



Characterization of Generalized Color Complements of $G_{m,n}$ Graph

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ABSTRACT: In this paper, we study the generalized color complement of the graph $G_{m,n}$ for specific values of m and n . The graph $G_{m,n} = (V, E)$ is an undirected, simple graph defined on a finite subset of natural numbers. The vertex set of $G_{m,n}$ is $V = I_n = \{1, 2, \dots, n\}$, and two distinct vertices $a, b \in V$ are adjacent if and only if $a \neq b$ and $a + b$ is not divisible by m , where $m \in N$ and $m > 1$. The paper specifically focuses on the generalized color complement of $G_{m,n}$ graphs with respect to the equal degree partition. Additionally, we determine the 2-color complement of $G_{m,n}$ for $m = 2$ and for both even and odd values of n . Our aim is to analyze the color complement of the graphs with respect to the equal-degree partition for certain values of $m, n \in N$. We determine conditions under which the generalized color complement of $G_{m,n}$ results in specific types of graphs, such as a path graph and a disjoint union of K_2 and K_1 .

Keywords: Generalized color complement, k and $k(i)$ -color complement, $G_{m,n}$ graph.

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1. Introduction

Let $G = (V, E)$ be a simple, undirected graph with n vertices. Consider two vertices u and v in G . If $uv \in E$, then u is adjacent to v , denoted by $u \sim v$. If u is not adjacent to v , we denote this by $u \not\sim v$. The complement of G , denoted by \overline{G} , shares the same vertex set as G , and two vertices u and v are adjacent in \overline{G} if and only if they are not adjacent in G . A graph G is called self-complementary if G is isomorphic to \overline{G} . The degree of a vertex v in a graph, denoted by $d(v)$, is the number of edges incident to it. We denote the cycle graph, the complete bipartite graph, the complete graph, and star graph by $C_n, K_{s,t}, K_n$ and S_n , respectively. Cocktail party graph $K_{s \times 2}$ is a graph having vertex set $V = \bigcup_{i=1}^s \{u_i, v_i\}$ and the edge set $E = \{u_i u_j, v_i v_j : i \neq j\} \cup \{u_i v_j, v_i u_j : 1 \leq i < j \leq s\}$.

In a graph G , a coloring is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The chromatic number of G , denoted by $\chi(G)$, is the least number of colors needed to obtain a proper coloring of G . The notations and terminologies mentioned above are standard in graph theory, as discussed in references such as [9], [4].

In 1998, E. Sampathkumar et al. introduced the generalized complements of a graph, namely the k -complement G_k^P and the $k(i)$ -complement $G_{k(i)}^P$ of a graph G [7], [8]. Further investigations into generalized complements can be found in works such as [2], [3]. In 2020, Swati Nayak et al. [5], [6] extended this line of research by introducing generalized color complements of graphs, which involve vertex colorings in the context of generalized complements.

Generalized color complements of a graph have significant applications in various fields. In wireless communication networks, they help in frequency assignment by ensuring interference-free connections

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based on color constraints. In bioinformatics, they model complex biological interactions, such as protein-protein interactions, where different functional groups must be distinguished. Social network analysis benefits from these complements by identifying indirect relationships and studying community structures with different attributes. In combinatorial optimization, they aid in efficient task scheduling and resource allocation, considering category-based constraints. Additionally, in cybersecurity, they help in designing secure network architectures by analyzing alternative secure pathways. These applications highlight the importance of generalized color complements in real-world problem-solving.

2. Preliminaries

Definition 2.1 [5] Let $G_c = (V, E)$ be a colored graph. The complement of G_c , denoted by $\overline{G_c}$, is obtained by keeping the same vertex coloring as in G_c and following these rules:

- Two vertices u and v are adjacent in $\overline{G_c}$ if they are non-adjacent in G_c and $c(u) \neq c(v)$.
- Two vertices u and v are non-adjacent in $\overline{G_c}$ if they are non-adjacent in G_c and $c(u) = c(v)$.
- Two vertices u and v are non-adjacent in $\overline{G_c}$ if they are adjacent in G_c .

Definition 2.2 [5] Let $G_c = (V, E)$ be a colored graph and $P = \{V_1, V_2, \dots, V_k\}$ be a partition of vertex set $V(G_c)$ of order $k \geq 1$. The k -color complement $(G_c)_k^P$ of G_c is defined as follows: For all V_i and V_j in P , $i \neq j$, remove the edges between V_i and V_j and add the edges which are not in G_c such that end vertices have different colors.

- The graph G_c is k -self color complementary (k -s.c.c.) with respect to P if $(G_c)_k^P \cong G_c$.
- G_c is k -co-self color complementary (k -co-s.c.c.) with respect to P if $(G_c)_k^P \cong \overline{G_c}$.

Definition 2.3 [5] For each subset V_r in the partition P , remove the edges of G_c that exist within V_r and add the edges of $\overline{G_c}$ joining the vertices of V_r . The resulting graph $(G_c)_{k(i)}^P$ is known as $k(i)$ -color complement of G_c with respect to the partition P of V .

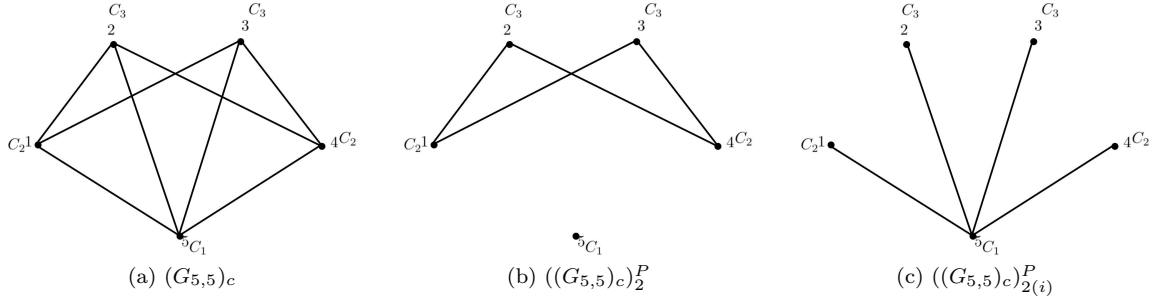
- The graph G_c is $k(i)$ -self color complementary ($k(i)$ -s.c.c.) if $(G_c)_{k(i)}^P \cong G_c$ for some partition P of order k .
- Furthermore, G_c is $k(i)$ -co-self color complementary ($k(i)$ -co-s.c.c.) if $(G_c)_{k(i)}^P \cong \overline{G_c}$.

Definition 2.4 [5] For a graph G_c ,

- i. indegree of vertex v denoted by $i_d(v)$ is defined as number of edges connected to the vertex v inside the partite.
- ii. outdegree of vertex v denoted by $o_d(v)$ is defined as number of edges connected to the vertex v outside the partite.

Definition 2.5 [1] The graph $G_{m,n} = (V, E)$ is an undirected, simple graph defined on a finite subset of natural numbers. The vertex set of $G_{m,n}$ is $V = I_n = \{1, 2, \dots, n\}$, and two distinct vertices $a, b \in V$ are adjacent if and only if $a \neq b$ and $a + b$ is not divisible by m , where $m \in \mathbb{N}$ and $m > 1$.

Example: Given the color graph $(G_{5,5})_c$ with vertex set $V = \{1, 2, 3, 4, 5\}$, and the partition $P = \{V_1, V_2\}$, where $V_1 = \{1, 2, 3, 4\}$ and $V_2 = \{5\}$.


 Figure 1: Graph $(G_{5,5})_c$ and its 2-color and $2(i)$ -color complement

For convenience, we denote the color graph $(G_{m,n})_c$ by G_c throughout the paper.

Proposition 2.1 *i. Graph G_c is k -co-s.c.c and $k(i)$ -s.c.c with respect to the partition of same color class.*

ii. For any graph G_c of order n with partition set $P = \{V_1, V_2, \dots, V_k\}$, then

a. $(G_c)_{k(i)}^P \cong G_c$ and $(G_c)_k^P \cong \overline{G_c}$, for $k = n$.

b. $(G_c)_{k(i)}^P \cong \overline{G_c}$ and $(G_c)_k^P \cong G_c$, for $k = 1$.

3. k -Color Complement of G_c

In this section, we analyze the nature of the k -color complement of G_c with respect to an equal degree partition. Specifically, we consider the partition P of the vertex set $V(G_c)$, where $P = \{V_1, V_2, \dots, V_k\}$ and each set in P contains the vertices of G_c that have the same degree.

Theorem 3.1 *For any graph G_c with $m = n$, the graph $(G_c)_k^P$ is disconnected. Furthermore,*

i. For odd n , $(G_c)_k^P$ is a disjoint union of K_1 and $(n-3)$ -regular graph.

ii. For even n , $(G_c)_k^P$ is a disjoint union of K_2 and $(n-4)$ -regular graph.

Proof: Let G_c be a graph with vertex set $V = \{1, 2, \dots, n\}$ and $m = n$.

i. Let n be an odd integer. Then in G_c , the degree of the vertex n is $d(n) = n-1$, and the degree of each vertex l is $d(l) = n-2$, where $l = 1, 2, \dots, n-1$. Thus, G_c can be partitioned into two sets $V_1 = \{1, 2, \dots, n-1\}$ and $V_2 = \{n\}$. Vertex n becomes isolated in $(G_c)_2^P$, as it is adjacent to all other vertices in G_c . Thus, $\langle V_2 \rangle$ forms K_1 . Hence $(G_c)_2^P$ is disconnected.

The number of vertices in V_1 is $n-1$. The vertex $l \in V_1$ is not adjacent to vertex labeled $n-l$. As a result, each vertex $l \in V_1$ has a degree $n-3$ in $(G_c)_2^P$, because it is not adjacent to $n-l$ and n . The subgraph induced by V_1 is therefore an $(n-3)$ -regular graph. Consequently, $(G_c)_2^P$ is disjoint union of K_1 and $(n-3)$ -regular graph, with respect to equal degree partition.

ii. Let n be an even integer. In G_c , the vertices n and $\frac{n}{2}$ are of degree $n-1$, as they are adjacent to every other vertices in the graph. All other vertices $l \neq n, \frac{n}{2} \in V(G_c)$ have degree $n-2$. Since the partition is based on equal degrees, we have $k = 2$, with $V_1 = \{n, \frac{n}{2}\}$ and $V_2 = V \setminus V_1$. The vertices n and $\frac{n}{2}$, being directly connected in G_c , remain adjacent in $(G_c)_2^P$ and non-adjacent with other partite vertices. Thus, $V_1 = \{n, \frac{n}{2}\}$ induces a subgraph isomorphic to K_2 .

For the vertices in $\langle V_2 \rangle$, note that in G_c , each vertex $l \in V_2$ is adjacent to both n and $\frac{n}{2}$ and they are not adjacent to the vertex labeled $n-l$. As a result in $(G_c)_2^P$, the vertices in V_2 are not adjacent to the vertices n and $\frac{n}{2}$. Thus, the degree of each vertex in V_2 reduces by 2, resulting in $d(l) = (n-2) - 2 = n-4$. Therefore, $(G_c)_2^P$ is disconnected, consisting of two components K_2 , formed by V_1 , and a graph with regularity $n-4$, formed by V_2 .

□

Theorem 3.2 For $m = 2$,

- i. G_c is k -self color complementary if n is an even integer.
- ii. G_c is k -co-self color complementary if n is an odd integer.

Proof: Let G_c be a graph with vertex set $V = \{1, 2, \dots, n\}$ and $m = 2$.

- i. Let n be an even integer. Since all the vertices in G_c have degree $\frac{n}{2}$, there exists only one partition of $V(G_c)$, $P = \{V\}$. From Proposition 2.1, we have $G_c \cong (G_c)_k^P$.
- ii. Let n be an odd integer, $\chi(G_c) = 2$. The vertices of G_c can be divided into two subsets based on their labels. Number of vertices labeled with an even integer is $\lfloor \frac{n}{2} \rfloor$, and they are adjacent to all vertices labeled with odd integers. The other subset contains $\lfloor \frac{n}{2} \rfloor + 1$ vertices labeled with odd integers, and are adjacent to all vertices labeled with even integers. However, there are no edges between any two vertices within the same subset, as all the vertices in the same subsets share the same color. This implies that the degree of each vertex in the even-labeled subset is $\lfloor \frac{n}{2} \rfloor + 1$, and the degree of each vertex in the odd-labeled subset is $\lfloor \frac{n}{2} \rfloor$. Thus $V(G_c)$ is partitioned as $P = \{V_1, V_2\}$, where $V_1 = \{l \in V : l \text{ is even}\}$ and $V_2 = \{l \in V : l \text{ is odd}\}$. In G_c , vertices of V_1 are adjacent only to those V_2 . Since $\chi(V_1) = \chi(V_2) = 1$, all vertices in V_1 receive one color and those in V_2 receive another. Hence, $\overline{G_c}$ is a null graph and no edges exist in $(G_c)_k^P$, the graph G_c is k -co-self color complementary.

□

Theorem 3.3 For any graph G_c with $m = n + 1$,

- i. If n is even, then G_c is k -self color complementary.
- ii. If n is odd, then $(G_c)_k^P$ is disjoint union of K_1 and $K_{s \times 2}$, where $s = \frac{m}{2} - 1$.

Proof: Let G_c be a graph with $V = \{1, 2, \dots, n\}$ and $m = n + 1$. The proof proceeds in two cases based on whether n is even or odd.

- i. Suppose n is an even integer. Then G_c is an $(n - 2)$ -regular graph in which each vertex $l \in V$ is non-adjacent to the vertex $m - l$ and both lie in the same color class. As a result, for an equal-degree partition, we have $k = 1$. By Proposition 2.1, it follows that $G_c \cong (G_c)_k^P$.
- ii. Suppose n is an odd integer. In G_c , the vertex labeled $\frac{n+1}{2}$ is of degree $n - 1$, as it is adjacent to every other vertices in the graph. All other vertices $l \neq \frac{n+1}{2} \in V(G_c)$ have degree $n - 2$. Since the partition is with respect to equal degree, we partition the vertex set $V(G_c)$ into 2 subsets i.e, $V_1 = \{\frac{n+1}{2}\}$ and $V_2 = V \setminus V_1$. In $(G_c)_2^P$, the vertex of V_1 remains isolated, yielding the induced subgraph K_1 . Consequently, $(G_c)_2^P$ is disconnected.

The set V_2 contains $n - 1$ vertices. For each vertex $l \in V_2$, the vertex l is non-adjacent to the vertex labeled $n - l + 1$ in G_c . Hence, in $(G_c)_2^P$, each $l \in V_2$ is non-adjacent to both $n - l + 1$ and $\frac{n+1}{2}$, giving degree $n - 3$. Thus, the subgraph induced by V_2 is a $(n - 3)$ -regular graph and isomorphic to the cocktail party graph $K_{s \times 2}$, where $s = \frac{m}{2} - 1$. Therefore, for $k = 2$, $(G_c)_k^P \cong K_1 \cup K_{s \times 2}$, $s = \frac{m}{2} - 1$.

□

Theorem 3.4 For any graph G_c with $m = n - 1$, the graph $(G_c)_k^P$ is disconnected. Furthermore,

- i. For even n , $(G_c)_k^P$ is a disjoint union of K_1 and a graph G' , where G' is a graph obtained by $(S_{n-2})_c$ with center n while other vertices labeled as $\{1, 2, \dots, n-3, n-2\}$ and $a, b \in \{1, 2, \dots, n-3\}$ are adjacent if and only if $a \neq b$, $m \nmid a+b$ and vertex $n-2$ is not adjacent to these vertices.
- ii. For odd n , $(G_c)_k^P$ is a disjoint union of K_2 and a graph G' , where G' is a graph obtained by $(S_{n-2})_c$ with central vertex n while other vertices labeled as $\{1, 2, \dots, n-3, n-2\}$ excluding $\frac{n-1}{2}$ and $a, b \in \{1, 2, \dots, n-3\}$ are adjacent if and only if $a \neq b$, $m \nmid a+b$ and vertex $n-2$ is not adjacent to these vertices.

Proof: Let G_c be a graph with $V = \{1, 2, \dots, n\}$ and $m = n-1$.

- i. When n is even, the degrees of the vertices in G_c are $d(n-2) = n-3$, $d(n-1) = n-1$ and $d(v) = n-2$, where $v \neq n-2, n-1$. Therefore, the graph G_c is partitioned into three sets $P = \{V_1, V_2, V_3\}$, where $V_1 = \{n-2\}$, $V_2 = \{n-1\}$ and $V_3 = V \setminus \{V_1, V_2\}$. In the graph $(G_c)_3^P$, the vertex labeled $(n-1)$, which was originally adjacent to all other vertices in G_c , becomes isolated. As a result, the subgraph induced by V_2 forms K_1 . Hence $(G_c)_3^P$ is disconnected.

Let G' be the subgraph induced by V_3 together with V_1 . In $(G_c)_3^P$, two vertices $a, b \in V_3$ are adjacent if and only if $a+b$ is not divisible by m . The vertex $n-2 \in V_1$ is adjacent only to the vertex labeled $n \in V_3$, since it is non-adjacent to vertex labeled 1 (same color as $n-2$) and to n in G_c . Therefore, $(G_c)_3^P$ is a disjoint union of K_1 and G' .

- ii. When n is odd, the degrees of the vertices in G_c are $d(n-2) = n-3$, $d(n-1) = d(\frac{n-1}{2}) = n-1$ and $d(v) = n-2$ for all $v \neq \{n-2, n-1, \frac{n-1}{2}\}$. Thus G_c consists of 3 partition $P = \{V_1, V_2, V_3\}$ such that $V_1 = \{n-2\}$, $V_2 = \{\frac{n-1}{2}, n-1\}$ and $V_3 = V \setminus \{V_1, V_2\}$. In the graph $(G_c)_3^P$, the vertices $n-1$ and $\frac{n-1}{2}$ are of degree $n-1$, as they are adjacent to every other vertices in the graph. Both these vertices being directly connected in G_c , they remain adjacent in $(G_c)_3^P$ and non-adjacent with other partite vertices. Thus, $\langle V_2 \rangle$ induces a subgraph isomorphic to K_2 . Hence $(G_c)_3^P$ is disconnected.

The graph G' is formed from $\langle V_3 \rangle$ and $\langle V_1 \rangle$, where two vertices $a, b \in V_3$ are adjacent if and only if $a+b$ is not divisible by m . The vertex $n-2 \in V_1$ is adjacent only to the vertex labeled $n \in V_3$, as it is not adjacent to vertex labeled 1 and n , and the vertex labeled 1 and $n-2$ share the same color. Therefore, $(G_c)_3^P$ is a disjoint union of K_2 and G' .

□

Theorem 3.5 Let $m = n$ with n even. Define the partition $P = \{V_1, V_2, \dots, V_{\frac{n}{2}}\}$, $V_i = \{i, n-1-i\}$ for $1 \leq i \leq \frac{n}{2}-1$ and $V_{\frac{n}{2}} = \{n-1, n\}$. Then $(G_c)_k^P$ is the disjoint union of $\frac{n}{2}$ copies of K_2 .

Proof: Let G_c be a graph with vertex set $V = \{1, 2, \dots, n\}$, where n is even and $m = n$. In G_c , the vertices n and $\frac{n}{2}$ have degree $n-1$, being adjacent to every other vertex, while all remaining vertices $l \neq n, \frac{n}{2}$ have degree $n-2$. Consider the partition $P = \{V_1, V_2, \dots, V_{\frac{n}{2}}\}$ of V such that, $V_i = \{i, n-1-i\}$ for $i = 1, 2, \dots, \frac{n}{2}-1$ and $V_{\frac{n}{2}} = \{n-1, n\}$. In G_c , vertices in V_i are adjacent to each other, hence also adjacent in $(G_c)_k^P$. Similarly, the vertices n and $n-1$ are adjacent in G_c , remains adjacent in $(G_c)_k^P$. The vertices i and $n-i$ for $1 \leq i \leq \frac{n}{2}-1$ are not adjacent in G_c , share the same color, and therefore remain non-adjacent in $(G_c)_k^P$. Moreover, although vertices n and $\frac{n}{2}$ are adjacent to all other vertices in G_c , in $(G_c)_k^P$, they are adjacent only to $n-1$ and $\frac{n}{2}-1$, respectively. Thus, every vertex in $(G_c)_k^P$ has degree one and the graph decomposes into the disjoint union of $\frac{n}{2}$ copies of K_2 . □

Theorem 3.6 For $m = n$ with n odd, the graph $(G_c)_k^P$ is the disjoint union of $\frac{n-1}{2}$ copies of K_2 together with an isolated vertex K_1 , where the partition $P = \{V_1, V_2, \dots, V_{\frac{n+1}{2}}\}$, is defined by $V_i = \{i, n-1-i\}$ for $1 \leq i \leq \frac{n-3}{2}$, $V_{\frac{n-1}{2}} = \{\frac{n-1}{2}\}$ and $V_{\frac{n+1}{2}} = \{n-1, n\}$.

Proof: Let $m = n$ and n be an odd integer. The graph G_c is bi-regular, where $d(n) = n - 1$ and $d(l) = n - 2$ with $l = 1, 2, \dots, n - 1$. Consider the partition $P = \{V_1, V_2, \dots, V_{\frac{n+1}{2}}\}$, where $V_i = \{i, n - 1 - i\}$ for $1 \leq i \leq \frac{n-3}{2}$, $V_{\frac{n-1}{2}} = \{\frac{n-1}{2}\}$ and $V_{\frac{n+1}{2}} = \{n - 1, n\}$. Vertices within each partite set remain adjacent in $(G_c)_k^P$, in particular, i and $n - 1 - i$ ($1 \leq i \leq \frac{n-3}{2}$) as well as $n - 1$ and n . The vertex $\frac{n-1}{2}$ is adjacent to all except the vertex labeled as $\frac{n+1}{2}$ in G_c , but since these two share the same color, they remain non-adjacent in $(G_c)_k^P$. Thus, the vertex $\frac{n-1}{2}$ becomes isolated in $(G_c)_k^P$. Hence, $(G_c)_k^P$ is the disjoint union of $\frac{n-1}{2}$ copies of K_2 and an isolated vertex K_1 . \square

4. $k(i)$ -Color Complement of G_c

In this section, we analyze the nature of the $k(i)$ -color complement of G_c with respect to equal degree partition.

Theorem 4.1 For any graph G_c and $m = n$,

- i. If n is odd, then $(G_c)_{k(i)}^P \cong (S_n)_c$.
- ii. If n is even, then $(G_c)_{k(i)}^P \cong (K_{s,t})_c$ with $s = |V_1|$ and $t = |V_2|$ and vice-versa.

Proof: Let G_c be a graph with vertex set $V = \{1, 2, \dots, n\}$ and $m = n$.

- i. Let n be an odd. In the graph G_c , the vertex set $V = \{1, 2, \dots, n\}$, the degrees are $d(n) = n - 1$ and $d(l) = n - 2$ for $1 \leq l \leq n - 1$. Define the partition $P = \{V_1, V_2\}$, with $V_1 = \{1, 2, \dots, n - 1\}$ and $V_2 = \{n\}$. In G_c , the vertex labeled n is adjacent to all vertices and this adjacency is preserved in $(G_c)_{2(i)}^P$.

Since $|V_1| = n - 1$, each vertex $l \in V_1$, is non-adjacent to the vertex labeled $n - l$, as both share the same color. Therefore, $l \approx n - l$ in $(G_c)_{2(i)}^P$. Thus, every $l \in V_1$ has degree 1, being adjacent only to n . Hence, $(G_c)_{k(i)}^P$ is isomorphic to a star graph $(S_n)_c$, $k = 2$.

- ii. Let n be an even integer. In the graph G_c , the vertices n and $\frac{n}{2}$ each have degree $n - 1$, being adjacent to every other vertex, while all remaining vertices $l (\neq n, \frac{n}{2}) \in V(G_c)$ have degree $n - 2$. Define the partition P , where $V_1 = \{n, \frac{n}{2}\}$ and $V_2 = V \setminus V_1$. Since n and $\frac{n}{2}$ are adjacent in G_c , they become non-adjacent in $(G_c)_{2(i)}^P$, and thus each vertex of V_1 has degree $n - 2$ in $(G_c)_{2(i)}^P$.

For the vertices in $\langle V_2 \rangle$, observe that in G_c , each vertex $l \in V_2$ is adjacent to both n and $\frac{n}{2}$, but not to the vertex labeled $n - l$. Since l and $n - l$ share the same color, they remain non-adjacent in $(G_c)_{2(i)}^P$. Hence, each $l \in V_2$ is adjacent only to the vertices of V_1 , giving degree 2. Therefore, $(G_c)_{k(i)}^P$ is isomorphic to complete bipartite graph, $(K_{s,t})_c$, where $s = |V_1|$ and $t = |V_2|$ and vice-versa, for $k = 2$. \square

Theorem 4.2 For the graph G_c with $m = 2$,

- i. If n is even, then G_c is $k(i)$ -co-self color complementary.
- ii. If n is odd, then G_c is $k(i)$ -self color complementary.

Proof: Let G_c be a graph with vertex set $V = \{1, 2, \dots, n\}$ and $m = 2$.

- i. For even n , every vertex in G_c has degree $\frac{n}{2}$. Hence, the entire vertex set forms a single partition $P = V$. Under this partition, no edges exist between any two vertices in $(G_c)_{k(i)}^P$, which implies that $\overline{G_c} \cong (G_c)_{k(i)}^P$.

- ii. From Theorem 3.2, it follows that G_c is k -co-self color complementary when n is odd. Since the partition $P = \{V_1, V_2\}$ is defined based on equal degree, it can equivalently be viewed as a partition by color classes. Hence, by Proposition 2.1, G_c is both $k(i)$ -self color complementary and k -co-self color complementary for odd n .

□

Theorem 4.3 For any graph G_c and $m = n + 1$,

- i. G_c is $k(i)$ -co-self color complementary, if n is an even integer.
- ii. $(G_c)_{k(i)}^P \cong (S_n)_c$, if n is an odd integer.

Proof: Let G_c be a graph with vertex set $V = \{1, 2, \dots, n\}$ and $m = n + 1$.

- i. If n is even, then G_c is an $(n-2)$ -regular graph, implies $k = 1$. For each vertex $l \in V$ is non-adjacent to the vertex $m - l$, and since they share the same color, they remain non-adjacent in $(G_c)_{k(i)}^P$. From Theorem 3.3, it follows that G_c is k -self color complementary for even n . Furthermore, by Proposition 2.1, we have $(G_c)_{k(i)}^P \cong \overline{G_c}$ and $(G_c)_k^P \cong G_c$, for $k = 1$. Therefore, we conclude that for even n , G_c is both k -self color complementary and $k(i)$ -co-self color complementary with respect to $k = 1$.
- ii. If n is odd, then in G_c , the vertex labeled $\frac{n+1}{2}$ has degree $n - 1$, as it is adjacent to every other vertex in the graph. All other vertices $l (\neq \frac{n+1}{2}) \in V(G_c)$ have degree $n - 2$. Based on equal degree, we choose $k = 2$ such that $V_1 = \{\frac{n+1}{2}\}$ and $V_2 = V \setminus V_1$. The number of vertices in V_2 is $n - 1$. For any $l \in V_2$, we note that l is not adjacent to the vertex labeled $n - l + 1$, and since both share the same color, they remain non-adjacent in $(G_c)_{2(i)}^P$. Consequently, each vertex $l \in V_2$ is adjacent only to the vertex in V_1 . This implies that every vertex in V_2 has degree 1 in $(G_c)_{2(i)}^P$. Therefore, $(G_c)_{k(i)}^P$ is isomorphic to star graph $(S_n)_c$ for $k = 2$.

□

Theorem 4.4 For $m = n = 5$, it holds that $(G_c)_{k(i)}^P \cong (P_n)_c$ under the partition $P = \{V_1, V_2\}$, where $V_1 = \{1, 2, \dots, n\}$ and $V_2 = \{3, 4\}$.

Proof: Consider the graph G_c with $m = n = 5$ and the partition $P = \{V_1, V_2\}$, where $V_1 = \{1, 2, 5\}$ and $V_2 = \{3, 4\}$. In G_c , the vertex degrees are $d(i) = 3$ for $i = 1, 2, 3, 4$ and $d(5) = 4$. We have $o_d(5) = i_d(i) = 2$ for $i = 1, 2, 5$. For vertices $i = 3, 4$, $i_d(i) = 1$, $o_d(i) = 2$. Thus o_d of each vertex remains same in $(G_c)_{k(i)}^P$. Consequently, $(G_c)_{k(i)}^P$ retains the structure of a path, and is thus isomorphic to path $(P_5)_c$ for $k = 2$.

□

5. Conclusion

This paper explores the generalized color complement of the graph $G_{m,n}$, taking into account both vertex partitions and assigned vertex colors. We establish specific conditions on the values of m, n and the corresponding vertex partitions V under which the generalized color complements of $G_{m,n}$ exhibit distinct structural forms. In particular, we identify the cases where the graph is self color complementary, co-self color complementary, or isomorphic to well-known graph classes such as the cocktail party graph, complete bipartite graph, path and star graph.

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