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Conformal η-Ricci-Yamabe Solitons on LP-Kenmotsu Manifolds

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ABSTRACT: The aim of the present paper is to study conformal η-Ricci-Yamabe solitons (CERYS) on Lorentzian-para Kenmotsu n-manifolds (in brief, $(LPK)_n$) with certain curvature conditions. Moreover, the existence of CERYS has been proved by constructing a non-trivial example of $(LPK)_3$.

Keywords: Conformal Ricci-Yamabe solitons, projective curvature tensor, Einstein manifolds, η-Einstein manifolds, Lorentzian para-Kenmotsu manifolds.

Contents

1. Introduction

In 1982, the concept of Ricci flow was proposed by Hamilton [14] to find a canonical metric on a smooth Riemannian manifold $\mathbb{M}$ and is defined by the relation for mertics $g(t)$ of the form $\frac{\partial}{\partial t}g(t) = -2\mathcal{S}(g(t))$ whose solution is known as Ricci soliton defined by

$$\pounds_K g + 2\mathcal{S} + 2\Lambda g = 0,$$

where $\mathcal{S}$ is the Ricci tensor, $\pounds_K$ is the Lie derivative operator along the vector field K (called the soliton vector field) on $\mathbb{M}$ and Λ is a real number.

Hamilton [11] also proposed the notion of Yamabe flow on $\mathbb{M}$ and is defined as the evolution of the Riemannian (or semi-Riemannian) metric g_0 in time t to $g = g(t)$ by the relation $\frac{\partial}{\partial t}g(t) = -rg,\ g(0) = g_0$, here $r(t)$ is the scalar curvature of the metric $g(t)$.

For $n = 2$, the Ricci and Yamabe flows are equivalent. However, for $n > 2$, there is no such an equivalence (since the conformal class of the metric is preserved by Yamabe flow but not by Ricci flow, in general).

On a Riemannian manifold $\mathbb{M}$ admitting a vector field K, the Yamabe soliton is defined by [12]

$$\pounds_K g + 2(\Lambda - r)g = 0.$$

A scalar combination of Ricci and Yamabe flows was proposed by the authors Güler and Crasmareanu [6]. This new class of geometric flows called Ricci-Yamabe (RY) flow of type (σ, ρ) and it is defined by

$$\frac{\partial}{\partial t}g(t) + 2\sigma\mathcal{S}(g(t)) + \rho r(t)g(t) = 0, \qquad g(0) = g_0,$$

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for some scalars σ and ρ. A solution to the RY flow is called a Ricci-Yamabe soliton (RYS) if it depends only on one parameter group of diffeomorphism and scaling.

A Riemannian (or semi-Riemannian) manifold $\mathbb{M}$ is said to admit a RYS if [4]

$$\pounds_K g + 2\sigma\mathcal{S} + (2\Lambda - \rho r)g = 0. \tag{1.1}$$

The concept of conformal Ricci flow was introduced by Fischer [5], which is defined on $\mathbb{M}$ by the relations

$$\frac{\partial g}{\partial t} = -2(\mathcal{S} + \frac{g}{n}) - pg, \quad r(g) = -1, \tag{1.2}$$

where p defines a time dependent non-dynamical scalar field (also called the conformal pressure). The term $-pg$ plays a role of constraint force to maintain r in (1.2).

Basu and Bhattacharyya [3] in 2015, proposed the concept of conformal Ricci soliton and is defined by the relation

$$\pounds_K g + 2\mathcal{S} + (2\Lambda - (p + \frac{2}{n}))g = 0. \tag{1.3}$$

An $\mathbb{M}$ is said to have a conformal Ricci-Yamabe soliton (CRYS) if [23]

$$\pounds_K g + 2\sigma\mathcal{S} + (2\Lambda - \rho r - (p + \frac{2}{n}))g = 0, \tag{1.4}$$

here $\sigma, \rho, \Lambda \in \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers..

As a generalization of conformal Ricci-Yamabe solitons, conformal η-Ricci-Yamabe soliton on a manifold $\mathbb{M}$ is defined by

$$\pounds_K g + 2\sigma\mathcal{S} + (2\Lambda - \rho r - (p + \frac{2}{n}))g + 2\mu\eta \otimes \eta = 0, \tag{1.5}$$

where $\mu \in \mathbb{R}$.

Also, we recommend the papers [1,2,8,9,10,17,19,20,21,22] and the references therein for more details about the related work.

In this article, we study CERYS on $(LPK)_n$. The article is organized in the following ways: In Section 2, we describe some basic definitions and results of $(LPK)_n$. Section 3 deals with study of CERYS in $(LPK)_n$. The Ricci semi-symmetric $(LPK)_n$ admitting CERYS have been studied in Section 4. In Section 5, it is shown that $(LPK)_n$ endowed with CERYS satisfying the curvature conditions: $\mathcal{P}(\mathcal{U}, \xi) \cdot \mathcal{S} = 0$, $\mathcal{R}(K, \xi) \cdot \mathcal{P} = 0$, and $\mathcal{S}(K, \xi) \cdot \mathcal{P} = 0$ are Einstein manifolds. In Section 6, we also study CERYS on $(LPK)_n$ admitting Codazzi type Ricci tensor and cyclic parallel Ricci tensor. In Section 7, the existence of CERYS has been proved by constructing a non-trivial example of $(LPK)_3$.

2. Preliminaries

A differentiable manifold $\mathbb{M}$ (dimension of $\mathbb{M} = n$) with the structure (φ, ξ, η) is named a Lorentzian almost paracontact manifold, where φ, ξ and η represent a $(1, 1)$ type tensor field, a contravariant vector field, and a 1-form, respectively on $\mathbb{M}$, satisfying

$$\eta(\xi) = -1 \text{ and } \varphi^2 = \eta \otimes \xi + I, \tag{2.1}$$

which infer that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \text{rank}(\varphi) = n - 1. \tag{2.2}$$

Let g (the Lorentzian metric) of $\mathbb{M}$ satisfies

$$g(\cdot, \xi) = \eta(\cdot) \text{ and } g(\varphi\,\cdot, \varphi\,\cdot) = g(\cdot\,, \cdot) + \eta(\cdot)\eta(\cdot), \tag{2.3}$$

then (φ, ξ, η, g) is named an almost paracontact structure, and $\mathbb{M}$ is termed as an almost paracontact metric manifold.
Define Φ (the second fundamental form) as:

$$\Phi(\mathcal{U}, \mathcal{V}) = \Phi(\mathcal{V}, \mathcal{U}) = g(\mathcal{U}, \varphi\mathcal{V}) \tag{2.4}$$

for any vector fields $\mathcal{U}, \mathcal{V} \in \mathfrak{X}(\mathbb{M})$, the Lie algebra of vector fields on $\mathbb{M}$. If $d\eta(\mathcal{U}, \mathcal{V}) = \Phi(\mathcal{U}, \mathcal{V})$, here d is an exterior derivative, then $(\mathbb{M}, \varphi, \xi, \eta, g)$ is termed as a paracontact metric manifold.

Definition 2.1 *A Lorentzian almost paracontact manifold $\mathbb{M}$ is termed an LP-Kenmotsu manifold (LPK) if [15,16]*

$$(\nabla_{\mathcal{U}}\varphi)\mathcal{V} = -g(\varphi\mathcal{U}, \mathcal{V})\xi - \eta(\mathcal{V})\varphi\mathcal{U}, \tag{2.5}$$

for any $\mathcal{U}, \mathcal{V}$ on $\mathbb{M}$.

In an $(LPK)_n$, we have

$$\nabla_{\mathcal{U}}\xi + \mathcal{U} + \eta(\mathcal{U})\xi = 0, \tag{2.6}$$

$$(\nabla_{\mathcal{U}}\eta)\mathcal{V} + g(\mathcal{U}, \mathcal{V}) + \eta(\mathcal{U})\eta(\mathcal{V}) = 0, \tag{2.7}$$

where ∇ stands for the Levi-Civita connection with respect to g.

Furthermore, in an $(LPK)_n$, the following relations hold [15,16]:

$$g(\mathcal{R}(\mathcal{U}, \mathcal{V})\mathcal{Z}, \xi) = \eta(\mathcal{R}(\mathcal{U}, \mathcal{V})\mathcal{Z}) = g(\mathcal{V}, \mathcal{Z})\eta(\mathcal{U}) - g(\mathcal{U}, \mathcal{Z})\eta(\mathcal{V}), \tag{2.8}$$

$$\mathcal{R}(\xi, \mathcal{U})\mathcal{V} = -\mathcal{R}(\mathcal{U}, \xi)\mathcal{V} = g(\mathcal{U}, \mathcal{V})\xi - \eta(\mathcal{V})\mathcal{U}, \tag{2.9}$$

$$\mathcal{R}(\mathcal{U}, \mathcal{V})\xi = \eta(\mathcal{V})\mathcal{U} - \eta(\mathcal{U})\mathcal{V}, \tag{2.10}$$

$$\mathcal{R}(\xi, \mathcal{U})\xi = \mathcal{U} + \eta(\mathcal{U})\xi, \tag{2.11}$$

$$S(\mathcal{U}, \xi) = (n-1)\eta(\mathcal{U}), \quad S(\xi, \xi) = -(n-1), \tag{2.12}$$

$$\mathcal{Q}\xi = (n-1)\xi, \tag{2.13}$$

for any $\mathcal{U}, \mathcal{V}, \mathcal{Z}$ on $\mathbb{M}$. Here $\mathcal{R}$ indicates the curvature tensor and $\mathcal{Q}$ indicates the Ricci operator.

Definition 2.2 *An $(LPK)_n$ is said to be η-Einstein if its Ricci tensor $\mathcal{S}(\neq 0)$ is of the form*

$$\mathcal{S}(\mathcal{U}, \mathcal{V}) = Ag(\mathcal{U}, \mathcal{V}) + B\eta(\mathcal{U})\eta(\mathcal{V}), \tag{2.14}$$

where A and B are smooth functions on $(LPK)_n$.

Remark 2.1 In an (LPK_n), we have [13]

$$\xi(r) = 2(r - n(n-1)). \tag{2.15}$$

Remark 2.2 From the relation (2.15), it is noticed that if an $(LPK)_n$ possesses the constant scalar curvature, then $r = n(n-1)$.

3. CERYS on $(LPK)_n$

Let the metric of an $(LPK)_n$ be a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$, then we have

$$\pounds_{\xi}g(\mathcal{U}, \mathcal{V}) + 2\sigma\mathcal{S}(\mathcal{U}, \mathcal{V}) + (2\Lambda - \rho r - (p + \frac{2}{n}))g(\mathcal{U}, \mathcal{V}) + 2\mu\eta(\mathcal{U})\eta(\mathcal{V}) = 0. \tag{3.1}$$

As we know that

$$(\pounds_{\xi}g)(\mathcal{U}, \mathcal{V}) = -2g(\mathcal{U}, \mathcal{V}) - 2\eta(\mathcal{U})\eta(\mathcal{V}), \tag{3.2}$$

for any $\mathcal{U}$, $\mathcal{V}$ on $(LPK)_n$. By using (3.2) in (3.1) we have

$$\mathcal{S}(\mathcal{U}, \mathcal{V}) = \frac{1}{\sigma}[1 - \Lambda + \frac{\rho r}{2} + \frac{1}{2}(p + \frac{2}{n})]g(\mathcal{U}, \mathcal{V}) + \frac{(1-\mu)}{\sigma}\eta(\mathcal{U})\eta(\mathcal{V}), \quad (\sigma \neq 0), \tag{3.3}$$

which is of the form $S(\mathcal{U},\mathcal{V}) = Ag(\mathcal{U},\mathcal{V}) + B\eta(\mathcal{U})\eta(\mathcal{V})$, where $A = \frac{1}{\sigma}[1-\Lambda+\frac{\rho r}{2}+\frac{1}{2}(p+\frac{2}{n})]$ and $B = \frac{(1-\mu)}{\sigma}$, $\sigma \neq 0$.

Now, putting $\mathcal{V} = \xi$ in (3.3), we have

$$\mathcal{S}(\mathcal{U},\xi) = A_1\eta(\mathcal{U}), \tag{3.4}$$

where $A_1 = \frac{1}{\sigma}[\mu - \Lambda + \frac{\rho r}{2} + \frac{1}{2}(p + \frac{2}{n})]$.

From (2.12) and (3.4), we obtain

$$\Lambda - \mu = \frac{\rho r}{2} + \frac{1}{2}(p + \frac{2}{n}) - \sigma(n-1). \tag{3.5}$$

Thus, we have

Theorem 3.1 *If an $(LPK)_n$ admits a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$, then the manifold is an η-Einstein manifold; and the scalars Λ are μ are related by $\Lambda - \mu = \frac{\rho r}{2} + \frac{1}{2}(p + \frac{2}{n}) - \sigma(n-1)$.*

4. Ricci semi-symmetric $(LPK)_n$ admitting CERYS

In 1992, Mirzoyan [18] introduced the notion of Ricci semi-symmetry for the Riemann spaces. In this section we consider a CERYS in an $(LPK)_n$ which satisfies Ricci semi-symmetric condition, i.e., $\mathcal{R}(\xi,\mathcal{U})\cdot\mathcal{S} = 0$. This leads to

$$\mathcal{S}(\mathcal{R}(\xi,\mathcal{U})\mathcal{V},\mathcal{Z}) + S(\mathcal{V},\mathcal{R}(\xi,\mathcal{U})\mathcal{Z}) = 0, \tag{4.1}$$

for $\mathcal{U},\mathcal{V},\mathcal{Z}$ on $(LPK)_n$. By using (2.9) in (4.1), we have

$$S(\xi,\mathcal{Z})g(\mathcal{U},\mathcal{V}) - \eta(\mathcal{V})\mathcal{S}(\mathcal{U},\mathcal{Z}) + \mathcal{S}(\mathcal{V},\xi)g(\mathcal{U},\mathcal{Z}) - \eta(\mathcal{Z})\mathcal{S}(\mathcal{V},\mathcal{U}) = 0. \tag{4.2}$$

By putting $\mathcal{Z} = \xi$ and using (3.4), the foregoing equation leads to

$$\mathcal{S}(\mathcal{U},\mathcal{V}) = \frac{1}{\sigma}[\mu - \Lambda + \frac{\rho r}{2} + \frac{1}{2}(p + \frac{2}{n})]g(\mathcal{U},\mathcal{V}), \quad \sigma \neq 0. \tag{4.3}$$

Now, from (2.3), (3.3) and (4.3), it follows that

$$\frac{(1-\mu)}{\sigma}g(\varphi\mathcal{U},\varphi\mathcal{V}) = 0, \quad \sigma \neq 0. \tag{4.4}$$

This gives $\mu = 1$, where $g(\varphi\mathcal{U},\varphi\mathcal{V}) \neq 0$.

Thus, (4.3) turns to

$$\mathcal{S}(\mathcal{U},\mathcal{V}) = \frac{1}{\sigma}[1 - \Lambda + \frac{\rho r}{2} + \frac{1}{2}(p + \frac{2}{n})]g(\mathcal{U},\mathcal{V}). \tag{4.5}$$

Thus, we have the following result;

Theorem 4.1 *Let an $(LPK)_n$ be Ricci semi-symmetric endowed with a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$. Then $(LPK)_n$ is an Einstein manifold.*

5. Projective curvature tensor in $(LPK)_n$ admitting CERYS

The projective curvature tensor $\mathcal{P}$ in an $(LPK)_n$ is defined by

$$\mathcal{P}(\mathcal{U},\mathcal{V})\mathcal{Z} = \mathcal{R}(\mathcal{U},\mathcal{V})\mathcal{Z} - \frac{1}{n-1}\{\mathcal{S}(\mathcal{V},\mathcal{Z})\mathcal{U} - \mathcal{S}(\mathcal{U},\mathcal{Z})\mathcal{V}\}, \tag{5.1}$$

for all $\mathcal{U},\mathcal{V}$ and $\mathcal{Z}$ on $(LPK)_n$.

In this section, we study $(LPK)_n$ admitting a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$ satisfying certain curvature conditions on $\mathcal{P}$.

First, we consider an $(LPK)_n$ admitting a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$ which satisfies the condition $\mathcal{P}(\mathcal{U}, \xi) \cdot \mathcal{S} = 0$. Thus, we have

$$\mathcal{S}(\mathcal{P}(\mathcal{U}, \xi)\mathcal{V}, \mathcal{Z}) + \mathcal{S}(\mathcal{V}, \mathcal{P}(\mathcal{U}, \xi)\mathcal{Z}) = 0. \tag{5.2}$$

From (2.9), (3.4) and (5.1), we find

$$P(\mathcal{U}, \xi)\mathcal{V} = -g(\mathcal{U}, \mathcal{V})\xi + (1 - \frac{A_1}{n-1})\eta(\mathcal{V})\mathcal{U} + \frac{1}{n-1}S(\mathcal{U}, \mathcal{V})\xi. \tag{5.3}$$

Plugging (5.3) into (5.2), we have

$$\eta(\mathcal{V})\mathcal{S}(\mathcal{U}, \mathcal{Z}) + \eta(\mathcal{Z})S(\mathcal{U}, \mathcal{V}) - A_1 g(\mathcal{U}, \mathcal{V})\eta(\mathcal{Z}) - A_1 g(\mathcal{U}, \mathcal{Z})\eta(\mathcal{V}) = 0,$$

which by putting $\mathcal{V} = \xi$ and then using (2.1) and (3.4) reduces to $S(\mathcal{U}, \mathcal{Z}) = A_1 g(\mathcal{U}, \mathcal{Z})$. By using (3.5) it takes the form

$$S(\mathcal{U}, \mathcal{Z}) = (n-1)g(\mathcal{U}, \mathcal{Z}). \tag{5.4}$$

On contracting (5.4), we obtain $r = n(n-1)$. Thus, (3.5) leads to $\Lambda - \mu = \frac{\rho n(n-1)}{2} + \frac{1}{2}(p + \frac{2}{n}) - \sigma(n-1)$. Now, we state the following result:

Theorem 5.1 *Let an $(LPK)_n$ be Ricci semi-symmetric endowed with a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$, then $(LPK)_n$ is an Einstein manifold. Moreover, Λ and μ are related by $\Lambda - \mu = \frac{\rho n(n-1)}{2} + \frac{1}{2}(p + \frac{2}{n}) - \sigma(n-1)$.*

Next, we consider an $(LPK)_n$ admitting a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$ which satisfies the condition $\mathcal{R}(\mathcal{U}, \xi) \cdot \mathcal{P} = 0$. Thus, we have

$$\begin{aligned} &\mathcal{R}(\mathcal{U}, \xi)\mathcal{P}(\mathcal{X}, \mathcal{V})\mathcal{W} - \mathcal{P}(\mathcal{R}(\mathcal{U}, \xi)\mathcal{X}, \mathcal{V})\mathcal{W} \\ &-\mathcal{P}(\mathcal{X}, \mathcal{R}(\mathcal{U}, \xi)\mathcal{V})\mathcal{W} - \mathcal{P}(\mathcal{X}, \mathcal{V})\mathcal{R}(\mathcal{U}, \xi)\mathcal{W} = 0, \end{aligned} \tag{5.5}$$

for any $\mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X} \in \chi(\mathbb{M})$.

By fixing $\mathcal{X} = \mathcal{W} = \xi$ in (5.5), we have

$$\begin{aligned} &\mathcal{R}(\mathcal{U}, \xi)\mathcal{P}(\xi, \mathcal{V})\xi - \mathcal{P}(\mathcal{R}(\mathcal{U}, \xi)\xi, \mathcal{V})\xi \\ &-\mathcal{P}(\xi, \mathcal{R}(\mathcal{U}, \xi)\mathcal{V})\xi - \mathcal{P}(\xi, \mathcal{V})\mathcal{R}(\mathcal{U}, \xi)\xi = 0. \end{aligned} \tag{5.6}$$

From (2.10), (3.4) and (5.1), we find

$$\mathcal{P}(\mathcal{U}, \mathcal{V})\xi = (1 - \frac{A_1}{n-1})(\eta(\mathcal{V})\mathcal{U} - \eta(\mathcal{U})\mathcal{V}), \tag{5.7}$$

$$\mathcal{P}(\xi, \mathcal{V})\mathcal{U} = -(1 - \frac{A_1}{n-1})\eta(\mathcal{U})\mathcal{V} + g(\mathcal{U}, \mathcal{V})\xi - \frac{1}{n-1}S(\mathcal{U}, \mathcal{V})\xi. \tag{5.8}$$

In view of (2.9), (5.7) and (5.8), after some steps calculation (5.6) gives $\mathcal{S}(\mathcal{U}, \mathcal{V})\xi = A_1 g(\mathcal{U}, \mathcal{V})\xi$, which by taking the inner product with ξ and using (3.5) leads to

$$\mathcal{S}(\mathcal{U}, \mathcal{V}) = (n-1)g(\mathcal{U}, \mathcal{V}). \tag{5.9}$$

On contracting (5.9), we obtain $r = n(n-1)$. Thus, (3.5) turns to $\Lambda - \mu = \frac{\rho n(n-1)}{2} + \frac{1}{2}(p + \frac{2}{n}) - \sigma(n-1)$. Now, we state the following result:

Theorem 5.2 *Let an $(LPK)_n$ admit a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$ and satisfies the condition $\mathcal{R}(\mathcal{U}, \xi) \cdot \mathcal{P} = 0$. Then $(LPK)_n$ is an Einstein manifold. Moreover, Λ and μ are related by $\Lambda - \mu = \frac{\rho n(n-1)}{2} + \frac{1}{2}(p + \frac{2}{n}) - \sigma(n-1)$.*

Further, we consider an $(LPK)_n$ admitting a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$ and satisfies the condition $\mathcal{S}(\xi, \mathcal{U}) \cdot \mathcal{P} = 0$. Then, we have

$$\begin{aligned} &\mathcal{S}(\mathcal{U}, \mathcal{P}(\mathcal{X}, \mathcal{V})\mathcal{W})\xi - \mathcal{S}(\xi, \mathcal{P}(\mathcal{X}, \mathcal{V})\mathcal{W})\mathcal{U} + \mathcal{S}(\mathcal{U}, \mathcal{X})\mathcal{P}(\xi, \mathcal{V})\mathcal{W} \\ &-\mathcal{S}(\xi, \mathcal{X})\mathcal{P}(\mathcal{U}, \mathcal{V})\mathcal{W} + \mathcal{S}(\mathcal{U}, \mathcal{V})\mathcal{P}(\mathcal{X}, \xi)\mathcal{W} - \mathcal{S}(\xi, \mathcal{V})\mathcal{P}(\mathcal{X}, \mathcal{U})\mathcal{W} \\ &+\mathcal{S}(\mathcal{U}, \mathcal{W})\mathcal{P}(\mathcal{X}, \mathcal{V})\xi - \mathcal{S}(\xi, \mathcal{W})\mathcal{P}(\mathcal{X}, \mathcal{V})\mathcal{U} = 0, \end{aligned} \tag{5.10}$$

for all $\mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X} \in \chi(M)$. Putting $\mathcal{X} = \mathcal{W} = \xi$ in (5.10), we have

$$\begin{aligned} &\mathcal{S}(\mathcal{U}, \mathcal{P}(\xi, \mathcal{V})\xi)\xi - \mathcal{S}(\xi, \mathcal{P}(\xi, \mathcal{V})\xi)\mathcal{U} + \mathcal{S}(\mathcal{U}, \xi)\mathcal{P}(\xi, \mathcal{V})\xi - \mathcal{S}(\xi, \xi)\mathcal{P}(\mathcal{U}, \mathcal{V})\xi \\ &+\mathcal{S}(\mathcal{U}, \mathcal{V})\mathcal{P}(\xi, \xi)\xi - \mathcal{S}(\xi, \mathcal{V})\mathcal{P}(\xi, \mathcal{U})\xi + \mathcal{S}(\mathcal{U}, \xi)\mathcal{P}(\xi, \mathcal{V})\xi - \mathcal{S}(\xi, \xi)\mathcal{P}(\xi, \mathcal{V})\mathcal{U} = 0, \end{aligned}$$

which in view of (3.4), (5.7), (5.8) and $\eta(\mathcal{P}(\xi, \mathcal{V})\xi) = 0$ reduces to

$$A_1 g(\mathcal{U}, \mathcal{V})\xi + 2A_1(1 - \frac{A_1}{n-1})\eta(\mathcal{U})\eta(\mathcal{V})\xi + (1 - 2\frac{A_1}{n-1})S(\mathcal{U}, \mathcal{V})\xi = 0.$$

By taking the inner product of the foregoing equation with ξ, then using (2.1), (2.3) and (3.5) it follows that

$$\mathcal{S}(\mathcal{U}, \mathcal{V}) = (n-1)g(\mathcal{U}, \mathcal{V}). \tag{5.11}$$

On contracting (5.11), we obtain $r = n(n-1)$. Thus, (3.5) can be expressed as $\Lambda - \mu = \frac{\rho n(n-1)}{2} + \frac{1}{2}(p + \frac{2}{n}) - \sigma(n-1)$. Now, we state the following result:

Theorem 5.3 *Let an $(LPK)_n$ admit a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$ and satisfies the condition $\mathcal{S}(\xi, \mathcal{U}) \cdot \mathcal{P} = 0$. Then $(LPK)_n$ is an Einstein manifold. Moreover, Λ and μ are related by $\Lambda - \mu = \frac{\rho n(n-1)}{2} + \frac{1}{2}(p + \frac{2}{n}) - \sigma(n-1)$.*

6. CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$ on $(LPK)_n$ admitting certain types of Ricci tensor

Definition 6.1 *An $(LPK)_n$ is said to have Codazzi type Ricci tensor $\mathcal{S}(\neq 0)$ of type $(0, 2)$ if it satisfies the following relation [7]:*

$$(\nabla_{\mathcal{Z}}\mathcal{S})(\mathcal{U}, \mathcal{V}) = (\nabla_{\mathcal{U}}\mathcal{S})(\mathcal{V}, \mathcal{Z}), \tag{6.1}$$

for all $\mathcal{U}, \mathcal{V}, \mathcal{Z} \in \chi(\mathbb{M})$.

Taking the covariant derivative of (3.3) with respect to $\mathcal{Z}$ and using (2.6), we get

$$(\nabla_{\mathcal{Z}}\mathcal{S})(\mathcal{U}, \mathcal{V}) = \frac{(1-\mu)}{\sigma}\{-g(\mathcal{Z}, \mathcal{U})\eta(\mathcal{V}) - g(\mathcal{Z}, \mathcal{V})\eta(\mathcal{U}) - 2\eta(\mathcal{U})\eta(\mathcal{V})\eta(\mathcal{Z})\}. \tag{6.2}$$

If the Ricci tensor $\mathcal{S}$ is of Codazzi type, then in view of (6.2), (6.1) leads to

$$\frac{(1-\mu)}{\sigma}\{g(\mathcal{U}, \mathcal{V})\eta(\mathcal{Z}) - g(\mathcal{Z}, \mathcal{V})\eta(\mathcal{U})\} = 0. \tag{6.3}$$

Putting $\mathcal{Z} = \xi$ in (6.3), we obtain

$$\frac{(1-\mu)}{\sigma} g(\varphi\mathcal{U}, \varphi\mathcal{V}) = 0, \quad \sigma \neq 0, \tag{6.4}$$

from which it gives $\mu = 1$, as $g(\varphi\mathcal{U}, \varphi\mathcal{V}) \neq 0$. Putting $\mu = 1$ in (3.3), it follows that

$$\mathcal{S}(\mathcal{U}, \mathcal{V}) = \frac{1}{\sigma}[1 - \Lambda + \frac{\rho r}{2} + \frac{1}{2}(p + \frac{2}{n})]g(\mathcal{U}, \mathcal{V}). \tag{6.5}$$

This relation shows that the manifold is an Einstein manifold. Thus, we have the following result:

Theorem 6.1 *An $(LPK)_n$ with the Codazzi type Ricci tensor admitting a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$ is an Einstein manifold of the form (6.5).*

Definition 6.2 *An $(LPK)_n$ is said to have cyclic parallel Ricci tensor, if its Ricci tensor S $(\neq 0)$ of type $(0, 2)$ satisfies the relation*

$$(\nabla_{\mathcal{Z}}\mathcal{S})(\mathcal{U},\mathcal{V}) + (\nabla_{\mathcal{U}}\mathcal{S})(\mathcal{V},\mathcal{Z}) + (\nabla_{\mathcal{V}}\mathcal{S})(\mathcal{U},\mathcal{Z}) = 0, \tag{6.6}$$

for all $\mathcal{U},\mathcal{V},\mathcal{Z} \in \chi(\mathbb{M})$.

Let an $(LPK)_n$ admiting a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$ has a cyclic parallel Ricci tensor, thus (6.6) holds. By taking the covariant derivative of (3.3) along $\mathcal{Z}$ and using (2.7), we easily find

$$(\nabla_{\mathcal{Z}}\mathcal{S})(\mathcal{U},\mathcal{V}) = \frac{(1-\mu)}{\sigma}\{-g(\mathcal{Z},\mathcal{U})\eta(\mathcal{V}) - g(\mathcal{Z},\mathcal{V})\eta(\mathcal{U}) - 2\eta(\mathcal{U})\eta(\mathcal{V})\eta(\mathcal{Z})\}. \tag{6.7}$$

Similarly, we have

$$(\nabla_{\mathcal{U}}\mathcal{S})(\mathcal{V},\mathcal{Z}) = \frac{(1-\mu)}{\sigma}\{-g(\mathcal{U},\mathcal{V})\eta(\mathcal{Z}) - g(\mathcal{U},\mathcal{Z})\eta(\mathcal{V}) - 2\eta(\mathcal{U})\eta(\mathcal{V})\eta(\mathcal{Z})\}, \tag{6.8}$$

and

$$(\nabla_{\mathcal{V}}\mathcal{S})(\mathcal{Z},\mathcal{U}) = \frac{(1-\mu)}{\sigma}\{-g(\mathcal{V},\mathcal{Z})\eta(\mathcal{U}) - g(\mathcal{V},\mathcal{U})\eta(\mathcal{Z}) - 2\eta(\mathcal{U})\eta(\mathcal{V})\eta(\mathcal{Z})\}. \tag{6.9}$$

Now using (6.7), (6.8) and (6.9) in (6.6), we lead to

$$\frac{(1-\mu)}{\sigma}\{g(\mathcal{U},\mathcal{V})\eta(\mathcal{Z}) + g(\mathcal{V},\mathcal{Z})\eta(\mathcal{U}) + g(\mathcal{Z},\mathcal{U})\eta(\mathcal{V}) + 3\eta(\mathcal{U})\eta(\mathcal{V})\eta(\mathcal{Z})\} = 0. \tag{6.10}$$

Putting $\mathcal{Z} = \xi$ in (6.10) and using (2.1) and (2.3), we obtain

$$\frac{(1-\mu)}{\sigma}g(\varphi\mathcal{U},\varphi\mathcal{Y}) = 0, \tag{6.11}$$

from which it follows that $\mu = 1$, as $g(\varphi\mathcal{U},\varphi\mathcal{Y}) \neq 0$. By using $\mu = 1$ in (3.3), we get

$$\mathcal{S}(\mathcal{U},\mathcal{V}) = \frac{1}{\sigma}[1 - \Lambda + \frac{\rho r}{2} + \frac{1}{2}(p + \frac{2}{n})]g(\mathcal{U},\mathcal{V}). \tag{6.12}$$

Thus, we have the following theorem:

Theorem 6.2 *If an $(LPK)_n$ admits a CERYS $(g, K = \xi, \sigma, \rho, \Lambda, \mu)$, and the manifold has a cyclic parallel Ricci tensor. Then, the manifold is an Einstein manifold of the form (6.12).*

7. Example

We consider a 3-dimensional manifold $\mathbb{M} = \{(t_1, t_2, t_3) \in R^3\}$, where (t_1, t_2, t_3) are the standard coordinates in R^3. Let ϱ_1, ϱ_2 and ϱ_3 be the vector fields on $\mathbb{M}$ given by

$$\varrho_1 = cosht_3\frac{\partial}{\partial t_1} + sinht_3\frac{\partial}{\partial t_2}, \quad \varrho_2 = sinht_3\frac{\partial}{\partial t_1} + cosht_3\frac{\partial}{\partial t_2}, \quad \varrho_3 = \frac{\partial}{\partial t_3} = \xi,$$

which are linearly independent at each point of $\mathbb{M}$. Let g be the metric (semi-Riemannian) defined by

$$g(\varrho_1, \varrho_1) = \quad g(\varrho_2, \varrho_2) = 1, \quad g(\varrho_3, \varrho_3) = -1, \quad g(\varrho_1, \varrho_2) = g(\varrho_1, \varrho_3) = g(\varrho_2, \varrho_3) = 0.$$

Let the 1-form η on $\mathbb{M}$ is defined by $\eta(\mathcal{U}) = g(\mathcal{U}, \varrho_3)$ for all $\mathcal{U} \in \chi(\mathbb{M})$. Let the $(1, 1)$ tensor field φ on $\mathbb{M}$ is defined by

$$\varphi\varrho_1 = -\varrho_2, \quad \varphi\varrho_2 = -\varrho_1, \quad \varphi\varrho_3 = 0.$$

The linearity of φ and g yields

$$\eta(\varrho_3) = -1, \quad \varphi^2\mathcal{U} = \mathcal{U} + \eta(\mathcal{U})\xi, \quad g(\varphi\mathcal{U}, \varphi\mathcal{V}) = g(\mathcal{U}, \mathcal{V}) + \eta(\mathcal{U})\eta(\mathcal{V}),$$

for all $\mathcal{U}, \mathcal{V} \in \chi(\mathbb{M})$.
Now, by direct computations, we obtain

$$[\varrho_1, \varrho_2] = 0, \quad [\varrho_2, \varrho_3] = -\varrho_1, \quad [\varrho_1, \varrho_3] = -\varrho_2.$$

By using Koszul's formula, we can easily calculate

$$\begin{array}{lll} \nabla_{\varrho_1}\varrho_1 = 0, & \nabla_{\varrho_2}\varrho_1 = -\varrho_3, & \nabla_{\varrho_3}\varrho_1 = 0, \\ \nabla_{\varrho_1}\varrho_2 = -\varrho_3, & \nabla_{\varrho_2}\varrho_2 = 0, & \nabla_{\varrho_3}\varrho_2 = 0, \\ \nabla_{\varrho_1}\varrho_3 = -\varrho_2, & \nabla_{\varrho_2}\varrho_3 = -\varrho_1, & \nabla_{\varrho_3}\varrho_3 = 0. \end{array}$$

Also, one can easily verify that

$$\nabla_{\mathcal{U}}\xi = -\mathcal{U} - \eta(\mathcal{U})\xi \quad \text{and} \quad (\nabla_{\mathcal{U}}\varphi)\mathcal{V} = -g(\varphi\mathcal{U}, \mathcal{V})\xi - \eta(\mathcal{V})\varphi\mathcal{U}.$$

Thus, the manifold $\mathbb{M}$ is an LP-Kenmotsu manifold. It is known that

$$\mathcal{R}(\mathcal{U}, \mathcal{V})\mathcal{Z} = \nabla_{\mathcal{U}}\nabla_{\mathcal{V}}\mathcal{Z} - \nabla_{\mathcal{V}}\nabla_{\mathcal{U}}\mathcal{Z} - \nabla_{[\mathcal{U}, \mathcal{V}]}\mathcal{Z}.$$

By using the above relations, we can easily obtain the components of $\mathcal{R}$ as follows:

$$\begin{array}{lll} \mathcal{R}(\varrho_1, \varrho_2)\varrho_1 = \varrho_2, & \mathcal{R}(\varrho_1, \varrho_2)\varrho_2 = -\varrho_1, & \mathcal{R}(\varrho_1, \varrho_2)\varrho_3 = 0, \\ \mathcal{R}(\varrho_2, \varrho_3)\varrho_1 = 0, & \mathcal{R}(\varrho_2, \varrho_3)\varrho_2 = -\varrho_3, & \mathcal{R}(\varrho_2, \varrho_3)\varrho_3 = -\varrho_2, \\ \mathcal{R}(\varrho_1, \varrho_3)\varrho_1 = -\varrho_3, & \mathcal{R}(\varrho_1, \varrho_3)\varrho_2 = 0, & \mathcal{R}(\varrho_1, \varrho_3)\varrho_3 = -\varrho_1. \end{array}$$

From these values of $\mathcal{R}$, we can easily calculate

$$\mathcal{S}(\varrho_1, \varrho_1) = \mathcal{S}(\varrho_2, \varrho_2) = 0, \quad \mathcal{S}(\varrho_3, \varrho_3) = -2 \implies r = 2. \tag{7.1}$$

Putting $\mathcal{U} = \mathcal{V} = \xi$ in (3.3) and using (7.1) and (2.12) it follows that

$$\Lambda - \mu = \rho - 2\sigma + \frac{1}{2}(p + \frac{2}{3}).$$

Hence Λ and μ satisfies (3.5), and so g defines a CERYS on the given $(LPK)_3$.

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