Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.78977

Generalized Forward-Backward System for Fuzzy SDEs with Delayed Coefficients

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ABSTRACT: The aim of this work is to propose a model of forward-backward fuzzy stochastic doubly differential equations with delay coefficients, highlighting the existence theory and the uniqueness of the solution to this system under a set of appropriate conditions through which the monotonically of the solution was discussed, then proving and discussing the maximum solution to this system of equations.

Keywords: Stochastic differential equations, forward-backward doubly stochastic differential equations, Stochastic fuzzy differential equations, delay differential equations, maximal solution.

Contents

1. Introduction

Suppose that (Ω, F, P) is a complete probability space with the natural filtration $F(t), t \geq 0$. Consider the following forward-backward stochastic differential equation (FBSDE):

$$\begin{cases} d\Upsilon(t) = f(t, \Upsilon(t), \Psi(t))d(t) + g(t, \Upsilon(t))dB(t) - \Psi(t)d\overleftarrow{W}(t), \\ dX(t) = \acute{f}(t, X(t), \Gamma(t)) + \acute{g}(t, X(t))d\overleftarrow{W}(t) - \Gamma(t)dB(t) \\ \Upsilon(0) = \eta, X(T) = l(\Upsilon(T), \end{cases} \tag{1.1}$$

where $W(t),) \leq t \leq T$ and $B(t),) \leq t \leq T$, are two mutually independent standard Brownian motions defined on (Ω, F, P) with the processes Υ, X, Ψ and Γ are defined on R^{α}, R^{λ}, $R^{\alpha\times\beta}$ and $R^{\lambda\times\beta}$, respectively, and $E\int_0^T ||\Upsilon(t)||^2 dt < \infty$ Also, the continuous function $f, \acute{f}, g, \acute{g}\,'$ and l are defined on R^{α}, R^{λ}, $R^{\alpha\times\beta}$, $R^{\lambda\times\beta}$ and R^{α}, respectively. From system (1.1), we notice that the forward equation is forward with respect to a standard stochastic integral driven by Brownian motion $W(t), 0 \leq t \leq T$, and backward with respect to a backward stochastic integral driven by Brownian motion $B(t), 0 \leq t \leq T$. Also, the backward equation is forward under the backward stochastic integral $B(t), 0 \leq t \leq T$ and backward under the forward one. By using Euclidean norm, we define $|\Upsilon| = \{tr(\Upsilon\Upsilon^{trac})\}^{\frac{1}{2}}$, where Υ^{trac} denotes the transpose. Therefore, the space $R^{\alpha\times\lambda}$ is a Hilbert space.

Fabio [1] proved in this research the existence and uniqueness of the solution to the backward stochastic differential equation after assuming a model of the differential equation in light of the theory of finance. In addition, he expanded the study to include the model of backward-forward equations, while proving the existence and uniqueness of the solution to this model of equations under appropriate assumptions. Jin et al. [2] proposed a general formula for forward-backward differential equations with the possibility of studying the numerical convergence of solving this type of equations. Moreover, Peng et al. [3] studied the strength of the convergence of the solution to these equations with standard coefficients. Feng et al. [4] proposed a model for the double forward-backward equations and studied the numerical solutions

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2020 *Mathematics Subject Classification:* 65Cxx.
Submitted September 15, 2025. Published March 19, 2026

with some applications of this model. Qingfen et al. [5] studied the existence and uniqueness of the solution to the same model of double forward-backward equations with random jumps. Evelyn's paper deals with the study of numerical solutions to delay differential equations [6]. Xuerong [7] also proposed a model for stochastic deceleration equations and highlighted numerical solutions for this type. Marek and Mariusz [8,9] studied the fuzzy theory and its application to stochastic differential equations, as well as studying the strong of the solution with some applications. Bernt [10] presented a different model for forward-backward stochastic differential equations and the possibility of studying the control of the optimal solution under the principle of maximum solution to minimize risks with some applications. Mohammed and Falah [11,12] expanded the study of the forward-backward fuzzy stochastic differential equation model under delayed coefficients. In this paper, we present another different model for the system of forward-backward fuzzy double stochastic differential equations with delayed coefficients. We study the monotonicity of the solution of this proposed model, with the possibility of studying the existence and uniqueness of the solution, in addition to shedding light on the maximum solution.

2. Basic Hypothesis

For all finite time horizon $T < \infty$, Assume that $W(t), 0 \leq t \leq T$ and $B(t), 0 \leq t \leq T$, are two mutually independent standard Brownian motions defined on the complete probability spaces (Ω_1, F_1, P_1) and (Ω_2, F_2, P_2) respectively. Suppose that N is the class of P-null sets of $F(t), t \in [0,T]$. Let $\Omega = \Omega_1 \times \Omega_2$, $F = F_1 \times F_2$ and $P = P_1 \times P_2$. For any $t \in [0,T]$, we define $F_t = F_t^W \vee F_{t,T}^B \vee N$, for any process X_t then $F_t^X = F_{0,t}^X, F_{a,t}^X = \sigma\{X_a - X_b; a \leq b \leq t\}$. It is clear that that the family of $\sigma-$fields $F(t), t \in [0,T]$, is neither increasing nor decreasing. Let $L^2(0,T;H)$ be the set of jointly measurable processes $\Upsilon(t), t \in [0,T]$, taking value from Euclidean space H. Therefore, the process $\Upsilon(t), t \in [0,T]$, is F_t-measurable such that $E\int_0^T |\Upsilon(t)|^2 dt < \infty$. Let us consider the following spaces:

i) Let $G^2(\Omega, F, P, R^\alpha)$ be the continuous space $\{F_v\}_{t \leq v \leq T}-$adapted process $\Upsilon : \Omega \times [t,T] \to R^\alpha$ such that $||\Upsilon||^2_{G^2} = E[\sup_{t \leq v \leq T} |\Upsilon|^2] < \infty$.

ii) Let $S^2(\Omega, F, P, R^{\alpha \times \lambda})$ be the space of $\{F_v\}_{t \leq v \leq T}-$progressively measurable process $\Psi : \Omega \times [t,T] \to R^{\alpha \times \lambda}$ such that $||\Psi||^2_{S^2} = E[\int_t^T |\Psi(v)|^2 dv] < \infty$.

iii) Let $H^2(\Omega, F, P, R^\alpha)$ be the space of $\{F_t\}_{0 \leq t \leq T}-$measurable variable $\Upsilon : \Omega \times [0,T] \to R^\alpha$ such that $E[|\Upsilon|^2] < \infty, t \in [0,T]$.

iv) Let $Q^2_{-T}(R^\alpha)$ be the space of measurable function $\Upsilon : \Omega \times [-T,0] \to R^\alpha$ such that $\sup_{-T \leq t \leq 0} |\Upsilon|^2 < \infty$.

v) Let $Q^2_{-T}(R^{\alpha \times \lambda})$ be the space of measurable function $\Psi : \Omega \times [-T,0] \to R^{\alpha \times \lambda}$ such that $\int_{-T}^0 |\Psi(t)|^2 < \infty$.

Suppose that $MO = MO([0,T], G^2(\Omega, F, P, R^\alpha))$ is the class of all mean continuous second order with defined the norm of stochastic process as: $||\Upsilon||_{MO} = \sup_{0 \leq t \leq T} = \sup_{0 \leq t \leq T}(E(\Upsilon)^2)^{\frac{1}{2}}$. Let us define the norms $||\Upsilon||^2_{G^2} = E[\sup_{t \leq v \leq T} |\Upsilon|^2]$ and $||\Upsilon||^2_{G^2} = E[\int_t^T |\Upsilon(v)|^2 dv], t \leq v \leq T$. Her, we consider the forward-backward stochastic doubly delay differential equation:

$$\begin{cases} d\Upsilon(t) = f(t, \Upsilon(t), \Psi(t), \Upsilon_t, \Psi_t)d(t) + g(t, \Upsilon(t), \Upsilon_t)dB(t) - \Psi(t)d\overleftarrow{W}(t), 0 \leq t \leq T \\ \Upsilon = \eta(\delta_0), -t \leq \delta \leq 0 \\ dX(t) = \acute{f}(t, X(t), \Gamma(t), X_t, \Gamma_t)d(t) + \acute{g}(t, X(t), X_t)d\overleftarrow{W}(v) - \Gamma(t)dB(t) \\ X_T = l(\delta_T), -T \leq \delta \leq 0, \end{cases} \tag{2.1}$$

where $f : \Omega \times [0,T] \times R^\alpha \times R^{\alpha \times \beta} \times Q^2_{-T}(R^\alpha) \times Q^2_{-T}(R^{\alpha \times \beta}) \to R^\alpha$, $\acute{f} : \Omega \times [0,T] \times R^\lambda \times R^{\lambda \times \beta} \times Q^2_{-T}(R^\lambda) \times Q^2_{-T}(R^{\lambda \times \beta}) \to R^\lambda$, $g : \Omega \times [0,T] \times R^\alpha \times Q^2_{-T}(R^\alpha) \to R^{\alpha \times \beta}$, $\acute{g} : \Omega \times [0,T] \times R^\lambda \times Q^2_{-T}(R^\lambda) \to R^{\lambda \times \beta}$ and $l : \Omega \times [0,T] \times R^\alpha \to R^\alpha$ are product measurable depend on the past values such that $\Upsilon_v = (\Upsilon(v + d)), -T \leq \delta \leq 0$, $X_v = (X(v + \delta)), -T \leq \delta \leq 0$ and $\Psi_v = (\Psi(v + \delta)), -T \leq \delta \leq 0$. Her, by taking the measure Υ as dirac measure with finitely valued measure δ supported on $[-T, 0]$ and for every

$v \in [0,T], (\Upsilon^1, \Psi^1), (\Upsilon^2, \Psi^2) \in R^\alpha \times R^{\alpha\times\beta}$ and $(\Upsilon^1_t, \Psi^1_t), (\Upsilon^2_t, \Psi^2_t) \in Q^2_{-T}(R^\alpha) \times Q^2_{-T}(R^{\alpha\times\beta})$, we consider the following hypotheses:
Hyp1: $\forall(\Upsilon, X, \Psi) \in R^\alpha \times R^\lambda \times R^{\alpha\times\beta}$ and for all constants $C_1, C_2, C_3, C_4, R_1, R_2, R_3, R_4 > 0$ such that:
i) $|f(t, \Upsilon^1(t), \Psi^1(t), \Upsilon^1_t, \Psi^1_t) - f(t, \Upsilon^2(t), \Psi^2(t), \Upsilon^2_t, \Psi^2_t)|^2 \leq C_1(|\Upsilon^1(t) - \Upsilon^2(t)|^2 + |\Psi^1(t) - \Psi^2(t)|^2) + R_1(\int_{-T}^0 |\Upsilon^1(t+\delta) - \Upsilon^2(t+\delta)|^2\gamma d\delta + \int_{-T}^0 |X^1(t+\delta) - X^2(t+\delta)|^2\gamma d\delta)$.
ii) $|\acute{f}(t, X^1(t), \Gamma^1(t), X^1_t, \Gamma^1_t) - \acute{f}(t, X^2(t), \Gamma^2(t), X^2_t, \Gamma^2_t)|^2 \leq C_2(|X^1(t) - X^2(t)|^2 + |\Gamma^1(t) - \Gamma^2(t)|^2) + R_2(\int_{-T}^0 |X^1(t+\delta) - X^2(t+\delta)|^2\gamma d\delta + \int_{-T}^0 |\Gamma^1(t+\delta) - \Gamma^2(t+\delta)|^2\gamma d\delta)$.
iii) $|g(t, \Upsilon^1(t), \Upsilon^1_t) - g(t, \Upsilon^2(t), \Upsilon^2_t)|^2 \leq C_3(|\Upsilon^1(t) - \Upsilon^2(t)|^2) + R_3(\int_{-T}^0 |\Upsilon^1(t+\delta) - \Upsilon^2(t+\delta)|^2\gamma d\delta)$.
iv) $|\acute{g}(t, X^1(t), X^1_t) - \acute{g}(t, X^2(t), X^2_t)|^2 \leq C_4(|X^1(t) - X^2(t)|^2) + R_4(\int_{-T}^0 |X^1(t+\delta) - X^2(t+\delta)|^2\gamma d\delta)$.
Hyp2. For all $C_5, C_6, C_7, C_8, C_9, R_5, R_6, R_7, R_8 > 0$ such that
i) $|f(t, \Upsilon(t), \Psi(t), \Upsilon_t, \Psi_t)|^2 \leq C_5(1 + |\Upsilon(t)|^2 + |\Psi(t)|^2) + R_5(\int_{-T}^0 |\Upsilon(t+\delta)|^2\gamma d\delta + \int_{-T}^0 |\Psi(t+\delta)|^2\gamma d\delta)$.
ii) $|\acute{f}(t, X(t), \Gamma(t), X_t, \Gamma_t)|^2 \leq C_6(1 + |X(t)|^2 + |\Gamma(t)|^2) + R_6(\int_{-T}^0 |X(t+\delta)|^2\gamma d\delta + \int_{-T}^0 |\Gamma(t+\delta)|^2\gamma d\delta)$.
iii) $|g(t, \Upsilon(t), \Upsilon_t)|^2 \leq C_7(1 + |\Upsilon(t)|^2) + R_7(\int_{-T}^0 |\Upsilon(t+\delta)|^2\gamma d\delta)$.
iv) $|\acute{g}(t, X(t), X_t)|^2 \leq C_8(1 + |X(t)|^2) + R_8(\int_{-T}^0 |X(t+\delta)|^2\gamma d\delta)$.
v) For all $-v \leq \delta < t \leq 0$ such that $E(|\eta(t) - \eta(\delta)|^2) \leq C_9(t - \delta)$ and $|l(\Upsilon(t))|^2 \leq C_{10}(1 + |\Upsilon(t)|^2) + R_9(\int_{-T}^0 |\Upsilon(t+\delta)|^2\gamma d\delta$.
Hyp3. $E(\int_0^T |f(v,0,0,0,0)|^2 dv) < \infty$, $E(\int_0^T |\acute{f}(v,0,0,0,0)|^2 dv) < \infty$, $E(\int_0^T |g(v,0,0)|^2 dv) < \infty$, $E(\int_0^T |\acute{g}(v,0,0)|^2 dv) < \infty$.
Hyp4. $f(t,\cdot,\cdot,\cdot,\cdot) = 0$, $\acute{f}(t,\cdot,\cdot,\cdot,\cdot) = 0$, $g(t,\cdot,\cdot) = 0$ and $\acute{g}(t,\cdot,\cdot) = 0$ for $t > 0$.
Hyp5. For any $C_{11}, C_{12}, C_{13}, C_{14}, R_{10}, R_{11}, R_{12}, R_{13}$ such that
i) $|f(t, \Upsilon(t), \Psi(t), \Upsilon_t, \Psi_t)| \leq C_{10}(1 + |\Upsilon(t)| + |\Psi(t)|) + R_{10}(\int_{-T}^0 |\Upsilon(t+\delta)|\gamma d\delta + \int_{-T}^0 |\Psi(t+\delta)|\gamma d\delta)$.
ii) $|\acute{f}(t, X(t), \Gamma(t), X_t, \Gamma_t)| \leq C_{12}(1 + |X(t)| + |\Gamma(t)|) + R_{11}(\int_{-T}^0 |X(t+\delta)|\gamma d\delta + \int_{-T}^0 |\Gamma(t+\delta)|\gamma d\delta)$.
iii) $|g(t, \Upsilon(t), \Upsilon_t)| \leq C_{13}(1 + |\Upsilon(t)|) + R_{12}(\int_{-T}^0 |\Upsilon(t+\delta)|\gamma d\delta)$.
iv) $|\acute{g}(t, X(t), X_t)| \leq C_{14}(1 + |X(t)|) + R_{13}(\int_{-T}^0 |X(t+\delta)|\gamma d\delta)$.
Hyp6. For any $n \geq 1$ such that
i) $f(t, \Upsilon(t), \Psi(t), \Upsilon_t, \Psi_t) \leq f^{n+1}(t, \Upsilon(t), \Psi(t), \Upsilon_t, \Psi_t) \leq f^n(t, \Upsilon(t), \Psi(t), \Upsilon_t, \Psi_t)$.
ii) $\acute{f}(t, X(t), \Gamma(t), X_t, \Gamma_t) \leq \acute{f}^{n+1}(t, X(t), \Gamma(t), X_t, \Gamma_t) \leq \acute{f}^n(t, X(t), \Gamma(t), X_t, \Gamma_t)$.

3. Preliminaries and Basic Definitions

Assume that $\prod(R^\alpha)$ is a family of all convex, compact and nonempty subsets R^α and $\Xi(R^\alpha)$ is a fuzzy set space of R^α. Therefore, we define the set of functions $\Lambda : R^\alpha \to [0,1]$ such that $[\Lambda]^r \in \prod(R^\alpha)$ for every $r \in [0,1]$ and $[\Lambda]^r = \{z \in R^\alpha : \Lambda(z) \geq r, r \in [0,1]\} = [\Lambda^r_L, \Lambda^r_U]$, where $\Lambda^r_L = \inf_{z\in R^\alpha}\{z \in [\Lambda]^r\}$ and $\Lambda^r_U = \sup_{z\in R^\alpha}\{z \in [\Lambda]^r\}$ and $[\Lambda]^0 = cl\{z \in R^\alpha : \Lambda(z) > 0\}$.

Definition 3.1. *Suppose that (Ω, F, P) is a complete probability space, we define $Z : \Omega \to \Xi(R^\alpha)$ is a fuzzy random variable, if for any $r \in [0,1]$, $[Z]^r : \Omega \to \prod(R^\alpha)$ is an $F-$measurable multifunction.*

Definition 3.2. *Suppose that (Ω, F, P) is a complete probability space, we define $Z : [0,T] \times \Omega \to \Xi(R^\alpha)$ is a fuzzy stochastic process if $Z(t,\cdot) = Z(t) : \Omega \to \Xi(R^\alpha), t \in [0,T]$, is a fuzzy random variable.*

Definition 3.3. *Fuzzy stochastic process Z is $b_\infty-$continuous, if the mappings $Z(\cdot, W) : [0,T] \to \Xi(R^\alpha)$ are $b_\infty-$continuous functions.*

Definition 3.4. *Let $\Upsilon(v)$ be a solution of stochastic differential equation. $\Upsilon(v)$ is a maximal solution if every solution $\Theta(v)$ such that $E(\Theta^2(v)) < E(\Upsilon^2(v))$.*

Definition 3.5. *Let $\Theta(v), \Upsilon(v) \in G^2([0,T], R^\alpha), v \in [0,T]$, such that $||\Theta(v)||^2 < ||\Upsilon(v)||^2$. A function $\Phi : G^2_{[0,T]}(\Omega, F, R^\alpha) \to G^2_{[0,T]}(\Omega, F, R^\alpha)$ is stochastically increasing if $||\Phi(v, \Theta(v))||^2 < ||\Phi(v, \Upsilon(v))||^2$.*

Definition 3.6. *Let $\Theta(v), \Upsilon(v) \in G^2([0,T], R^\alpha), v \in [0,T]$, such that $||\Theta(v)||^2 < ||\Upsilon(v)||^2$. A function $\Phi : G^2_{[0,T]}(\Omega, F, R^\alpha) \to G^2_{[0,T]}(\Omega, F, R^\alpha)$ is stochastically decreasing if $||\Phi(v, \Theta(v))||^2 > ||\Phi(v, \Upsilon(v))||^2$.*

Let's know the forward-backward fuzzy stochastic doubly delay differential equation (FBFSDDDE) in the following form:

$$\begin{cases} d\Upsilon(t) = f(t, \Upsilon(t), \Psi(t), \Upsilon_t, \Psi_t)d(t) + g(t, \Upsilon(t), \Upsilon_t)dB(t) - \Psi(t)d\overleftarrow{W}(t) \\ dX(t) = \acute{f}(t, X(t), \Gamma(t), X_t, \Gamma_t)d(t) + \acute{g}(t, X(t), X_t)d\overleftarrow{W}(v) - \Gamma(t)dB(t) \\ \Upsilon = \eta(\delta_0), -t \leq \delta \leq 0, X_T = l(\delta_T), -T \leq \delta \leq 0, \end{cases} \tag{3.1}$$

where the processes Υ, X, Ψ and Γ are defined on $R^\alpha, R^\lambda, R^{\alpha\times\beta}$ and $R^{\lambda\times\beta}$, respectively and $\eta(\delta_0)$ is a given F_1-measurable random variable with $E|\eta|^2 < \infty$ as well as the functions $f : \Omega\times[0, T]\times R^\alpha\times R^{\alpha\times\beta}\times Q^2_{-T}(R^\alpha)\times Q^2_{-T}(R^{\alpha\times\beta})\times\Xi(R^\alpha) \to \Xi(R^\alpha)$, $\acute{f} : \Omega\times[0, T]\times R^\lambda\times R^{\lambda\times\beta}\times Q^2_{-T}(R^\lambda)\times Q^2_{-T}(R^{\lambda\times\beta})\times\Xi(R^\lambda) \to \Xi(R^\lambda)$, $g : \Omega \times [0, T] \times R^\alpha \times Q^2_{-T}(R^\alpha) \times \Xi(R^\alpha) \to R^{\alpha\times\beta}$, $\acute{g} : \Omega \times [0, T] \times R^\lambda \times Q^2_{-T}(R^\lambda) \times \Xi(R^\lambda) \to R^{\lambda\times\beta}$ and $l : \Omega \times [0, T] \times R^\alpha \times \Xi(R^\alpha) \to \Xi(R^\alpha)$. Her, we introduce numerical formula: let $0 = v_0 < v_1 < \cdots < v_n = T, n \geq 1$ be a partition on interval $[0, T]$, let $\pi = \Delta v_{j+1} = v_{j+1} - v_j = \frac{T}{n}, \Delta W_{v_{j+1}} = W_{v_{j+1}} - W_{v_j}$, $\Delta B_{v_{j+1}} = B_{v_{j+1}} - B_{v_j}$ and $\Delta v = \max \Delta v_{j+1}$, for $j = 0, 1, \cdots, n-1, n \geq 1$. Let's know the FBFSDDDEs on the small interval $[v_j, v_{j+1}]$ in the following form:

$$\begin{cases} \Upsilon(v_j) = \eta + \int_{v_j}^{v_{j+1}} f(t, \Upsilon(t), \Psi(t), \Upsilon_t, \Psi_t)dt + \int_{v_j}^{v_{j+1}} g(t, \Upsilon(t), \Upsilon_t)dB(t) - \int_{v_j}^{v_{j+1}} \Psi(t)d\overleftarrow{W}(t) \\ X(v_j) = X(v_{j+1}) + \int_{v_j}^{v_{j+1}} \acute{f}(t, X(t), \Gamma(t), X_t, \Gamma_t)dt + \int_{v_j}^{v_{j+1}} \acute{g}(t, X(t), X_t)d\overleftarrow{W}(v) - \int_{v_j}^{v_{j+1}} \Gamma(t)dB(t) \end{cases} \tag{3.2}$$

Her, we consider approximation formula as follows:

$$\begin{cases} \Upsilon(v) = \Upsilon(0) + \int_0^v f(t, \Upsilon^n_j(t), \Psi^n_j(t), \Upsilon^n_{j\,t}, \Psi^n_{j\,t})dt + \int_0^v g(t, \Upsilon^n_j(t), \Upsilon^n_{j\,t})dB(t) - \int_0^v \Psi(t)d\overleftarrow{W}(t) \\ X(v) = X(T) + \int_v^T \acute{f}(t, X^n_j(t), \Gamma^n_j(t), X^n_{j\,t}, \Gamma^n_{j\,t})dt + \int_v^T \acute{g}(t, X^n_j(t), X^n_{j\,t})d\overleftarrow{W}(v) - \int_v^T \Gamma(t)dB(t) \end{cases} \tag{3.3}$$

where $t \in [0, T], j = 1, 2, \cdots, n$.

Lemma 3.7. *Under hypotheses (Hyp1-Hyp6), the FBFSDDDE system (3.3) has an unique solution* $(\Upsilon^n, \Psi^n, X^n, \Gamma^n)$*, and there exists a constant* $M > 0$ *such that*

$$||\Upsilon^n||_{G^2} + ||\Psi^n||_{S^2} + ||X^n||_{G^2} + ||\Gamma^n||_{S^2} \leq M.$$

Proof. For any $v \in [0, T]$, we take the forward of FBFSDDDE (3.3):

$$\Upsilon(v) = \Upsilon(0) + \int_0^v f(t, \Upsilon^n_j(t), \Psi^n_j(t), \Upsilon^n_{j\,t}, \Psi^n_{j\,t})dt + \int_0^v g(t, \Upsilon^n_j(t), \Upsilon^n_{j\,t})dB(t) - \int_0^v \Psi(t)d\overleftarrow{W}(t) \tag{3.4}$$

Her, we define the mapping: $J(q^n, y^n) = (\Upsilon^n, \Psi^n)$ and $(\Upsilon^{i,n}, \Psi^{i,n}) = J(q^{i,n}, y^{i,n}), i = 1, 2$. Let $(\bar{\Upsilon^n}, \bar{\Psi^n}) = (\Upsilon^{1,n} - \Upsilon^{2,n}, \Psi^{1,n} - \Psi^{2,n})$. By applying Ito's formula to $|\bar{\Upsilon^n}(v)|^2, v \in [0, T]$, we have

$$E[|\bar{\Upsilon^n}(v)|^2] + E[\int_0^v |\bar{\Upsilon^n}(t)|^2dt] + E[\int_0^v |\bar{\Psi^n}(t)|^2dt] = 2E[\int_0^v \bar{\Upsilon^n}(t)\bar{f}(t)dt] + E[\int_0^v |\bar{g}(t)|^2dt], \tag{3.5}$$

where

$$\bar{f}(t) = f(t, \Upsilon^{1,n}(t), \Psi^{1,n}(t), \Upsilon^{1,n}_t, \Psi^{1,n}_t) - f(t, \Upsilon^{2,n}(t), \Psi^{2,n}(t), \Upsilon^{2,n}_t, \Psi^{2,n}_t),$$

$$\bar{g}(t) = g(t, \Upsilon^{1,n}(t), \Upsilon^{1,n}_t) - g(t, \Upsilon^{2,n}(t), \Upsilon^{2,n}_t).$$

Applying inequality $2xy \leq \frac{1}{b}x^2 + by^2, b > 0$, we have

$$\begin{gathered} 2E[\int_0^v \bar{\Upsilon^n}(t)\bar{f}(t)dt] \leq 2E[\int_0^v |\bar{\Upsilon^n}(t)||\bar{f}(t)|dt] \leq \tfrac{1}{b}E[\int_0^v |\bar{\Upsilon^n}(t)|^2dt] + b2E[\int_0^v |\bar{f}(t)|^2dt] \\ \leq \tfrac{1}{b}E[\int_0^v |\bar{\Upsilon^n}(t)|^2dt] + bE[\int_0^v [C_1(|\bar{q^n}(t)|^2 + |\bar{y^n}(t)|^2) + R_1(\int_{-v}^0 |(\bar{q^n}(t+\delta)|^2\gamma d\delta + \int_{-v}^0 |(\bar{y^n}(t+\delta)|^2\gamma d\delta]dt] \\ \leq \tfrac{1}{b}E[\int_0^v |\bar{\Upsilon^n}(t)|^2dt] + bC_1E[\int_0^v |\bar{q^n}(t)|^2dt] + bC_1E[\int_0^v |\bar{y^n}(t)|^2dt] + bR_1E[\int_0^v (\int_{-v}^0 |\bar{q^n}(t+\delta)|^2\gamma d\delta)dt] + \\ bR_1E[\int_0^v (\int_{-v}^0 |(\bar{y^n}(t+\delta)|^2\gamma d\delta)dt]. \end{gathered}$$

By changing of integration order argument, we have

$$\int_0^v(\int_{-v}^0 |\bar{q^n}(t+\delta)|^2\gamma d\delta)dt = \int_{-v}^0(\int_0^v |\bar{q^n}(t+\delta)|^2\gamma d\delta)dt = \int_{-v}^0(\int_0^{v+\delta} |\bar{q^n}(t)|^2\gamma d\delta)dt \leq \mu\int_0^v |\bar{q^n}(t)|^2dt$$

and

$$\int_0^v(\int_{-v}^0 |\bar{y^n}(t+\delta)|^2\gamma d\delta)dt = \int_{-v}^0(\int_0^v |\bar{y^n}(t+\delta)|^2\gamma d\delta)dt = \int_{-v}^0(\int_0^{v+\delta} |\bar{y^n}(t)|^2\gamma d\delta)dt \leq \mu\int_0^v |\bar{y^n}(t)|^2dt,$$

where $\mu = \int_{-v}^0 \gamma d\delta$. Therefore, we have

$$\begin{gathered} 2E[\int_0^v \bar{\Upsilon}^n(t)\bar{f}(t)dt] \leq \tfrac{1}{b}E[\int_0^v |\bar{\Upsilon}^n(t)|^2dt] + bC_1E[\int_0^v |\bar{q^n}(t)|^2dt] + bC_1R_1\mu E[\int_0^v |\bar{y^n}(t)|^2dt] + \\ bC_1E[\int_0^v |\bar{q^n}(t)|^2dt] + bC_1R_1\mu E[\int_0^v |\bar{y^n}(t)|^2dt] = \\ \tfrac{1}{b}E[\int_0^v |\bar{\Upsilon}^n(t)|^2dt] + (bC_1+bR_1\mu)E[\int_0^v |\bar{q^n}(t)|^2dt] + (bC_1+bR_1\mu)E[\int_0^v |\bar{y^n}(t)|^2dt] \end{gathered}$$

and

$$\begin{gathered} E[\int_0^v |\bar{g}(t)|^2dt] \leq E[\int_0^v[C_6(1+|\bar{q^n}(t)|^2) + R_6(\int_{-v}^0 |\bar{q^n}(t+\delta)|^2\gamma d\delta)]dt] \leq \\ C_6E[\int_0^v |\bar{q^n}(t)|^2dt] + R_6E[\int_0^v(\int_{-v}^0 |\bar{q^n}(t+\delta)|^2\gamma d\delta)dt]. \end{gathered}$$

Therefore, we deduce

$$E[\int_0^v |\bar{g}(t)|^2dt] \leq C_6E[\int_0^v |\bar{\Upsilon}^n(t)|^2dt] + R_6\mu E[\int_0^v |\bar{\Upsilon}^n(t)|^2dt]$$

Consequently by (3.5), we have

$$\begin{gathered} E[|\bar{\Upsilon}^n(v)|^2] + E[\int_0^v |\bar{\Upsilon}^n(t)|^2dt] + E[\int_0^v |\bar{\Psi}^n(t)|^2dt] \leq \tfrac{1}{b}E[\int_0^v |\bar{\Upsilon}^n(t)|^2dt] + (bC_1+bR_1\mu)E[\int_0^v |\bar{q^n}(t)|^2dt] + \\ (bC_1+bR_1\mu)E[\int_0^v |\bar{y^n}(t)|^2dt] + C_6E[\int_0^v |\bar{\Upsilon}^n(t)|^2dt] + R_6\mu E[\int_0^v |\bar{\Upsilon}^n(t)|^2dt] = \\ (\tfrac{1}{b}+C_6+R_6\mu)E[\int_0^v |\bar{\Upsilon}^n(t)|^2dt] + (bC_1+bR_1\mu)E[\int_0^v |\bar{q^n}(t)|^2dt] + (bC_1+bR_1\mu)E[\int_0^v |\bar{y^n}(t)|^2dt]. \end{gathered}$$

Therefore, we have

$$\begin{gathered} (1-\tfrac{1}{b}-C_6-R_6\mu)E[\int_0^v |\bar{\Upsilon}^n(t)|^2dt] + E[\int_0^v |\bar{\Psi}^n(t)|^2dt] \leq \\ (bC_1+bR_1\mu)E[\int_0^v |\bar{q^n}(t)|^2dt] + (bC_1+bR_1\mu)E[\int_0^v |\bar{y^n}(t)|^2dt]. \end{gathered}$$

From contraction mapping theorem, we conclude the existence of a unique fixed point $(\Upsilon^n, \Psi^n) \in G^2(\Omega, F, P, R^\alpha)\times S^2(\Omega, F, P, R^{\alpha\times\beta})$ such that $J(\Upsilon^n, \Psi^n) = (\Upsilon^n, \Psi^n)$. There is a unique solution (Υ^n, Ψ^n) for the equation (3.4). Moreover, from a classical result for backward of FBFSDDDEs, then the existence and unique of solution (X^n, Γ^n) are fulfilled.

Part 1, we must prove that there exists a constant $M_1 > 0$ such that

$$||\Upsilon^n||_{G^2} + ||\Psi^n||_{S^2} \leq M_1$$

Taking Ito's formula for $|\Upsilon^n(t)|^2$, we deduce

$$\begin{cases} |\Upsilon^n(v)|^2 + \int_0^v |\Upsilon^n(t)|^2dt + \int_0^v |\Psi^n(t)|^2dt = |\eta|^2 + 2\int_0^v \Upsilon^n(t)f(t, \Upsilon^n(t), \Psi^n(t), \Upsilon^n_t, \Psi^n_t)dt \\ +2\int_0^v \Upsilon^n(t)g(t, \Upsilon^n(t), \Upsilon^n_t)dB(t) - 2\int_0^v \Upsilon^n(t)\Psi^n(t)d\overleftarrow{W}(t) + \int_0^v |g(t, \Upsilon^n(t), \Upsilon^n_t)|^2dt. \end{cases} \tag{3.6}$$

Applying inequality $2xy \leq \frac{1}{b}x^2 + by^2, b > 0$, we have

$$\begin{gathered} 2\int_0^v \Upsilon^n(t)f(t, \Upsilon^n(t), \Psi^n(t), \Upsilon^n_t, \Psi^n_t)dt \leq \tfrac{1}{b}\int_0^v |\Upsilon^n(t)|^2dt + b\int_0^v |f(t, \Upsilon^n(t), \Psi^n(t), \Upsilon^n_t, \Psi^n_t)|^2dt \leq \\ \tfrac{1}{b}\int_0^v |\Upsilon^n(t)|^2dt + b\int_0^v[C_4(1+|\Upsilon^n(t)|^2+|\Psi^n(t)|^2) + R_4(\int_{-v}^0 |\Upsilon^n(t+\delta)|^2\gamma d\delta + \int_{-v}^0 |\Psi^n(t+\delta)|^2\gamma d\delta)]dt \leq \\ \tfrac{1}{b}\int_0^v |\Upsilon^n(t)|^2dt + bC_4\int_0^v |f(t,0,0,0,0)|^2dt + bC_4\int_0^v |\Upsilon^n(t)|^2dt + bC_4\int_0^v |\Psi^n(t)|^2dt + bR_4\int_0^v[\int_{-v}^0 |\Upsilon^n(t+ \\ \delta)|^2\gamma d\delta]dt + bR_4\int_0^v[\int_{-v}^0 |\Psi^n(t+\delta)|^2\gamma d\delta]dt \end{gathered}$$

Again By changing of integration order argument, we have

$$\begin{cases} 2\int_0^v \Upsilon^n(t)f(t,\Upsilon^n(t),\Psi^n(t),\Upsilon^n_t,\Psi^n_t)dt \leq \frac{1}{b}\int_0^v |\Upsilon^n(t)|^2dt + bC_4\int_0^v |f(t,0,0,0,0)|^2dt \\ +bC_4\int_0^v |\Upsilon^n(t)|^2dt + bC_4\int_0^v |\Psi^n(t)|^2dt + \mu bR_4\int_0^v |\Upsilon^n(t)|^2dt + \mu bR_4\int_0^v |\Psi^n(t)|^2dt \\ = (\frac{1}{b} + bC_4 + \mu bR_4)\int_0^v |\Upsilon^n(t)|^2dt + (bC_4 + \mu bR_4)\int_0^v |\Psi^n(t)|^2dt + bC_4\int_0^v |f(t,0,0,0,0)|^2dt, \end{cases} \tag{3.7}$$

also

$$\begin{cases} 2\int_0^v \Upsilon^n(t)g(t,\Upsilon^n(t),\Upsilon^n_t)dB(t) \leq \frac{1}{b}\int_0^v |\Upsilon^n(t)|^2dt + b\int_0^v |g(t,\Upsilon^n(t),\Upsilon^n_t)|^2dt \\ \leq \frac{1}{b}\int_0^v |\Upsilon^n(t)|^2dt + b\int_0^v [C_6(1+|\Upsilon^n|^2) + R_6(\int_{-v}^0 |\Upsilon^n(t+\delta)|^2\gamma d\delta)]dt \\ \leq \frac{1}{b}\int_0^v |\Upsilon^n(t)|^2dt + bC_6\int_0^v |g(t,0,0)|^2dt + bC_6\int_0^v |\Upsilon^n(t)|^2dt + bR_6\int_0^v [\int_{-v}^0 |\Upsilon^n(t+\delta)|^2\gamma d\delta]dt \\ \leq \frac{1}{b}\int_0^v |\Upsilon^n(t)|^2dt + bC_6\int_0^v |g(t,0,0)|^2dt + bC_6\int_0^v |\Upsilon^n(t)|^2dt + bR_6\mu\int_0^v |\Upsilon^n(t)|^2dt \\ = (\frac{1}{b} + bC_6 + bR_6\mu)\int_0^v |\Upsilon^n(t)|^2dt + bC_6\int_0^v |g(t,0,0)|^2dt, \end{cases} \tag{3.8}$$

consequently,

$$2\int_0^v \Upsilon^n(t)\Psi^n(t)d\overleftarrow{W}(t) \leq \frac{1}{b}\int_0^v |\Upsilon^n(t)|^2dt + b\int_0^v |\Psi^n(t)|^2dt, \tag{3.9}$$

and

$$\begin{cases} \int_0^v |g(t,\Upsilon^n(t),\Upsilon^n_t)|^2dt \leq \int_0^v [C_6(1+|\Upsilon^n(t)|^2) + R_6(\int_{-v}^0 |\Upsilon^n(t+\delta)|^2\gamma d\delta)]dt \\ \leq C_6\int_0^v |g(t,0,0)|^2dt + C_6\int_0^v |\Upsilon^n(t)|^2dt + R_6\mu\int_0^v |\Upsilon^n(t)|^2dt \\ = (C_6 + R_6\mu)\int_0^v |\Upsilon^n(t)|^2dt + C_6\int_0^v |g(t,0,0)|^2dt \end{cases} \tag{3.10}$$

Combining (3.7), (3.8), (3.9) and (3.10) with (3.6), we have

$$\begin{cases} |\Upsilon^n(v)|^2 + \int_0^v |\Upsilon^n(t)|^2dt + \int_0^v |\Psi^n(t)|^2dt \leq |\eta|^2 + (\frac{1}{b} + bC_4 + \mu bR_4 + \frac{1}{b} + bC_6 + \mu bR_6 \\ -\frac{1}{b} + C_4 + \mu R_4)\int_0^v |\Upsilon^n(t)|^2dt + (bC_4 + \mu bR_4 - b)\int_0^v |\Psi^n(t)|^2dt + bC_4\int_0^v |f(t,0,0,0,0)|^2dt \\ +(bC_6 + C_6)\int_0^v |g(t,0,0)|^2dt. \end{cases} \tag{3.11}$$

By taking the expectation, we have

$$E[|\Upsilon^n(v)|^2] + D_1E[\int_0^v |\Upsilon^n(t)|^2dt] + D_2E[\int_0^v |\Psi^n(t)|^2dt] \leq$$
$$E[|\eta|^2] + bC_4E[\int_0^v |f(t,0,0,0,0)|^2dt] + (bC_6 + C_6)E[\int_0^v |g(t,0,0)|^2dt],$$

where $D_1 = \frac{b-1-b^2C_4-\mu b^2R_4-b^2C_6-\mu b^2R_6-bC_6-\mu bR_6}{b}$ and $D_2 = 1-bC_4-\mu bR_4+b$. For choosing $D_1, D_2 > 0$. Therefore, there exists a constant $N > 0$ depending on D_1 and D_2 such that

$$E[|\Upsilon^n(v)|^2] + E[\int_0^v |\Upsilon^n(t)|^2dt] + E[\int_0^v |\Psi^n(t)|^2dt] \leq$$
$$N[E[|\eta|^2] + E[\int_0^v |f(t,0,0,0,0)|^2dt] + E[\int_0^v |g(t,0,0)|^2dt]],$$

Applying the Burkholder-Davis-Gundy inequality and Young's inequality, then there exists a constant $M_1 > 0$ such that

$$E[\sup_{0\leq t\leq v} |\Upsilon^n(v)|^2] + E[\int_0^v |\Psi^n(t)|^2dt] \leq N[E[|\eta|^2] + E[\int_0^v |f(t,0,0,0,0)|^2dt] + E[\int_0^v |g(t,0,0)|^2dt]] \leq M_1.$$

Thus

$$||\Upsilon^n||_{G^2} + ||\Psi^n||_{S^2} \leq M_1. \tag{3.12}$$

Part 2, we prove the backward of FBFSDDDE (3.3):

$$X(v) = X(T) + \int_v^T \acute{f}(t,X^n_j(t),\Gamma^n_j(t),X^n_{j\,t},\Gamma^n_{j\,t})dt + \int_v^T \acute{g}(t,X^n_j(t),X^n_{j\,t})d\overleftarrow{W}(v) - \int_v^T \Gamma(t)dB(t)$$

has an unique solution (X^n,Γ^n), and for $M_2 > 0$ we have

$$||X^n||_{G^2} + ||\Gamma^n||_{S^2} \leq M_2.$$

From existence and uniqueness of the solution of the forward of FBFSDDDE (3.3). The backward of FBFSDDDE (3.3) becomes a classical equation terminal condition $X(T)$ and coefficients $\acute{f}(t, X^n(t), \Gamma^n(t), X^n_t, \Gamma^n_t)$ and $\acute{g}(t, X^n(t), X^n_t)$. Therefore, we conclude that the backward of FBFSDDDE (3.3) has a unique solution. Therefore, we can use the same technique of forward of FBFSDDDE (3.3), we have

$$||X^n||_{G^2} + ||\Gamma^n||_{S^2} \leq M_2, \tag{3.13}$$

where $M_2 > 0$. Let $M = M_1 + M_2$ and by combining (3.12) and (3.13), we have

$$||\Upsilon^n||_{G^2} + ||\Psi^n||_{S^2} + ||X^n||_{G^2} + ||\Gamma^n||_{S^2} \leq M.$$

□

4. Main Results

Lemma 4.1. *Suppose that* (Υ^1, Ψ^1) *and* (Υ^2, Ψ^2) *are two solutions of the forward equation of FBFSDDDE system (3.3) with data* (η^1, f^1, g) *and* (η^2, f^2, g). *Then, if* $f^1 \leq f^2$, *it holds that also* $\Upsilon^1 \leq \Upsilon^2$.

Proof. Let $\bar{\eta} = \eta^1 - \eta^2, (\bar{\Upsilon}, \bar{\Psi}) = (\Upsilon^1 - \Upsilon^2, \Psi^1 - \Psi^2)$. Because $\eta^1 \leq \eta^2$, then $\bar{\eta}^+ = 0$. From Ito's formula and inequality $2xy \leq \frac{1}{b}x^2 + by^2, b > 0$, we deduce

$$\begin{gathered}
E(|\Upsilon^n(v)|^2) + E[\int_0^v |\Upsilon^n(t)|^2 dt] = E(\bar{\eta})^2 + 2E[\int_0^v \bar{\Upsilon}(t)(f^1(t, \Upsilon^1(t), \Psi^1(t), \Upsilon^1_t, \Psi^1_t) - \\
f^2(t, \Upsilon^2(t), \Psi^2(t), \Upsilon^2_t, \Psi^2_t))dt] + E[\int_0^v |g(t, \Upsilon^1(t), \Upsilon^1_t) - g(t, \Upsilon^2(t), \Upsilon^2_t)|^2 dt] \leq \frac{1}{b}E[\int_0^v |\Upsilon^n(t)|^2 dt] + \\
bE[\int_0^v |(f^1(t, \Upsilon^1(t), \Psi^1(t), \Upsilon^1_t, \Psi^1_t) - f^2(t, \Upsilon^2(t), \Psi^2(t), \Upsilon^2_t, \Psi^2_t)|^2)dt] + E[\int_0^v |g(t, \Upsilon^1(t), \Upsilon^1_t) - \\
g(t, \Upsilon^2(t), \Upsilon^2_t)|^2 dt] \leq \frac{1}{b}E[\int_0^v |\Upsilon^n(t)|^2 dt] + bE[\int_0^v |(f^1(t, \Upsilon^1(t), \Psi^1(t), \Upsilon^1_t, \Psi^1_t) - f^1(t, \Upsilon^2(t), \Psi^2(t), \Upsilon^2_t, \Psi^2_t) + \\
f^1(t, \Upsilon^2(t), \Psi^2(t), \Upsilon^2_t, \Psi^2_t) - f^2(t, \Upsilon^2(t), \Psi^2(t), \Upsilon^2_t, \Psi^2_t)|^2)dt] + E[\int_0^v |g(t, \Upsilon^1(t), \Upsilon^1_t) - g(t, \Upsilon^2(t), \Upsilon^2_t)|^2 dt] \leq \\
\frac{1}{b}E[\int_0^v |\Upsilon^n(t)|^2 dt] + bC_1 E[\int_0^v |\Upsilon^n(t)|^2 dt] + bC_1 E[\int_0^v |\Psi^n(t)|^2 dt] + \\
bR_1 \mu E[\int_0^v |\Upsilon^n(t)|^2 dt] R_1 \mu E[\int_0^v |\Psi^n(t)|^2 dt] + C_3 E[\int_0^v |\Upsilon^n(t)|^2 dt] + R_3 \mu E[\int_0^v |\Upsilon^n(t)|^2 dt] \leq \\
(\frac{1}{b} + bC_1 + bR_1\mu + C_3 + \mu R_3)E[\int_0^v |\Upsilon^n(t)|^2 dt] + (bC_1 + R_1\mu)E[\int_0^v |\Psi^n(t)|^2 dt].
\end{gathered}$$

From Gronwall's inequality, $E(\bar{\Upsilon}(v))^2 = 0, v \in [0, T]$, ie., $\Upsilon^1(v) \leq \Upsilon^2(v), v \in [0, T]$. Hence $\Upsilon^1 \leq \Upsilon^2$. □

Lemma 4.2. *Suppose that* (X^1, Γ^1) *and* (X^2, Γ^2) *are two solutions of the backward equation of FBFSDDDE system (3.3) with data* $(l, \acute{f}^1, \acute{g})$ *and* $(l, \acute{f}^2, \acute{g})$. *Then, if* $\acute{f}^1 \leq \acute{f}^2$, *it holds that also* $X^1 \leq X^2$.

Proof. Let us define $(\bar{X}, \bar{\Gamma}) = (X^1 - X^2, \Gamma^1 - \Gamma^2)$. From Ito's formula and inequality $2xy \leq \frac{1}{b}x^2 + by^2, b > 0$, we deduce

$$\begin{gathered}
E(|\bar{X}(v)|^2) + E[\int_0^v |\bar{\Gamma}(t)|^2 dt] = 2E[\int_0^v \bar{X}(t)(\acute{f}^1(t, X^1(t), \Gamma^1(t), X^1_t, \Gamma^1_t) - \acute{f}^2(t, X^2(t), \Gamma^2(t), X^1_t, \Gamma^2_t))dt] + \\
E[\int_0^v |\acute{g}(t, X^1(t), X^1_t) - \acute{g}(t, X^2(t), X^2_t)|^2 dt] \leq \frac{1}{b}E[\int_0^v |\bar{X}^n(t)|^2 dt] + bE[\int_0^v |(\acute{f}^1(t, X^1(t), \Gamma^1(t), X^1_t, \Gamma^1_t) - \\
\acute{f}^2(t, X^2(t), \Gamma^2(t), X^2_t, \Gamma^2_t))|^2 dt] + E[\int_0^v |\acute{g}(t, X^1(t), X^1_t) - \acute{g}(t, X^2(t), X^2_t)|^2 dt] \leq \\
\frac{1}{b}E[\int_0^v |\bar{X}^n(t)|^2 dt] + bE[\int_0^v |(\acute{f}^1(t, X^1(t), \Gamma^1(t), X^1_t, \Gamma^1_t) - \acute{f}^1(t, X^2(t), \Gamma^2(t), X^2_t, \Gamma^2_t) + \\
\acute{f}^1(t, X^2(t), \Gamma^2(t), X^2_t, \Gamma^2_t) - \acute{f}^2(t, X^2(t), \Gamma^2(t), X^2_t, \Gamma^2_t))|^2 dt] + E[\int_0^v |\acute{g}(t, X^1(t), X^1_t) - \acute{g}(t, X^2(t), X^2_t)|^2 dt] \leq \\
\frac{1}{b}E[\int_0^v |\bar{X}(t)|^2 dt] + bC_1 E[\int_0^v |\bar{X}(t)|^2 dt] + bC_2 E[\int_0^v |\bar{\Gamma}(t)|^2 dt] + bR_2\mu E[\int_0^v |\bar{X}(t)|^2 dt] + \\
bR_2\mu E[\int_0^v |\bar{\Gamma}(t)|^2 dt] + C_3 E[\int_0^v |\bar{X}(t)|^2 dt] + R_3\mu E[\int_0^v |\bar{\Gamma}(t)|^2 dt] = \\
(\frac{1}{b} + bC_2 + bR_2\mu + C_3)E[\int_0^v |\bar{X}(t)|^2 dt] + (bC_2 + bR_2\mu + R_3\mu)E[\int_0^v |\bar{\Gamma}(t)|^2 dt].
\end{gathered}$$

From Gronwall's inequality, $E(\bar{X}(v))^2 = 0, v \in [0, T]$, ie., $X^1(v) \leq X^2(v), v \in [0, T]$. Hence $X^1 \leq X^2$. □

Theorem 4.3. *Suppose that* $f, \acute{f}, g$ *and* $\acute{g}$ *are stochastically increasing functions. Under the hypotheses (Hyp 1- Hyp 6), the FBFSDDDEs system (3.3) has a maximal solution* $(\Upsilon, \Psi, X, \Gamma)$.

Proof. We will prove that $(\Upsilon^n, \Psi^n, X^n, \Gamma^n)$ is monotonic and its limit verifies system (3.3). Firstly, we construct the start point of FBFSDDDEs system. The following it is the two general forward equations of FBFSDDDEs system:

$$\tilde{\Upsilon}^0(v) = \eta + \int_o^v f^1(t, \tilde{\Upsilon}^0(v), \tilde{\Psi}^0(t), \tilde{\Upsilon}^0_t, \tilde{\Psi}^0_t)dt + \int_0^v g(t, \tilde{\Upsilon}^0(v), \tilde{\Upsilon}^0_t)dB(t) - \int_0^v \tilde{\Psi}^0(t)d\overleftarrow{W}(t), \tag{4.1}$$

$$\Upsilon^0(v) = \eta + \int_o^v f^2(t, \Upsilon^0(v), \Psi^0(t), \Upsilon^0_t, \Psi^0_t)dt + \int_0^v g(t, \Upsilon^0(v), \Upsilon^0_t)dB(t) - \int_0^v \Psi^0(t)d\overleftarrow{W}(t). \tag{4.2}$$

From lemma (3.7), the equations (4.1) and (4.2) have unique solutions $(\tilde{\Upsilon}^0, \tilde{\Psi}^0)$ and (Υ^0, Ψ^0), respectively, which satisfy $||\tilde{\Upsilon}^0||_{G^2} + ||\tilde{\Psi}^0||_{S^2} \leq M_1$ and $||\Upsilon^0||_{G^2} + ||\Psi^0||_{S^2} \leq M_1$. By lemma (4.1), we have $\tilde{\Upsilon}^0 \leq \Upsilon^0$. Also, we consider the following two general backward equations of FBFSDDEs system:

$$\tilde{X}^0(v) = l(\tilde{\Upsilon}^0(T)) + \int_v^T \acute{f}^1(t, \tilde{X}^0(t), \tilde{\Gamma}^0(t), \tilde{X}^0_t, \tilde{\Gamma}^0_t)dt + \int_v^T \acute{g}(t, \tilde{X}^0(t), \tilde{X}^0_t)d\overleftarrow{W}(v) - \int_v^T \tilde{\Gamma}^0(t)dB(t), \tag{4.3}$$

$$X^0(v) = l(\Upsilon^0(T)) + \int_v^T \acute{f}^2(t, X^0(t), \Gamma^0(t), X^0_t, \Gamma^0_t)dt + \int_v^T \acute{g}(t, X^0(t), X^0_t)d\overleftarrow{W}(v) - \int_v^T \Gamma^0(t)dB(t). \tag{4.4}$$

From lemma (3.7), the equations (4.3) and (4.4) have unique solutions $(\tilde{X}^0, \tilde{\Gamma}^0)$ and (X^0, Γ^0), respectively, which satisfy $||\tilde{X}^0||_{G^2} + ||\tilde{\Gamma}^0||_{S^2} \leq M_2$ and $||X^0||_{G^2} + ||\Gamma^0||_{S^2} \leq M_2$. By lemma (4.2), we have $\tilde{X}^0 \leq X^0$. Her, we will prove that the following backward of FBFSDDDEs system has a solution (X^1, Γ^1):

$$X^1(v) = l(\Upsilon^0(T)) + \int_v^T \acute{f}(t, X^1(t), \Gamma^1(t), X^1_t, \Gamma^1_t)dt + \int_v^T \acute{g}(t, X^1(t), X^1_t)d\overleftarrow{W}(v) - \int_v^T \Gamma^1(t)dB(t). \tag{4.5}$$

By using Lemma (3.7), we deduce the backward of FBFSDDDEs system has an unique solution $(X^{1,n}, \Gamma^{1,n}), n \geq 1$.

$$X^{1,n}(v) = l(\Upsilon^0(T)) + \int_v^T \acute{f}^{1,n}(t, X^{1,n}(t), \Gamma^{1,n}(t), X^{1,n}_t, \Gamma^{1,n}_t)dt + \int_v^T \acute{g}(t, X^{1,n}(t), X^{1,n}_t)d\overleftarrow{W}(v) - \int_v^T \Gamma^{1,n}(t)dB(t). \tag{4.6}$$

From lemma (4.2), it follows that $\tilde{X}^0(v) \leq X^{1,n+1}(v) \leq X^{1,n}(v) \leq X^0(v), v \in [0, T]$. Apply Ito's formula to $|X^{1,n}(v)|^2$, we have

$$\begin{cases} |X^{1,n}(v)|^2 + \int_v^T |\Gamma^{1,n}(t)|^2 dt = |l(\Upsilon^0(T))|^2 + 2\int_v^T X^{1,n}(t)\acute{f}^{1,n}(t, X^{1,n}(t), \Gamma^{1,n}(t), X^{1,n}_t, \Gamma^{1,n}_t)dt \\ -2\int_v^T X^{1,n}(t)\Gamma^{1,n}(t)dB(t) + \int_v^T X^{1,n}(t)\acute{g}(t, X^{1,n}(t), X^{1,n}_t)d\overleftarrow{W}(v) + \int_v^T |\acute{g}(t, X^{1,n}(t), X^{1,n}_t)|^2 dt. \end{cases} \tag{4.7}$$

By taking expectations and applying inequality $2xy \leq \frac{1}{b}x^2 + by^2, b > 0$, we have

$$\begin{gathered} E[|X^{1,n}(v)|^2] + E[\int_v^T |\Gamma^{1,n}(t)|^2 dt] = E[|l(\Upsilon^0(T))|^2] + 2E[\int_v^T X^{1,n}(t)\acute{f}^{1,n}(t, X^{1,n}(t), \Gamma^{1,n}(t), X^{1,n}_t, \Gamma^{1,n}_t)dt] - \\ 2E[\int_v^T X^{1,n}(t)\Gamma^{1,n}(t)dB(t)] + E[\int_v^T X^{1,n}(t)\acute{g}(t, X^{1,n}(t), X^{1,n}_t)d\overleftarrow{W}(v)] + E[\int_v^T |\acute{g}(t, X^{1,n}(t), X^{1,n}_t)|^2 dt] \leq \\ E[|l(\Upsilon^0(T))|^2] + \frac{1}{b}E[\int_v^T |X^{1,n}(t)|^2] + bE[\int_v^T |\acute{f}^{1,n}(t, X^{1,n}(t), \Gamma^{1,n}(t), X^{1,n}_t, \Gamma^{1,n}_t)|^2 dt] - \frac{1}{b}E[\int_v^T |X^{1,n}(t)|^2 dt] - \\ b\frac{1}{b}E[\int_v^T |\Gamma^{1,n}(t)|^2] + \frac{1}{b}E[\int_v^T |X^{1,n}(t)|^2] + bE[\int_v^T |\acute{g}(t, X^{1,n}(t), X^{1,n}_t)|^2 dt] + E[\int_v^T |\acute{g}(t, X^{1,n}(t), X^{1,n}_t)|^2 dt] \leq \\ E[|l(\Upsilon^0(T))|^2] + \frac{1}{b}E[\int_v^T |X^{1,n}(t)|^2] + bC_6E[\int_v^T (1 + |X^{1,n}(t)|^2 + \Gamma^{1,n}(t)|^2)dt] + bR_6\mu E[\int_v^T |X^{1,n}(t)|^2 dt] + \\ bR_6\mu E[\int_v^T |\Gamma^{1,n}(t)|^2 dt] - \frac{1}{b}E[\int_v^T |X^{1,n}(t)|^2 dt] - bE[\int_v^T |\Gamma^{1,n}(t)|^2 dt] + \frac{1}{b}E[\int_v^T |X^{1,n}(t)|^2 dt] + \\ bE[\int_v^T |X^{1,n}(t)|^2 dt] + b\mu E[\int_v^T |X^{1,n}(t)|^2 dt] \leq \\ E[|l(\Upsilon^0(T))|^2] + (\frac{1}{b} + bC_6 + bR_6\mu - \frac{1}{b} + \frac{1}{b} + b + b\mu)E[\int_v^T |X^{1,n}(t)|^2 dt] + (bC_6 + bR_6\mu - b)E[\int_v^T |\Gamma^{1,n}(t)|^2 dt] \leq \\ N(1 + E[\int_v^T |X^{1,n}(t)|^2 dt]) + (bC_6 + bR_6\mu - b)E[\int_v^T |\Gamma^{1,n}(t)|^2 dt], \end{gathered}$$

where N depends on C_6, R_6, b and μ. Therefore, for any $v \in [0, T]$, we have

$$E[|X^{1,n}(v)|^2]+] + (1 - bC_6 - bR_6\mu + b)E[\int_v^T |\Gamma^{1,n}(t)|^2 dt] \leq N(1 + E[\int_v^T |X^{1,n}(t)|^2 dt]).$$

According to Gronwall's inequality, we deduce

$$\begin{gathered}\sup_{v\in[0,T]} E[|X^{1,n}(v)|^2] \leq N,\\ E[\int_v^T |\Gamma^{1,n}(t)|^2 dt] \leq N.\end{gathered}$$

From (4.7), let us take the supremum and then take the expectation

$$\begin{gathered}
E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2] \leq E[|l(\Upsilon^0(T))|^2] + 2E[\int_0^T X^{1,n}(t)\acute{f}^{1,n}(t, X^{1,n}(t), \Gamma^{1,n}(t), X_t^{1,n}, \Gamma_t^{1,n})dt] -\\
(E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2])^{\frac{1}{2}}(E[\sup_{v\in[0,T]} | \int_v^T \Gamma^{1,n}(v)dB(t)|^2])^{\frac{1}{2}} +\\
(\sup_{v\in[0,T]} E[|X^{1,n}(v)|^2])^{\frac{1}{2}}(E[\sup_{v\in[0,T]} | \int_v^T \acute{g}(t, X^{1,n}(t), X_t^{1,n})d\overleftarrow{W}(v)|^2])^{\frac{1}{2}} + \int_0^T |\acute{g}(t, X^{1,n}(t), X_t^{1,n})|^2 dt \leq\\
C_10E(1 + |\Upsilon^0(T)|^2) + R_9 E(\int_{-T}^0 |\Upsilon^0(t+\delta)|^2 \gamma d\delta + \frac{1}{b}E[\int_0^T |X^{1,n}(t)|^2 dt] +\\
bE[\int_0^T |\acute{f}^{1,n}(t, X^{1,n}(t), \Gamma^{1,n}(t), X_t^{1,n}, \Gamma_t^{1,n})|^2 dt] - \frac{1}{b}E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2] - bN_1 E[\int_0^T |\Gamma^{1,n}(t)|^2 dt] +\\
\frac{1}{b}E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2] + bN_1 E[\int_0^T |\acute{g}(t, X^{1,n}(t), X_t^{1,n})|^2 dt] + E[\int_0^T |\acute{g}(t, X^{1,n}(t), X_t^{1,n})|^2 dt] \leq\\
C_{10}E(1 + \Upsilon^0(T)|^2) + R_9\mu + \frac{1}{b}E[\int_0^T |X^{1,n}(t)|^2 dt] + bE[\int_0^T |X^{1,n}(t)|^2 dt] + bE[\int_0^T |\Gamma^{1,n}(t)|^2 dt] +\\
b\mu E[\int_0^T |X^{1,n}(t)|^2 dt] + b\mu E[\int_0^T |\Gamma^{1,n}(t)|^2 dt] - \frac{1}{b}E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2] - bN_1 E[\int_0^T |\Gamma^{1,n}(t)|^2 dt] +\\
\frac{1}{b}E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2] + bN_1 E[\int_0^T |\acute{g}(t, X^{1,n}(t), X_t^{1,n})|^2 dt] + E[\int_v^T |\acute{g}(t, X^{1,n}(t), X_t^{1,n})|^2 dt] \leq C_{10}E(1 +\\
\Upsilon^0(T)|^2) + R_9\mu + \frac{1}{b}E[\int_0^T |X^{1,n}(t)|^2 dt] + bE[\int_0^T |X^{1,n}(t)|^2 dt] + bE[\int_0^T |\Gamma^{1,n}(t)|^2 dt] + b\mu E[\int_0^T |X^{1,n}(t)|^2 dt] +\\
b\mu E[\int_0^T |\Gamma^{1,n}(t)|^2 dt] - \frac{1}{b}E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2] - bN_1 E[\int_0^T |\Gamma^{1,n}(t)|^2 dt] + \frac{1}{b}E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2] +\\
bN_1 E[\int_0^T |X^{1,n}(t)|^2 dt] + bN_1\mu E[\int_0^T |\Gamma^{1,n}(t)|^2 dt] + E[\int_0^T |\Gamma^{1,n}(t)|^2 dt] + \mu \leq \frac{2}{b}E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2] +\\
(\frac{1}{b} + b + b\mu + bN_1\mu + bN_1 + 1)E[\int_0^T |X^{1,n}(t)|^2 dt] + (b + b\mu + bN_1)E[\int_0^T |\Gamma^{1,n}(t)|^2 dt] + D_1.
\end{gathered}$$

Then,

$$E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2] \leq KE[\int_0^T |X^{1,n}(t)|^2 dt] + N.$$

From (4.7), we have

$$E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2] \leq D_1.$$

By the monotone bounded convergence theorem, then $\lim_{n\to\infty} X^{1,n} = X^1$ and Fatou's theorem then $E[\sup_{v\in[0,T]} |X^{1,n}(v)|^2] \leq D_1$. Using the theorems of dominated convergence, we deduce $E[\sup_{v\in[0,T]} |X^{1,n}(v) - X^1(v)|^2] \to 0, n \to \infty$. Applying Ito's formula to $|X^{1,i}(v) - \Psi^{1,n}(v)|^2$ and by Hölder's inequality, we have

$$\begin{gathered}
E[|X^{1,i}(0) - X^{1,n}(0)|^2] + E[\int_0^T |\Gamma^{1,i}(t) - \Gamma^{1,n}(t)|^2 dt] \leq\\
2(E[\int_0^T |X^{1,i}(t) - X^{1,n}(t)|^2 dt])^{\frac{1}{2}}(E[\int_0^T |\acute{f}^{1,i}(t, X^{1,i}(t), \Gamma^{1,i}(t), X_t^{1,i}, \Gamma_t^{1,i}) -\\
\acute{f}^{1,n}(t, X^{1,n}(t), \Gamma^{1,n}(t), X_t^{1,n}, \Gamma_t^{1,n})|dt])^{\frac{1}{2}} + E[\int_0^T |\acute{g}(t, X^{1,i}(t), X_t^{1,i}) - \acute{g}(t, X^{1,n}(t), X_t^{1,n})|^2 dt] \leq\\
\frac{1}{b}E[\int_0^T |X^{1,i}(t) - X^{1,n}(t)|^2 dt] + bE[\int_0^T |\acute{f}^{1,i}(t, X^{1,i}(t), \Gamma^{1,i}(t), X_t^{1,i}, \Gamma_t^{1,i}) -\\
\acute{f}^{1,n}(t, X^{1,n}(t), \Gamma^{1,n}(t), X_t^{1,n}, \Gamma_t^{1,n})|^2 dt] + E[\int_0^T |\acute{g}(t, X^{1,i}(t), X_t^{1,i}) - \acute{g}(t, X^{1,n}(t), X_t^{1,n})|^2 dt] \leq\\
\frac{1}{b}E[\int_0^T |X^{1,i}(t) - X^{1,n}(t)|^2 dt] + bC_1 E[\int_0^T |X^{1,i}(t) - X^{1,n}(t)|^2 dt] + bC_1 E[\int_0^T |\Gamma^{1,i}(t) - \Gamma^{1,n}(t)|^2 dt] +\\
bR_2\mu E[\int_0^T |X^{1,i}(t) - X^{1,n}(t)|^2 dt] + bR_2\mu E[\int_0^T |\Gamma^{1,i}(t) - \Gamma^{1,n}(t)|^2 dt] + C_4 E[\int_0^T |X^{1,i}(t) - X^{1,n}(t)|^2 dt] +\\
R_4\mu E[\int_0^T |X^{1,i}(t) - X^{1,n}(t)|^2 dt] =\\
(\frac{1}{b} + bC_2 + bR_2\mu + C_4 + R_4\mu)E[\int_0^T |X^{1,i}(t) - X^{1,n}(t)|^2 dt] + (bC_2 + bR_2\mu)E[\int_0^T |\Gamma^{1,i}(t) - \Gamma^{1,n}(t)|^2 dt].
\end{gathered}$$

Because $\{\Gamma^{1,n}\}_{n\geq 1}$ is a Cauchy sequence in $S^2(\Omega, F, P, R^{\alpha\times\lambda})$, then

$$E[\int_0^T (|X^{1,i}(t) - X^{1,n}(t)|^2 + |\Gamma^{1,i}(t) - \Gamma^{1,n}(t)|^2)dt] \to 0 \text{ as } i, n \to \infty,$$

this means $\lim_{n\to\infty} \Gamma^{1,n} = \Gamma^1$. Hence, (X^1, Γ^1) is a solution of backward of FBFSDDDEs system. From above we can prove (Υ^1, Ψ^1) is a solution of forward of FBFSDDDEs system. Therefore, the following backward of FBFSDDEs system has a unique solution $(X^{2,n}, \Gamma^{2,n}), n \geq 1$:

$$X^{2,n}(v) = l(\Upsilon^1(T)) + \int_0^T \acute{f}^{2,n}(t, X^{2,n}(t), \Gamma^{2,n}(t), X_t^{2,n}, \Gamma_t^{2,n})dt + \int_0^T \acute{g}(t, X^{2,n}(t), X_t^{2,n})d\overleftarrow{W}(v) - \int_0^T \Gamma^{2,n}(t)dB(t), v \in [0,T].$$

We kow $\Upsilon^1(T) \leq \Upsilon^0(T)$, we deduce $l(\Upsilon^1(T)) \leq l(\Upsilon^1(T))$. From Lemma (4.2), we have that $X^{2,n}(v) \leq X^{1,n}(v)$, then $X^2(v) \leq X^1(v)$ as $n \to \infty$. Similarly, we prove that (Υ^2, Ψ^2) is the solution of forward of FBFSDDDEs system:

$$\Upsilon^2(v) = \eta + \int_o^v f(t, \Upsilon^2(v), \Psi^2(t), \Upsilon_t^2, \Psi_t^2)dt + \int_0^v g(t, \Upsilon^2(v), \Upsilon_t^2)dB(t) - \int_0^v \Psi^2(t)d\overleftarrow{W}(t),$$

and $\tilde{\Upsilon}^0(v) \leq \Upsilon^2(v) \leq \Upsilon^1(v) \leq \Upsilon^0(v)$, for all $v \in [0,T]$. Therefore, we deduce the existence of a sequence $(\Upsilon^n, \Psi^n, X^n, \Gamma^n)$ which satisfies (3.3). Therefore, for any $v \in [0,T]$ such that

$$\tilde{\Upsilon}^0(v) \leq \cdots \leq \Upsilon^{n+1}(v) \leq \Upsilon^n(v) \leq \cdots \leq \Upsilon^2(v) \leq \Upsilon^1(v) \leq \Upsilon^0(v),$$

$$\tilde{X}^0(v) \leq \cdots \leq X^{n+1}(v) \leq X^n(v) \leq \cdots \leq X^2(v) \leq X^1(v) \leq X^0(v).$$

Therefore, we deduce $\Upsilon^1 = \lim_{n\to\infty} \Upsilon^{1,n}$ and $X^1 = \lim_{n\to\infty} X^{1,n}$. Therefore, we conclude $\Psi^1 = \lim_{n\to\infty} \Psi^{1,n}$ and $\Gamma^1 = \lim_{n\to\infty} \Gamma^{1,n}$. Hence, we deduce that $(\Upsilon, \Psi, X, \Gamma) \in G^2(\Omega, F, P, R^\alpha) \times G^2(\Omega, F, P, R^\lambda) \times S^2(\Omega, F, P, R^{\alpha\times\beta}) \times S^2(\Omega, F, P, R^{\lambda\times\beta})$ is a solution of FBFSDDDEs system (3.1). Finally, Suppose that $(\ddot{\Upsilon}, \ddot{\Psi}, \ddot{X}, \ddot{\Gamma}) \in G^2(\Omega, F, P, R^\alpha) \times G^2(\Omega, F, P, R^\lambda) \times S^2(\Omega, F, P, R^{\alpha\times\beta}) \times S^2(\Omega, F, P, R^{\lambda\times\beta})$ is an arbitrary solution of the system (3.1). From lemma (4.1), we conclude $\tilde{\Upsilon}^0(v) \leq \ddot{\Upsilon}^0(v) \leq \Upsilon^0(v), v \in [0,T]$. From $l(\ddot{\Upsilon}(T)) \leq l(\Upsilon^0(T))$, and then using lemma (4.2), we deduce immediately that $\tilde{X}^0(v) \leq \ddot{X}^0(v) \leq X^0(v), v \in [0,T]$. Therefore, we deduce $X^1(v) \leq X^0(v), v \in [0,T]$. Again the same procedure, we deduce

$$\tilde{\Upsilon}^0(v) \leq \ddot{\Upsilon}^0(v) \leq \cdots \leq \Upsilon^n(v) \leq \cdots \leq \Upsilon^2(v) \leq \Upsilon^1(v) \leq \Upsilon^0(v),$$

$$\tilde{X}^0(v) \leq \ddot{X}^0(v) \leq \cdots \leq X^n(v) \leq \cdots \leq X^2(v) \leq X^1(v) \leq X^0(v).$$

It implies that $\ddot{\Upsilon} \leq \Upsilon$ and $\ddot{X} \leq X$, that is the system of FBFSDDDEs (3.1) has a maximal solution $(\Upsilon, \Psi, X, \Gamma) \in G^2(\Omega, F, P, R^\alpha) \times G^2(\Omega, F, P, R^\lambda) \times S^2(\Omega, F, P, R^{\alpha\times\beta}) \times S^2(\Omega, F, P, R^{\lambda\times\beta})$. □

Theorem 4.4. *Under the hypotheses (Hyp 1- Hyp 6), the FBFSDDDEs system (3.1) has a unique solution* $(\Upsilon, \Psi, X, \Gamma)$.

Proof. Let $(\Upsilon^1, \Psi^1, X^1, \Gamma^1)$ and $(\Upsilon^2, \Psi^2, X^2, \Gamma^2)$ be two solution of FBFSDDDEs system (3.1). We set $(\bar{\Upsilon}, \bar{\Psi}, \bar{X}, \bar{\Gamma}) = (\Upsilon^1 - \Upsilon^2, \Psi^1 - \Psi^2, X^1 - X^2, \Gamma^1 - \Gamma^2)$. Applying Ito's formula to $\bar{\Upsilon}, \bar{X}$ and by using inequality $xy \leq \frac{1}{2b}x^2 + \frac{b}{2}y^2, b > 0$, we have

$$E[\bar{\Upsilon}^n(T)l(\bar{\Upsilon}^n(T))] \leq E[\int_0^T \bar{X}^n(t)(f(t, \Upsilon^{2,n}(t), \Psi^{2,n}(t), \Upsilon_t^{2,n}, \Psi_t^{2,n}) - f(t, \Upsilon^{1,n}(t), \Psi^{1,n}(t), \Upsilon_t^{1,n}, \Psi_t^{1,n}))dt] + E[\int_0^T \bar{\Gamma}^n(t)(g(t, \Upsilon^{2,n}(t), \Upsilon_t^{2,n}) - g(t, \Upsilon^{1,n}(t), \Upsilon_t^{1,n}))dt] - E[\int_0^T \bar{\Upsilon}^n(t)(\acute{f}(t, X^{2,n}(t), \Gamma^{2,n}(t), X_t^{2,n}, \Gamma_t^{2,n}) - \acute{f}(t, X^{1,n}(t), \Gamma^{1,n}(t), X_t^{1,n}, \Gamma_t^{1,n}))dt] + E[\int_0^T \bar{\Psi}^n(t)(\acute{g}(t, X^{2,n}(t), X_t^{2,n}) - \acute{g}(t, X^{1,n}(t), X_t^{1,n}))dt] \leq \frac{1}{2b}E[\int_0^T |\bar{X}^n(t)|^2dt] + \frac{b}{2}E[\int_0^T |f(t, \Upsilon^{2,n}(t), \Psi^{2,n}(t), \Upsilon_t^{2,n}, \Psi_t^{2,n}) - f(t, \Upsilon^{1,n}(t), \Psi^{1,n}(t), \Upsilon_t^{1,n}, \Psi_t^{1,n})|^2dt] + \frac{1}{2b}E[\int_0^T |\bar{\Gamma}^n(t)|^2dt] + \frac{b}{2}E[\int_0^T |g(t, \Upsilon^{2,n}(t), \Upsilon_t^{2,n}) - g(t, \Upsilon^{1,n}(t), \Upsilon_t^{1,n})|^2dt] - \frac{1}{2b}E[\int_0^T |\bar{\Upsilon}^n(t)|^2dt] - \frac{b}{2}E[\int_0^T |\acute{f}(t, X^{2,n}(t), \Gamma^{2,n}(t), X_t^{2,n}, \Gamma_t^{2,n}) - \acute{f}(t, X^{1,n}(t), \Gamma^{1,n}(t), X_t^{1,n}, \Gamma_t^{1,n})|^2dt] + \frac{1}{2b}E[\int_0^T |\bar{\Psi}^n(t)|^2dt] + \frac{b}{2}E[\int_0^T |(\acute{g}(t, X^{2,n}(t), X_t^{2,n}) - \acute{g}(t, X^{1,n}(t), X_t^{1,n})|^2dt] \leq (\frac{cb}{2} + \frac{bR_1\mu}{2} + \frac{bC_3}{2} + \frac{bR_3}{2} - \frac{1}{2b})E[\int_0^T |\bar{\Upsilon}^n(t)|^2dt] + (\frac{1}{2b} - \frac{bC_2}{2} - \frac{bR_2\mu}{2} + \frac{bC_3}{2} + \frac{bR_4}{2})E[\int_0^T |\bar{X}^n(t)|^2dt] + (\frac{bC_1}{2} + \frac{bR_1\mu}{2} + \frac{1}{2b})E[\int_0^T |\bar{\Psi}^n(t)|^2dt] + (\frac{1}{2b} - \frac{bC_2}{2} - \frac{bR_2\mu}{2})E[\int_0^T |\bar{\Gamma}^n(t)|^2dt].$$

By Cauchy sequence, we have

$$E\int_0^T |\bar{\Upsilon}^n(t)|^2dt + \int_0^T |\bar{X}^n(t)|^2dt + \int_0^T |\bar{\Psi}^n(t)|^2dt + \int_0^T |\bar{\Gamma}^n(t)|^2dt] \to 0,$$

as $n \to \infty$. That is mean

$$E[\int_0^T |(\Upsilon^1(t), \Psi^1(t), X^1(t), \Gamma^1(t)) - (\Upsilon^2(t), \Psi^2(t), X^2(t), \Gamma^2(t))|^2dt] = 0.$$

Hence

$$(\Upsilon^1(t), \Psi^1(t), X^1(t), \Gamma^1(t)) = (\Upsilon^2(t), \Psi^2(t), X^2(t), \Gamma^2(t)).$$

□

5. Conclusion

We noticed by placing appropriate conditions on the forward-backward system of equations that there is a solution and a unique solution. This system of equations is also characterized by the presence of a hierarchy in the solution, which leads to a maximum solution.

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