_s\}$ is termed \mathfrak{I}^* -convergent to $v \in \mathcal{U}$ if and only if there can be found a set $\mathcal{C} = \{m_1 < m_2 < \dots < m_k < \dots\} \subseteq \mathbb{N}$,

$\mathcal{C} \in \mathcal{F}(\mathcal{I})$ such that $\mathcal{A} - \lim_{k \rightarrow \infty} \zeta_{m_k} = v$, i.e., the subsequence $\{\zeta_s\}_{s \in \mathcal{C}}$ is convergent to $v \in \mathcal{U}$ in relation to \mathcal{A} . In this scenario, we express $\zeta_s \xrightarrow{\mathcal{I}^* - \mathcal{A}} v$.

Definition 2.4 Consider $\{\zeta_s\}$ to be a sequence in an \mathcal{A} -metric space $(\mathcal{U}, \mathcal{A})$. Then, $\{\zeta_s\}$ is referred to as \mathcal{I}^* -Cauchy sequence if there can be found a set $\mathcal{C} = \{m_1 < m_2 < \dots < m_k < \dots\} \in \mathcal{F}(\mathcal{I})$ such that the subsequence $\{\zeta_s\}_{s \in \mathcal{C}}$ is a Cauchy sequence in \mathcal{U} with respect to \mathcal{A} , i.e., for each preassigned $\varpi > 0$ there exists $k_0 \in \mathbb{N}$ such that $\mathcal{A}(\zeta_{m_k}, \zeta_{m_k}, \dots, \zeta_{m_k}, \zeta_{m_t}) < \varpi$ for all $k, t \geq k_0$.

Definition 2.5 Consider $\{\zeta_s\}$ to be a sequence in an \mathcal{A} -metric space $(\mathcal{U}, \mathcal{A})$. Then, $\{\zeta_s\}$ is defined as \mathcal{I} -localized sequence in the set $\mathcal{H} \subseteq \mathcal{U}$ with respect to \mathcal{A} ($\mathcal{I}^{\mathcal{A}}$ -localized) if the number sequence $\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)$ is \mathcal{I} -convergent for each $v \in \mathcal{H}$.

Guided by the nature of the subset \mathcal{H} , we introduce the notions below.

1. The maximal subset \mathcal{H} of \mathcal{U} on which the sequence $\{\zeta_s\}$ is \mathcal{I} -localized in relation to \mathcal{A} , is defined as \mathcal{I} -locator of $\{\zeta_s\}$. Henceforth, it will be denoted by $loc_{\mathcal{I}^{\mathcal{A}}}(\zeta_s)$.
2. The sequence $\{\zeta_s\}$ in \mathcal{U} is defined as \mathcal{I} -localized everywhere if $loc_{\mathcal{I}^{\mathcal{A}}}(\zeta_s)$ is the whole set \mathcal{U} .
3. The sequence $\{\zeta_s\}$ in \mathcal{U} is termed \mathcal{I} -localized in itself in relation to \mathcal{A} if the set $\{s \in \mathbb{N} : s \in loc_{\mathcal{I}^{\mathcal{A}}}(\zeta_s)\} \in \mathcal{F}(\mathcal{I})$.

Definition 2.6 Consider $\{\zeta_s\}$ to be a sequence in an \mathcal{A} -metric space $(\mathcal{U}, \mathcal{A})$. Then, $\{\zeta_s\}$ is referred to as \mathcal{I}^* -localized sequence in the set \mathcal{U} if the number sequence $\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)$ is \mathcal{I}^* -convergent for each $v \in \mathcal{U}$ with respect to \mathcal{A} .

Next, we explore the connections between \mathcal{I} -convergence, \mathcal{I} -localized sequences, and \mathcal{I} -Cauchy sequences in the context of \mathcal{A} -metric spaces.

Proposition 2.1 In an \mathcal{A} -metric space \mathcal{U} , every \mathcal{I} -Cauchy sequence is \mathcal{I} -localized everywhere.

Proof: Let $\{\zeta_s\}$ is an $\mathcal{I}^{\mathcal{A}}$ -Cauchy sequence. Then, for every $\varpi > 0$ there can be found $t = t(\varpi) \in \mathbb{N}$ such that the set $\mathcal{C} = \left\{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \zeta_t) \geq \frac{\varpi}{n-1}\right\} \in \mathcal{I}$, $n(\neq 1) \in \mathbb{N}$. Let $v \in \mathcal{U}$ be arbitrary. From Lemma 1.3, we obtain:

$$|\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \mathcal{A}(v, v, \dots, v, \zeta_t)| \leq (n-1)\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \zeta_t).$$

This turns into

$$\{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \mathcal{A}(v, v, \dots, v, \zeta_t)| \geq \varpi\} \subseteq \mathcal{C}.$$

As, $\mathcal{C} \in \mathcal{I}$ the set $\{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \mathcal{A}(v, v, \dots, v, \zeta_t)| \geq \varpi\} \in \mathcal{I}$. As a result, the number sequence $\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)$ is \mathcal{I} -convergent, since $\mathcal{A}(v, v, \dots, v, \zeta_t)$ remains constant. Given that $v \in \mathcal{U}$ is arbitrary, it follows that the sequence $\{\zeta_s\}$ is \mathcal{I} -localized everywhere. This completes the proof. \square

Theorem 2.1 Every $\mathcal{I}^{\mathcal{A}}$ -convergent sequence $\{\zeta_s\}$ in \mathcal{U} is also $\mathcal{I}^{\mathcal{A}}$ -Cauchy.

Proof: Suppose that $\{\zeta_s\}$ is $\mathcal{I}^{\mathcal{A}}$ -convergent to $v \in \mathcal{U}$. Let $n \in \mathbb{N}$ be fixed. Then, for every $\varpi > 0$ the set $\mathcal{C} = \left\{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) < \frac{\varpi}{n}\right\} \in \mathcal{F}(\mathcal{I})$. So, we can choose an element t from \mathcal{C} . Thus, $\mathcal{A}(\zeta_t, \zeta_t, \dots, \zeta_t, v) < \frac{\varpi}{n}$. Using Lemma 1.1 and 1.3, we obtain:

$$\begin{aligned} \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \zeta_t) &\leq (n-1)\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) + \mathcal{A}(v, v, \dots, v, \zeta_t) \\ &= (n-1)\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) + \mathcal{A}(\zeta_t, \zeta_t, \dots, \zeta_t, v) \\ &< (n-1)\frac{\varpi}{n} + \frac{\varpi}{n} \\ &= \varpi. \end{aligned}$$

This results in $\mathcal{C} \subseteq \{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \zeta_t) < \varpi\} \in \mathcal{F}(\mathcal{I})$. Hence $\{\zeta_s\}$ is an $\mathcal{I}^{\mathcal{A}}$ -Cauchy sequence. This completes the proof. \square

Corollary 2.1 *Combining Proposition 2.1 with Theorem 2.1, we conclude that every $\mathfrak{I}^{\mathcal{A}}$ -convergent sequence is \mathfrak{I} -localized everywhere.*

Given that \mathfrak{I} is an admissible ideal, it follows that every localized sequence is \mathfrak{I} -localized in the context of \mathcal{A} .

We now introduce the notion of \mathfrak{I} -boundedness in \mathcal{A} -metric spaces, which plays a pivotal role in the forthcoming result, and establish a significant connection with \mathfrak{I} -localized sequences.

Definition 2.7 *A sequence $\{\zeta_s\}$ in an \mathcal{A} -metric space \mathcal{U} is defined as \mathfrak{I} -bounded ($\mathfrak{I}^{\mathcal{A}}$ -bounded) if there can be found $v \in \mathcal{U}$ and $\varpi > 0$ in such a way that $\{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) > \varpi\} \in \mathfrak{I}$.*

Proposition 2.2 *Every $\mathfrak{I}^{\mathcal{A}}$ -localized sequence is an $\mathfrak{I}^{\mathcal{A}}$ -bounded in \mathcal{U} .*

Proof: Let $\{\zeta_s\}$ is an $\mathfrak{I}^{\mathcal{A}}$ -localized sequence in \mathcal{U} . Then, the number sequence $\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)$ is $\mathfrak{I}^{\mathcal{A}}$ -convergent for each $v \in \mathcal{U}$. Let the number sequence is $\mathfrak{I}^{\mathcal{A}}$ -convergent to $\sigma \in [0, \infty)$. Also, let $\varpi > 0$ be given. Then, by the condition the set $\{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \sigma| > \varpi\} \in \mathfrak{I}$. This yields that

$$\{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \sigma > \varpi\} \cup \{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \sigma < -\varpi\} \in \mathfrak{I}.$$

This turns into $\{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) > \sigma + \varpi\} \in \mathfrak{I}$. Hence, $\{\zeta_s\}$ is an $\mathfrak{I}^{\mathcal{A}}$ -bounded sequence in \mathcal{U} . This completes the proof. \square

We now proceed to explore certain set-theoretic property of the set $loc_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$, as presented in the following theorem.

Theorem 2.2 *Suppose \mathfrak{I} satisfies the condition (AP). Consider $\{\zeta_s\}$ to be a sequence in an \mathcal{A} -metric space \mathcal{U} . Let $\tau \in \mathcal{U}$ be selected in such a way that for any $\varpi > 0$ there can be found $v \in loc_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$ for which the set*

$$\{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau)| \geq \varpi\} \in \mathfrak{I}. \quad (2.1)$$

Then, $\tau \in loc_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$.

Proof: Suppose that $\mathcal{H} = loc_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$ and $v \in loc_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$ satisfying the Equation (2.1). Also, let

$$\mathcal{C} = \{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau)| \geq \varpi\}.$$

Then, complement of \mathcal{C} in \mathbb{N} , i.e., $\mathcal{B} = \mathbb{N} \setminus \mathcal{C} \in \mathcal{F}(\mathcal{I})$. Hence, $\mathcal{B} \neq \emptyset$. Thus, for all $s \in \mathcal{B}$,

$$|\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau)| < \varpi.$$

As $v \in loc_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$, the number sequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)\}_{s \in \mathbb{N}}$ is \mathfrak{I} -convergent. Hence, it is \mathfrak{I} -Cauchy sequence. Due to the condition (AP) satisfied by \mathfrak{I} , the number sequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)\}_{s \in \mathbb{N}}$ is \mathfrak{I}^* -Cauchy sequence. Then there exists a set $\mathcal{M} \subseteq \mathbb{N}$, $\mathcal{M} \in \mathcal{F}(\mathcal{I})$ such that the number sequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)\}_{s \in \mathcal{M}}$ is an ordinary Cauchy sequence, i.e., there exists $s_0 \in \mathbb{N}$ such that

$$|\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \mathcal{A}(\zeta_t, \zeta_t, \dots, \zeta_t, v)| < \varpi$$

for all $s, t \geq s_0$ and $s, t \in \mathcal{M}$. Let $\mathcal{D} = \mathcal{B} \cap \mathcal{M}$. So, $\mathcal{D} \in \mathcal{F}(\mathcal{I})$. Now, for $a, b \in \mathcal{D}$ and $a, b \geq s_0$ we obtain:

$$\begin{aligned} & |\mathcal{A}(\zeta_a, \zeta_a, \dots, \zeta_a, \tau) - \mathcal{A}(\zeta_b, \zeta_b, \dots, \zeta_b, \tau)| \\ & \leq |\mathcal{A}(\zeta_a, \zeta_a, \dots, \zeta_a, \tau) - \mathcal{A}(\zeta_a, \zeta_a, \dots, \zeta_a, v)| + |\mathcal{A}(\zeta_a, \zeta_a, \dots, \zeta_a, v) - \mathcal{A}(\zeta_b, \zeta_b, \dots, \zeta_b, v)| \\ & \quad + |\mathcal{A}(\zeta_b, \zeta_b, \dots, \zeta_b, v) - \mathcal{A}(\zeta_b, \zeta_b, \dots, \zeta_b, \tau)| \\ & < \varpi + \varpi + \varpi \\ & = 3\varpi. \end{aligned}$$

Therefore, the number subsequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau)\}_{s \in \mathcal{D}}$ is an ordinary Cauchy sequence. So, it is convergent. Hence, the number sequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau)\}_{s \in \mathbb{N}}$ is \mathfrak{I}^* -convergent. Due to the admissibility of \mathfrak{I} , the number sequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau)\}_{s \in \mathbb{N}}$ is \mathfrak{I} -convergent. Thus, the sequence $\{\zeta_s\}_{s \in \mathbb{N}}$ is \mathfrak{I} -localized and $\tau \in \text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$. This completes the proof. \square

We now turn our attention to defining the concepts of \mathfrak{I} -limit points, \mathfrak{I} -cluster points, as well as \mathfrak{I} -thin and non-thin subsequences within the framework of \mathcal{A} -metric spaces.

Definition 2.8 Consider $\{\zeta_s\}$ to be a sequence in an \mathcal{A} -metric space \mathcal{U} . Let $v \in \mathcal{U}$. Then, v

1. is defined as \mathfrak{I} -limit point of $\{\zeta_s\}$ if there can be found a set $\mathcal{C} \subseteq \mathbb{N}$, $\mathcal{C} \notin \mathfrak{I}$ such that the subsequence $\{\zeta_s\}_{s \in \mathcal{C}}$ is convergent to v with respect to \mathcal{A} .
2. is termed \mathfrak{I} -cluster point of $\{\zeta_s\}$ if, for every $\varpi > 0$ the set $\{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) < \varpi\} \notin \mathfrak{I}$.

In what follows, we use the notations $\Lambda_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$ and $\Gamma_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$ to denote the set of all \mathfrak{I} -limit points and \mathfrak{I} -cluster points, respectively, of the sequence $\{\zeta_s\}$ in the space \mathcal{U} .

Definition 2.9 Let $\mathcal{C} = \{m_1 < m_2 < \dots < m_s < \dots\} \subseteq \mathbb{N}$. If $\mathcal{C} \in \mathfrak{I}$, then the subsequence $\{\zeta_{m_s}\}$ is called an \mathfrak{I} -thin subsequence of $\{\zeta_s\}$ with respect to \mathcal{A} . On the other hand, if $\mathcal{C} \notin \mathfrak{I}$, the subsequence is referred to as an \mathfrak{I} -nonthin subsequence of $\{\zeta_s\}$ in the space \mathcal{U} .

We now establish a significant result concerning the sets $\Lambda_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$ and $\Gamma_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$, as well as the notion of $\mathfrak{I}^{\mathcal{A}}$ -convergence, an essential step that will serve as a foundation for developing our core results.

Theorem 2.3 Consider $\{\zeta_s\}$ to be a sequence in an \mathcal{A} -metric space \mathcal{U} such that $\mathfrak{I}^{\mathcal{A}} - \lim \zeta_s = v$. Then, $\Lambda_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s) = \Gamma_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s) = \{v\}$.

Proof: Assume, if possible, that $\tau \in \Lambda_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$ with $\tau \neq v$. Then, by the given condition, there exist two subsets of \mathbb{N} defined as follows:

$$\begin{aligned} \mathcal{C}_1 &= \{m_1 < m_2 < \dots < m_s < \dots\} \quad \text{with } \mathcal{C}_1 \notin \mathfrak{I}, \\ \mathcal{C}_2 &= \{m_1 < m_2 < \dots < m_t < \dots\} \quad \text{with } \mathcal{C}_2 \notin \mathfrak{I}, \end{aligned}$$

such that the corresponding subsequences $\{\zeta_{m_s}\}$ and $\{\zeta_{m_t}\}$ converge to v and τ , respectively, with respect to \mathcal{A} . Choose $\varpi > 0$. Then, there can be found a positive integer t_0 in such a way that

$$\mathcal{A}(\zeta_{m_t}, \zeta_{m_t}, \dots, \zeta_{m_t}, \tau) < \varpi \quad \text{for all } t > t_0.$$

Define

$$\mathcal{M}_1 = \{m_t \in \mathcal{C}_2 : \mathcal{A}(\zeta_{m_t}, \zeta_{m_t}, \dots, \zeta_{m_t}, \tau) \geq \varpi\}.$$

Then, $\mathcal{M}_1 \subseteq \{m_1, m_2, \dots, m_{t_0}\}$. Due to admissibility of \mathfrak{I} , $\mathcal{M}_1 \in \mathfrak{I}$. Define

$$\mathcal{M}_2 = \{m_t \in \mathcal{C}_2 : \mathcal{A}(\zeta_{m_t}, \zeta_{m_t}, \dots, \zeta_{m_t}, \tau) < \varpi\}.$$

Clearly, $\mathcal{M}_2 \notin \mathfrak{I}$. For, if $\mathcal{M}_2 \in \mathfrak{I}$, then $\mathcal{C}_2 = \mathcal{M}_1 \cup \mathcal{M}_2$ would also belong to \mathfrak{I} , a contradiction to our assumption. It is given that $\mathfrak{I}^{\mathcal{A}} - \lim \zeta_s = v$. Then, the set

$$\mathcal{H} = \{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) \geq \varpi\} \in \mathfrak{I}.$$

Since $\tau \neq v$, it follows that $\mathcal{M}_2 \cap \mathcal{H}^c = \emptyset$, and hence $\mathcal{M}_2 \subseteq \mathcal{H}$. Given that $\mathcal{H} \in \mathfrak{I}$, we deduce $\mathcal{M}_2 \in \mathfrak{I}$, a contradiction to our earlier conclusion. Therefore, we conclude that

$$\Lambda_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s) = \{v\}.$$

Next, suppose that $\sigma \in \Gamma_{\mathfrak{J}}^{\mathcal{A}}(\zeta_s)$ with $\sigma \neq v$. Then, consider the sets

$$\begin{aligned}\mathcal{B}_1 &= \left\{ s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) < \frac{\varpi}{2(n-1)} \right\} \quad (n \neq 1), \\ \mathcal{B}_2 &= \left\{ s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \sigma) < \frac{\varpi}{2} \right\}.\end{aligned}$$

By assumption, both \mathcal{B}_1 and \mathcal{B}_2 do not belong to \mathfrak{J} . Now, since $\sigma \neq v$, the sets \mathcal{B}_1 and \mathcal{B}_2 are disjoint; that is,

$$\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset.$$

Assume the contrary: suppose there exists some $t \in \mathcal{B}_1 \cap \mathcal{B}_2$. Then, we obtain

$$\begin{aligned}& \mathcal{A}(v, v, \dots, v, \sigma) \\ & \leq (n-1)\mathcal{A}(v, v, \dots, v, \zeta_t) + \mathcal{A}(\sigma, \sigma, \dots, \sigma, \zeta_t), \quad \text{by Lemma 1.2} \\ & = (n-1)\mathcal{A}(\zeta_t, \zeta_t, \dots, \zeta_t, v) + \mathcal{A}(\zeta_t, \zeta_t, \dots, \zeta_t, \sigma), \quad \text{by Lemma 1.1} \\ & < (n-1)\frac{\varpi}{2(n-1)} + \frac{\varpi}{2} \\ & = \frac{\varpi}{2} + \frac{\varpi}{2} \\ & = \varpi.\end{aligned}$$

Since $\varpi > 0$ is arbitrary, it follows that $\mathcal{A}(v, v, \dots, v, \sigma) = 0$. Hence, $v = \sigma$, which contradicts the assumption $\sigma \neq v$. Therefore, $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ implies $\mathcal{B}_2 \subseteq \mathcal{B}_1^c$. Given that $\mathcal{B}_1^c \in \mathfrak{J}$, it follows that $\mathcal{B}_2 \in \mathfrak{J}$, which contradicts the initial assumption that $\mathcal{B}_2 \notin \mathfrak{J}$. Thus, we conclude that

$$\Gamma_{\mathfrak{J}}^{\mathcal{A}}(\zeta_s) = \{v\}.$$

This completes the proof. \square

We now proceed to establish the relationship between an \mathfrak{J} -limit point of a sequence in an \mathcal{A} -metric space and the corresponding real-valued sequence induced by the \mathcal{A} -metric. A similar connection is also explored in the context of an \mathfrak{J} -cluster point.

Lemma 2.1 *Let $\{\zeta_s\}$ be a sequence in an \mathcal{A} -metric space \mathcal{U} . If $v \in \mathcal{U}$ is an \mathfrak{J} -limit point of the sequence $\{\zeta_s\}$ with respect to the \mathcal{A} -metric, then for every $\tau \in \mathcal{U}$, the real number $\mathcal{A}(v, v, \dots, v, \tau)$ is an \mathfrak{J} -limit point of the real sequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau)\}_{s \in \mathbb{N}}$.*

Proof: Suppose that v is an \mathfrak{J} -limit point of the sequence $\{\zeta_s\}$. Then, there exists a set $\mathcal{C} = \{m_1 < m_2 < \dots < m_s < \dots\}$ with $\mathcal{C} \notin \mathfrak{J}$ such that the subsequence $\{\zeta_{m_s}\}$ converges to v with respect to the \mathcal{A} -metric. That is, for every $\varpi > 0$, there exists $s_0 \in \mathbb{N}$ such that

$$\mathcal{A}(\zeta_{m_s}, \zeta_{m_s}, \dots, \zeta_{m_s}, v) < \frac{\varpi}{n-1}, \quad (n \neq 1) \quad \text{for all } s \geq s_0. \quad (2.2)$$

Now, let $\tau \in \mathcal{U}$. By applying Lemma 1.3 and using the Equation (2.2), we obtain

$$|\mathcal{A}(\zeta_{m_s}, \zeta_{m_s}, \dots, \zeta_{m_s}, \tau) - \mathcal{A}(\tau, \tau, \dots, \tau, v)| \leq (n-1)\mathcal{A}(\zeta_{m_s}, \zeta_{m_s}, \dots, \zeta_{m_s}, v) < \varpi \quad \forall s \geq s_0.$$

Hence, the real sequence $\{\mathcal{A}(\zeta_{m_s}, \zeta_{m_s}, \dots, \zeta_{m_s}, \tau)\}$ converges to $\mathcal{A}(\tau, \tau, \dots, \tau, v)$. Furthermore, by Lemma 1.1, we have

$$\mathcal{A}(\tau, \tau, \dots, \tau, v) = \mathcal{A}(v, v, \dots, v, \tau).$$

Therefore, $\mathcal{A}(v, v, \dots, v, \tau)$ is an \mathfrak{J} -limit point of the real sequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau)\}_{s \in \mathbb{N}}$. Thus, the proof is complete. \square

Lemma 2.2 *Let $\{\zeta_s\}$ be a sequence in an \mathcal{A} -metric space \mathcal{U} . If $v \in \mathcal{U}$ is an \mathfrak{I} -cluster point of $\{\zeta_s\}$ with respect to the \mathcal{A} -metric, then for every $\tau \in \mathcal{U}$, the real number $\mathcal{A}(v, v, \dots, v, \tau)$ serves as an \mathfrak{I} -cluster point of the real sequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau)\}_{s \in \mathbb{N}}$.*

Proof: Suppose that v is an \mathfrak{I} -cluster point of the sequence $\{\zeta_s\}$. Then, for every $\varpi > 0$, the following set is not a member of \mathfrak{I} :

$$\left\{ s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) < \frac{\varpi}{n-1} \right\} \notin \mathfrak{I}, \quad (n \neq 1). \quad (2.3)$$

Let $\tau \in \mathcal{U}$. By Lemma 1.3, we obtain:

$$|\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau) - \mathcal{A}(\tau, \tau, \dots, \tau, v)| \leq (n-1)\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v).$$

Therefore, for any $\varpi > 0$, we have:

$$\left\{ s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) < \frac{\varpi}{n-1} \right\} \subseteq \{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau) - \mathcal{A}(\tau, \tau, \dots, \tau, v)| < \varpi\}.$$

In view of Equation (2.3), it follows that:

$$\{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau) - \mathcal{A}(\tau, \tau, \dots, \tau, v)| < \varpi\} \notin \mathfrak{I}. \quad (2.4)$$

Moreover, by Lemma 1.1, we have:

$$\mathcal{A}(\tau, \tau, \dots, \tau, v) = \mathcal{A}(v, v, \dots, v, \tau).$$

Hence, from (2.4) we conclude that the real number $\mathcal{A}(v, v, \dots, v, \tau)$ is an \mathfrak{I} -cluster point of the real sequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \tau)\}_{s \in \mathbb{N}}$. Thus, the proof is complete. \square

In the following two results, we explore key results concerning \mathfrak{I} -limit points and \mathfrak{I} -cluster points of \mathfrak{I} -localized sequences, focusing on their relationship with the \mathfrak{I} -locator of the sequence in the context of an \mathcal{A} -metric space. Henceforth, we denote by $\Lambda_{\mathfrak{I}}(\zeta_s)$ and $\Gamma_{\mathfrak{I}}(\zeta_s)$ the sets of all \mathfrak{I} -limit points and \mathfrak{I} -cluster points, respectively, of the real number sequence $\{\zeta_s\}$.

Theorem 2.4 *Let τ and v be two \mathfrak{I} -limit points of an \mathfrak{I} -localized sequence $\{\zeta_s\}$ in an \mathcal{A} -metric space \mathcal{U} . Then, for every $\sigma \in \text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$, we have*

$$\mathcal{A}(\tau, \tau, \dots, \tau, \sigma) = \mathcal{A}(v, v, \dots, v, \sigma).$$

Proof: Let τ and v be two \mathfrak{I} -limit points of an \mathfrak{I} -localized sequence $\{\zeta_s\}$ in an \mathcal{A} -metric space \mathcal{U} . Suppose $\sigma \in \text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$ and consider the real sequence $\{w_s\} = \{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \sigma)\}$. By Lemma 2.1, it follows that $\mathcal{A}(\tau, \tau, \dots, \tau, \sigma)$ and $\mathcal{A}(v, v, \dots, v, \sigma)$ are both \mathfrak{I} -limit points of the sequence $\{w_s\}$. In other words,

$$\mathcal{A}(\tau, \tau, \dots, \tau, \sigma), \quad \mathcal{A}(v, v, \dots, v, \sigma) \in \Lambda_{\mathfrak{I}}(w_s).$$

However, since $\{\zeta_s\}$ is \mathfrak{I} -localized and $\sigma \in \text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$, the sequence $\{w_s\}$ is \mathfrak{I} -convergent. Consequently, $\Lambda_{\mathfrak{I}}(w_s)$ contains exactly one element, implying that

$$\mathcal{A}(\tau, \tau, \dots, \tau, \sigma) = \mathcal{A}(v, v, \dots, v, \sigma).$$

This establishes the desired equality. \square

Remark 2.1 *An analogous result to Theorem 2.4 can be formulated in terms of \mathfrak{I} -cluster points, replacing \mathfrak{I} -limit points, by employing Lemma 2.2 in the framework of \mathcal{A} -metric spaces.*

Proposition 2.3 *Let $\{\zeta_s\}$ be a sequence in an \mathcal{A} -metric space \mathcal{U} . Then, the \mathfrak{I} -locator set $\text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$ contains at most one \mathfrak{I} -limit point of the sequence $\{\zeta_s\}$.*

Proof: Suppose, to the contrary, that there exist two distinct \mathfrak{I} -limit points $v_1, v_2 \in \text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$. Then, by Theorem 2.4, we have

$$\mathcal{A}(v_1, v_1, \dots, v_1, v_1) = \mathcal{A}(v_2, v_2, \dots, v_2, v_1).$$

Since \mathcal{A} is an \mathcal{A} -metric, it follows that

$$\mathcal{A}(v_1, v_1, \dots, v_1, v_1) = 0,$$

and hence,

$$\mathcal{A}(v_2, v_2, \dots, v_2, v_1) = 0.$$

By the property of \mathcal{A} -metric spaces, this implies that $v_1 = v_2$, contradicting our assumption. Therefore, $\text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$ cannot contain more than one \mathfrak{I} -limit point. This completes the proof. \square

Remark 2.2 *An analogous version of Proposition 2.3 can be established in terms of \mathfrak{I} -cluster points, in place of \mathfrak{I} -limit points, in the framework of \mathcal{A} -metric spaces.*

It is known from Theorem 2.3 that if the sequence $\{\zeta_s\} \in \mathcal{U}$ is $\mathfrak{I}^{\mathcal{A}}$ -convergent to $v \in \mathcal{U}$, then the \mathfrak{I} -limit point is unique. The converse, however, holds provided that the \mathfrak{I} -limit point belongs to the set $\text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$, as demonstrated in the following theorem.

Theorem 2.5 *Let $\{\zeta_s\}$ be a sequence in an \mathcal{A} -metric space \mathcal{U} . If $v \in \mathcal{U}$ is an \mathfrak{I} -limit point of $\{\zeta_s\}$ such that $v \in \text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$, then the sequence $\{\zeta_s\}$ is $\mathfrak{I}^{\mathcal{A}}$ -convergent to v .*

Proof: Let $v \in \mathcal{U}$ be an \mathfrak{I} -limit point of the sequence $\{\zeta_s\}$ such that $v \in \text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$. Then, by Lemma 2.1, the number $\mathcal{A}(v, v, \dots, v, v)$ is an \mathfrak{I} -limit point of the real sequence $\{w_s\} := \{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)\}$. Since $v \in \text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$, it follows that $\{w_s\}$ is \mathfrak{I} -convergent. Let $\{w_s\}$ converges to $\tau \in \mathbb{R}$ in the sense of \mathfrak{I} . Then,

$$\mathcal{A}(v, v, \dots, v, v) \in \Lambda_{\mathfrak{I}}(w_s) \quad \text{and} \quad \Lambda_{\mathfrak{I}}(w_s) = \{\tau\}.$$

Therefore, we conclude that $\mathcal{A}(v, v, \dots, v, v) = \tau = 0$, by the property of \mathcal{A} -metric. Hence, $\{w_s\}$ is \mathfrak{I} -convergent to 0. As a result, for every $\varpi > 0$, the set

$$\{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) \geq \varpi\} \in \mathfrak{I}.$$

which shows that the sequence $\{\zeta_s\}$ is $\mathfrak{I}^{\mathcal{A}}$ -convergent to v . This completes the proof. \square

We now introduce the notion of an \mathfrak{I} -barrier for a sequence in an \mathcal{A} -metric space. Utilizing this concept, we establish a connection between \mathfrak{I} -Cauchy sequences and \mathfrak{I} -localized sequences in the next two results.

Definition 2.10 *Let $\{\zeta_s\}$ be an \mathfrak{I} -localized sequence in an \mathcal{A} -metric space \mathcal{U} , and denote $\mathcal{H} = \text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$. Then, the quantity*

$$\Psi_{\mathcal{A}} = \inf_{v \in \mathcal{H}} \left(\mathfrak{I} - \lim_{s \rightarrow \infty} \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) \right)$$

is defined as the \mathfrak{I} -barrier of the sequence $\{\zeta_s\}$ with respect to \mathcal{A} (briefly, the $\mathfrak{I}^{\mathcal{A}}(\zeta_s)$ -barrier).

Theorem 2.6 *Let \mathfrak{I} be an ideal satisfying the condition (AP). Then, an \mathfrak{I} -localized sequence is \mathfrak{I} -Cauchy with respect to \mathcal{A} if and only if $\Psi_{\mathcal{A}} = 0$.*

Proof: Let $\{\zeta_s\}$ be an $\mathfrak{I}^{\mathcal{A}}$ -Cauchy sequence in \mathcal{U} . Since \mathfrak{I} satisfies the condition (AP), it follows that $\{\zeta_s\}$ is also an \mathfrak{I}^* -Cauchy sequence. Therefore, there exists a set $\mathcal{C} = \{m_1 < m_2 < \dots < m_s < \dots\} \subseteq \mathbb{N}$ such that $\mathcal{C} \in \mathcal{F}(\mathfrak{I})$ and

$$\lim_{s, t \rightarrow \infty} \mathcal{A}(\zeta_{m_s}, \zeta_{m_s}, \dots, \zeta_{m_s}, \zeta_{m_t}) = 0.$$

Hence, for each preassigned $\varpi > 0$ there exists $s_0 \in \mathbb{N}$ such that

$$\mathcal{A}(\zeta_{m_s}, \zeta_{m_s}, \dots, \zeta_{m_s}, \zeta_{m_{s_0}}) < \varpi \quad \text{for all } s \geq s_0.$$

Since, by the given condition and by applying Proposition 2.1, $\{\zeta_s\}$ is an $\mathcal{I}^{\mathcal{A}}$ -localized sequence everywhere, the limit

$$\mathcal{I} - \lim \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \zeta_{m_{s_0}})$$

exists. Therefore,

$$\mathcal{I} - \lim \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \zeta_{m_{s_0}}) \leq \varpi, \quad \text{i.e., } \Psi_{\mathcal{A}} \leq \varpi.$$

Since $\varpi > 0$ is arbitrary, it follows that $\Psi_{\mathcal{A}} = 0$.

Conversely, assume that $\Psi_{\mathcal{A}} = 0$. By the Definition 2.10, for each $\varpi > 0$, there exists an element $v \in \text{loc}_{\mathcal{I}^{\mathcal{A}}}(\zeta_s)$ such that

$$\xi(v) = \mathcal{I} - \lim_{s \rightarrow \infty} \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) < \varpi.$$

This implies that the set

$$\{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \xi(v)| \geq \varpi - \xi(v)\} \in \mathcal{I},$$

since $\varpi - \xi(v) > 0$. Now, we observe that

$$\begin{aligned} \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) &= |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \xi(v) + \xi(v)| \\ &\leq |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \xi(v)| + \xi(v), \end{aligned}$$

which leads to

$$\{s \in \mathbb{N} : \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) \geq \varpi\} \in \mathcal{I}.$$

Thus, the sequence $\{\zeta_s\}$ is $\mathcal{I}^{\mathcal{A}}$ -convergent, and consequently, by Theorem 2.1, it is an $\mathcal{I}^{\mathcal{A}}$ -Cauchy sequence. This completes the proof. \square

It is noted that from the proof of the above theorem, it follows that the converse holds even in the absence of the condition (AP).

We have observed that a $\mathcal{I}^{\mathcal{A}}$ -localized sequence becomes an $\mathcal{I}^{\mathcal{A}}$ -Cauchy sequence whenever its \mathcal{I} -barrier is zero. In what follows, we extend this relationship by establishing that a $\mathcal{I}^{\mathcal{A}}$ -localized sequence is $\mathcal{I}^{\mathcal{A}}$ -Cauchy even without assuming the condition $\Psi_{\mathcal{A}} = 0$.

Theorem 2.7 *Let $\{\zeta_s\}$ be a sequence in an \mathcal{A} -metric space \mathcal{U} . If $\{\zeta_s\}$ is $\mathcal{I}^{\mathcal{A}}$ -localized in itself and contains an \mathcal{I} -nonthin Cauchy subsequence, then the sequence $\{\zeta_s\}$ is $\mathcal{I}^{\mathcal{A}}$ -Cauchy sequence itself.*

Proof: Let $\{\tau_s\}$ be an \mathcal{I} -nonthin Cauchy subsequence of $\{\zeta_s\}$ with respect to \mathcal{A} . Without loss of generality, we may assume that all members of $\{\tau_s\}$ belong to $\text{loc}_{\mathcal{I}^{\mathcal{A}}}(\zeta_s)$. Let $\mathcal{H} = \text{loc}_{\mathcal{I}^{\mathcal{A}}}(\zeta_s)$. Since $\{\tau_s\}$ is an $\mathcal{I}^{\mathcal{A}}$ -Cauchy sequence, it follows from Theorem 2.6 that

$$\inf_{\tau_s \in \mathcal{H}} \left(\mathcal{I} - \lim_{t \rightarrow \infty} \mathcal{A}(\tau_t, \tau_t, \dots, \tau_t, \tau_s) \right) = 0.$$

Now, as $\{\zeta_s\}$ is $\mathcal{I}^{\mathcal{A}}$ -localized in itself, the sequence $\{\mathcal{A}(\zeta_t, \zeta_t, \dots, \zeta_t, \tau_s)\}_{t \in \mathbb{N}}$ is \mathcal{I} -convergent for each $\tau_s \in \mathcal{H}$. Therefore, we have

$$\mathcal{I} - \lim_{t \rightarrow \infty} \mathcal{A}(\zeta_t, \zeta_t, \dots, \zeta_t, \tau_s) = \mathcal{I} - \lim_{t \rightarrow \infty} \mathcal{A}(\tau_t, \tau_t, \dots, \tau_t, \tau_s) = 0.$$

This implies that

$$\inf_{v \in \mathcal{H}} \left(\mathcal{I} - \lim_{t \rightarrow \infty} \mathcal{A}(\zeta_t, \zeta_t, \dots, \zeta_t, v) \right) = 0,$$

i.e., $\Psi_{\mathcal{A}} = 0$. Hence, by Theorem 2.6, the sequence $\{\zeta_s\}$ is $\mathcal{I}^{\mathcal{A}}$ -Cauchy sequence itself. This completes the proof. \square

Let τ_0 be a point in an \mathcal{A} -metric space $(\mathcal{U}, \mathcal{A})$ and let $\varpi > 0$. Recall (cf, [37,46,48]) that a sequence $\{\zeta_s\}$ in \mathcal{U} is referred to as \mathfrak{I} -bounded with respect to \mathcal{A} (in short $\mathfrak{I}^{\mathcal{A}}$ -bounded) if there can be found a subset $\mathcal{C} = \{s_1 < s_2 < \dots < s_m < \dots\} \subseteq \mathbb{N}$ such that $\mathcal{C} \in \mathcal{F}(\mathcal{I})$ and the subsequence $\{\zeta_{s_m}\} \subseteq \mathcal{B}(\tau_0, \varpi)$, where $\mathcal{B}(\tau_0, \varpi)$ denotes open ball centered at τ_0 with radius ϖ . It is evident that $\{\zeta_{s_m}\}$ is bounded in \mathcal{U} and contains a subsequence that is localized in itself (see [25]; this result holds in \mathcal{A} -metric spaces [46]). Accordingly, the following statement holds.

Proposition 2.4 *Every $\mathfrak{I}^{\mathcal{A}}$ -bounded sequence in an \mathcal{A} -metric space \mathcal{U} has an $\mathfrak{I}^{\mathcal{A}}$ -localized in itself subsequence.*

In the following result, we identify the condition under which a bounded subset of \mathcal{U} becomes totally bounded.

Proposition 2.5 *If every $\mathfrak{I}^{\mathcal{A}}$ -localized in itself sequence in \mathcal{U} is $\mathfrak{I}^{\mathcal{A}}$ -Cauchy sequence, then every bounded set of \mathcal{U} is totally bounded.*

Proof: Let $\{\zeta_s\}$ be an $\mathfrak{I}^{\mathcal{A}}$ -localized in itself sequence in \mathcal{U} , and suppose that it is also an $\mathfrak{I}^{\mathcal{A}}$ -Cauchy sequence. Assume further that a set $\mathcal{C} \subseteq \mathcal{U}$ is bounded but not totally bounded. Following the approach of [46], there exists a constant $\varpi > 0$ and a sequence $\{\zeta_s\} \subseteq \mathcal{C}$ such that

$$\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \zeta_t) > \varpi \quad \text{for all } s \neq t.$$

By Proposition 2.2, the sequence $\{\zeta_s\}$ is $\mathfrak{I}^{\mathcal{A}}$ -bounded. Then, applying Proposition 2.4, it follows that $\{\zeta_s\}$ admits an $\mathfrak{I}^{\mathcal{A}}$ -localized in itself subsequence $\{\zeta'_s\}$. Consequently,

$$\mathcal{A}(\zeta'_s, \zeta'_s, \dots, \zeta'_s, \zeta'_t) > \varpi \quad \text{for all } s \neq t,$$

which implies that $\{\zeta'_s\}$ is not a Cauchy sequence, and hence not an $\mathfrak{I}^{\mathcal{A}}$ -Cauchy sequence—contradicting our original assumption. Therefore, the set \mathcal{C} must be totally bounded. This completes the proof. \square

We now recall the concept of a thick subset of \mathcal{U} and proceed to establish a fundamental result related to it.

Definition 2.11 (cf, [46]) *Let \mathcal{U} be an \mathcal{A} -metric space. An infinite subset $\mathcal{C} \subseteq \mathcal{U}$ is defined as thick relative to a nonempty subset $\mathcal{K} \subseteq \mathcal{U}$, if for every $\varpi > 0$, there exists a point $\xi \in \mathcal{K}$ such that the ball $\mathcal{B}(\xi, \varpi)$ contains infinitely many points of \mathcal{C} . In particular, if \mathcal{C} is thick relative to $\mathcal{K} \subseteq \mathcal{C}$, then \mathcal{C} is called thick in itself.*

Theorem 2.8 *If every bounded infinite subset of an \mathcal{A} -metric space \mathcal{U} is thick in itself, then every $\mathfrak{I}^{\mathcal{A}}$ -localized in itself sequence in \mathcal{U} is an $\mathfrak{I}^{\mathcal{A}}$ -Cauchy sequence.*

Proof: Let $\{\zeta_s\}$ be an $\mathfrak{I}^{\mathcal{A}}$ -localized in itself sequence in \mathcal{U} . By Proposition 2.2, it follows that $\{\zeta_s\}$ is an $\mathfrak{I}^{\mathcal{A}}$ -bounded in \mathcal{U} . Consequently, there exists an infinite subset $\mathcal{C} \subseteq \{\zeta_s\}$ which is a bounded in \mathcal{U} . By assumption, every bounded infinite set in \mathcal{U} is thick in itself. Hence, \mathcal{C} is thick in itself. This implies that for any $\varpi > 0$, there exists a point $\zeta_t \in \mathcal{C}$ such that the ball $\mathcal{B}(\zeta_t, \varpi)$ contains infinitely many points $\zeta'_1, \zeta'_2, \dots, \zeta'_s, \dots$ of \mathcal{U} . As a result, the sequence $\{\mathcal{A}(\zeta'_s, \zeta'_s, \dots, \zeta'_s, \zeta_t)\}$ is \mathfrak{I} -convergent and, satisfies

$$\mathfrak{I} - \lim_{s \rightarrow \infty} \mathcal{A}(\zeta'_s, \zeta'_s, \dots, \zeta'_s, \zeta_t) \leq \varpi.$$

Since $\varpi > 0$ was arbitrary, this yields $\Psi_{\mathcal{A}} = 0$. Therefore, the sequence $\{\zeta_s\}$ is an $\mathfrak{I}^{\mathcal{A}}$ -Cauchy sequence. Hence, the proof is complete. \square

We now move forward to explore the relationship between \mathfrak{I} -localized and \mathfrak{I}^* -localized sequences in the framework of an \mathcal{A} -metric space.

Theorem 2.9 *If a sequence $\{\zeta_s\}$ in an \mathcal{A} -metric space \mathcal{U} is \mathfrak{I}^* -localized on the subset $\mathcal{C} \subseteq \mathcal{U}$, then it is \mathfrak{I} -localized on \mathcal{C} and $\text{loc}_{\mathfrak{I}^*}^{\mathcal{A}}(\zeta_s) \subset \text{loc}_{\mathfrak{I}}^{\mathcal{A}}(\zeta_s)$.*

Proof: Let $\{\zeta_s\}$ be an \mathfrak{I}^* -localized sequence on \mathcal{C} . According to Definition 2.6, the sequence of real numbers $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)\}$ is \mathfrak{I}^* -convergent for every $v \in \mathcal{C}$. Now, since \mathfrak{I} is an admissible ideal, it follows that this sequence is also \mathfrak{I} -convergent for each $v \in \mathcal{C}$. Therefore, $\{\zeta_s\}$ is \mathfrak{I} -localized on \mathcal{C} . This completes the proof. \square

It is important to note that the converse of Theorem 2.9 does not hold in general, which is illustrated by the following example.

Example 2.1 *We begin by setting $\mathcal{U} = \mathbb{R}$ and defining an \mathcal{A} -metric on \mathcal{U} as described in Example 1.1. Consider a decomposition $\mathbb{N} = \bigcup_{a=1}^{\infty} \mathcal{W}_a$, where each \mathcal{W}_a is an infinite subset of \mathbb{N} , and the sets are pairwise disjoint, i.e., $\mathcal{W}_a \cap \mathcal{W}_b = \emptyset$, for $a \neq b$. Let \mathfrak{I} be the class of all subsets of \mathbb{N} that intersects only finitely many of the \mathcal{W}_a 's. Clearly, \mathfrak{I} forms an admissible ideal on \mathbb{N} . Next, define a sequence $\{\zeta_s\}$ in $(\mathcal{U}, \mathcal{A})$ by*

$$\zeta_s = \frac{1}{a}, \quad \text{whenever } s \in \mathcal{W}_a.$$

Let $\varpi > 0$ be arbitrary. Since the sequence $\{\frac{1}{a}\}_{a \in \mathbb{N}}$ converges to zero in \mathbb{R} , there exists $c \in \mathbb{N}$ such that

$$\left| \frac{1}{a} - 0 \right| < \frac{\varpi}{(n-1)^2} \quad \text{for all } a \geq c. \quad (2.5)$$

Let us fix $v = 0$. Then, for each $s \in \mathcal{W}_a$ we compute

$$\begin{aligned} \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) &= \mathcal{A}\left(\underbrace{\frac{1}{a}, \frac{1}{a}, \dots, \frac{1}{a}}_{n-1 \text{ times}}, 0\right) \\ &= \underbrace{\left| \frac{1}{a} - \frac{1}{a} \right| + \left| \frac{1}{a} - \frac{1}{a} \right| + \dots + \left| \frac{1}{a} - \frac{1}{a} \right|}_{n-2 \text{ times}} + \left| \frac{1}{a} - 0 \right| \\ &\quad + \underbrace{\left| \frac{1}{a} - \frac{1}{a} \right| + \left| \frac{1}{a} - \frac{1}{a} \right| + \dots + \left| \frac{1}{a} - \frac{1}{a} \right|}_{n-3 \text{ times}} + \left| \frac{1}{a} - 0 \right| \\ &\quad + \underbrace{\vdots \quad \dots \quad \vdots}_{(n-5) \text{ times}} \\ &\quad + \left| \frac{1}{a} - \frac{1}{a} \right| + \left| \frac{1}{a} - 0 \right| \\ &\quad + \left| \frac{1}{a} - 0 \right| \\ &= (n-1) \frac{1}{a} \\ &< (n-1) \cdot \frac{\varpi}{(n-1)^2}, \quad \text{using (2.5)} \\ &= \frac{\varpi}{n-1}, \quad \text{for all } a \geq c. \end{aligned}$$

Now, let $\xi \in \mathcal{U}$. By Lemma 1.3, we have

$$\begin{aligned} |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \xi) - \mathcal{A}(\xi, \xi, \dots, \xi, 0)| &\leq (n-1) \mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, 0) \\ &< (n-1) \cdot \frac{\varpi}{n-1} \\ &= \varpi, \quad \text{for all } a \geq c. \end{aligned}$$

Therefore, the set

$$\begin{aligned} & \{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \xi) - \mathcal{A}(\xi, \xi, \dots, \xi, 0)| \geq \varpi\} \\ & = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \dots \cup \mathcal{W}_c \in \mathfrak{I}. \end{aligned}$$

Hence, for every $\xi \in \mathcal{U}$ the sequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \xi)\}$ is \mathfrak{I} -convergent. Consequently, the sequence $\{\zeta_s\}$ is \mathfrak{I} -localized in the \mathcal{A} -metric space $(\mathcal{U}, \mathcal{A})$.

Now, we demonstrate that the sequence $\{\zeta_s\}$ is not \mathfrak{I}^* -localized in $(\mathcal{U}, \mathcal{A})$. Suppose, to the contrary, that $\{\zeta_s\}$ is \mathfrak{I}^* -localized in $(\mathcal{U}, \mathcal{A})$. Then, by definition, the sequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \xi)\}$ is \mathfrak{I}^* -convergent for each $\xi \in \mathcal{U}$.

Hence, there exists a set $\mathcal{P} \in \mathfrak{I}$ such that

$$\mathcal{K} := \mathbb{N} \setminus \mathcal{P} = \{m_1 < m_2 < \dots < m_s < \dots\} \in \mathcal{F}(\mathfrak{I}),$$

and the subsequence $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, \xi)\}_{s \in \mathcal{K}}$ is convergent. By the structure of the ideal \mathfrak{I} , there exists a positive integer k such that $\mathcal{P} \subseteq \mathcal{W}_1 \cup \mathcal{W}_2 \cup \dots \cup \mathcal{W}_k$. Consequently, for all $b \geq k+1$, $\mathcal{W}_b \subseteq \mathbb{N} \setminus \mathcal{P} = \mathcal{K}$. In particular, both \mathcal{W}_{k+1} and \mathcal{W}_{k+2} are both subsets of \mathcal{K} .

Since each \mathcal{W}_a 's is infinite, there are infinitely many s 's such that

$$\zeta_{m_s} = \begin{cases} \frac{1}{k+1}, & \text{if } m_s \in \mathcal{W}_{k+1} \\ \frac{1}{k+2}, & \text{if } m_s \in \mathcal{W}_{k+2}. \end{cases}$$

Therefore, for $\xi = 0$, we obtain:

$$\mathcal{A}(\zeta_{m_s}, \zeta_{m_s}, \dots, \zeta_{m_s}, 0) = \begin{cases} \mathcal{A}\left(\frac{1}{k+1}, \frac{1}{k+1}, \dots, \frac{1}{k+1}, 0\right) = (n-1)\frac{1}{k+1}, & \text{if } m_s \in \mathcal{W}_{k+1} \\ \mathcal{A}\left(\frac{1}{k+2}, \frac{1}{k+2}, \dots, \frac{1}{k+2}, 0\right) = (n-1)\frac{1}{k+2}, & \text{if } m_s \in \mathcal{W}_{k+2}. \end{cases}$$

Thus, the sequence $\{\mathcal{A}(\zeta_{m_s}, \zeta_{m_s}, \dots, \zeta_{m_s}, 0)\}_{s \in \mathbb{N}}$ contains infinitely many terms equal to $(n-1)\frac{1}{k+1}$ and $(n-1)\frac{1}{k+2}$, and hence cannot be convergent. This contradiction implies that our assumption was false. Therefore, the sequence $\{\zeta_s\}$ is not \mathfrak{I}^* -localized in $(\mathcal{U}, \mathcal{A})$.

Next, we investigate the condition under which an \mathfrak{I} -localized sequence becomes \mathfrak{I}^* -localized in an \mathcal{A} -metric space.

Theorem 2.10 Let $\{\zeta_s\}$ be a sequence in an \mathcal{A} -metric space \mathcal{U} , and suppose the ideal \mathfrak{I} satisfies the condition (AP). If $\{\zeta_s\}$ is $\mathfrak{I}^{\mathcal{A}}$ -localized on a subset $\mathcal{V} \subseteq \mathcal{U}$, then it is also \mathfrak{I}^* -localized on \mathcal{V} .

Proof: Since $\{\zeta_s\}$ is \mathfrak{I} -localized on \mathcal{V} , by definition, the sequence of real numbers $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)\}$ is \mathfrak{I} -convergent for each $v \in \mathcal{V}$. Let us denote the \mathfrak{I} -limit by $\ell = \ell(v) \in \mathbb{R}$. Then, for every $\varpi > 0$, the set

$$\mathcal{M} = \{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \ell| \geq \varpi\} \in \mathfrak{I}.$$

Now, define the disjoint sets:

$$\mathcal{M}_1 = \{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \ell| \geq 1\},$$

and for $i \geq 2$,

$$\mathcal{M}_i = \left\{s \in \mathbb{N} : \frac{1}{i} \leq |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \ell| < \frac{1}{i-1}\right\}.$$

Clearly, each $\mathcal{M}_i \in \mathfrak{I}$ for $i \in \mathbb{N}$, and the family $\{\mathcal{M}_i\}$ is pairwise disjoint. Since \mathfrak{I} satisfies the condition (AP), there exists a countable family of sets $\{\mathcal{C}_i\}_{i=1}^{\infty}$ such that $\mathcal{M}_i \Delta \mathcal{C}_i$ is finite for each $i \in \mathbb{N}$, and the union $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i \in \mathfrak{I}$.

We now demonstrate that $\{\zeta_s\}$ is \mathfrak{I}^* -localized. It suffices to show that for every $v \in \mathcal{V}$, the sequence

$\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)\}$ is \mathfrak{J}^* -convergent to ℓ . Let $\mathcal{K} = \mathbb{N} \setminus \mathcal{C}$. Then, $\mathcal{K} \in \mathcal{F}(\mathcal{I})$. Fix $\eta > 0$. Choose $i \in \mathbb{N}$ such that $\frac{1}{i+1} < \eta$. Then,

$$\{s \in \mathbb{N} : |\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \ell| \geq \eta\} \subseteq \bigcup_{p=1}^{i+1} \mathcal{M}_p.$$

Since each symmetric difference $\mathcal{M}_p \Delta \mathcal{C}_p$ is finite, there exists $s_0 \in \mathbb{N}$ such that for all $s > s_0$,

$$\left(\bigcup_{p=1}^{i+1} \mathcal{C}_p \right) \cap \{s \in \mathbb{N} : s > s_0\} = \left(\bigcup_{p=1}^{i+1} \mathcal{M}_p \right) \cap \{s \in \mathbb{N} : s > s_0\}.$$

Thus, for $s > s_0$ with $s \notin \mathcal{C}$, it follows that $s \notin \bigcup_{p=1}^{i+1} \mathcal{M}_p$, and hence

$$|\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v) - \ell| < \frac{1}{i+1} < \eta.$$

Therefore, $\{\mathcal{A}(\zeta_s, \zeta_s, \dots, \zeta_s, v)\}$ is \mathfrak{J}^* -convergent to ℓ for each $v \in \mathcal{V}$, which proves that the sequence $\{\zeta_s\}$ is \mathfrak{J}^* localized on \mathcal{V} . \square

3. Conclusion and future developments

\mathcal{A} -metric spaces, introduced by Abbas et al. [1], generalize classical metric spaces and have been studied mainly in fixed point theory, with limited progress in summability. Motivated by the concept of localized sequences [25], we develop the theory of \mathfrak{J} -localized sequences in \mathcal{A} -metric spaces. We introduce \mathfrak{J} -localized and \mathfrak{J} -Cauchy sequences, establish their fundamental properties, and demonstrate how these results extend classical sequence theory. This work connects summability theory with \mathcal{A} -metric structures, opening avenues for further study, such as \mathfrak{J}_2 -localized double sequences, extensions to partial b -metric and G -metric spaces, and also within intuitionistic fuzzy and neutrosophic metric spaces.

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