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A Diophantine Identity of Sophie Germain Type and a Telescoping Family of Series with Zeta Asymptotics

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Abstract: We study a Diophantine equation encoding a family of factorizations of Sophie Germain type for binomials of the form

$$(a_1x^n)^2 + (a_2y^n)^2.$$

A tractable subfamily of this equation is completely described and shown to generate infinite classes of identities analogous to the classical Sophie Germain identity

$$x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy).$$

Several Aurifeuillean factorizations arise naturally as special cases. As an application we introduce a family of infinite series derived from these identities. This family exhibits three distinct structural regimes. First, for a discrete arithmetic locus of the parameter, the corresponding terms factor through an Aurifeuillean decomposition leading to an exact telescoping mechanism and finite rational evaluations. Second, away from this locus the same series admits natural representations in terms of special functions, including digamma functions and quadratic resolvent sums. Third, the large-parameter asymptotic expansion of the series involves odd values of the Riemann zeta function. The resulting family therefore combines an arithmetic telescoping locus with a broader analytic structure governed by special–function identities and Mellin-type asymptotics. In particular, it provides an explicit example where a Diophantine factorization gives rise to a nontrivial analytic family of series linking rational evaluations, special functions, and zeta-value asymptotics.

Keywords: Sophie Germain identity, Aurifeuillean factorization, telescoping series, Mellin transform, digamma function.

Contents

1. Introduction

The arithmetic of sums of two squares has been studied since antiquity. A classical identity, attributed to Diophantus of Alexandria and later generalized by Brahmagupta, states that the product of two sums of two squares is again a sum of two squares:

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) = (a_1b_1 - a_2b_2)^2 + (a_1b_2 + a_2b_1)^2. \tag{1.1}$$

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This identity appears already in the *Arithmetica* of Diophantus and was subsequently placed in a general algebraic framework by Brahmagupta; see, for instance, [1,2,3]. In modern terms, (1.1) corresponds to the multiplicativity of the modulus in the complex numbers: if $\alpha = a_1 + ia_2$ and $\beta = b_1 + ib_2$, then

$$|\alpha\beta| = |\alpha|\,|\beta|.$$

Restricting attention to the integers, the study of sums of two squares is closely related to classical problems in number theory. A fundamental result, originating in the work of Fermat and completed by Euler, characterizes the integers representable as a sum of two squares in terms of their prime decomposition; see, for example, [4,5,6]. In particular, a prime p can be written as $p = x^2 + y^2$ with $x, y \in \mathbb{Z}$ if and only if $p \equiv 1 \pmod 4$.

Among the many identities involving sums of two squares, Sophie Germain's identity

$$x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy) \tag{1.2}$$

plays a distinguished role. Besides its intrinsic algebraic interest, (1.2) has found applications in elementary number theory, for instance in providing simple factorizations that imply the compositeness of certain integers arising in mathematical contests; see [7,8]. Several authors have also considered functional and algebraic generalizations of Sophie Germain's identity; see, for example, [9].

Motivated by (1.2), in this work we introduce and study a Diophantine equation encoding a family of factorizations of "Sophie Germain type" for binomials of the form

$$(a_1x^n)^2 + (a_2y^n)^2.$$

We identify a tractable subfamily of this equation whose integer solutions can be described explicitly. This description yields infinite families of identities analogous to (1.2), and several classical Aurifeuillean factorizations arise naturally as special cases; see [10].

Beyond their intrinsic algebraic interest, these identities also lead naturally to analytic applications. Series involving rational kernels of the form $1/(1 + r^2k^4)$ and related expressions arise frequently in analytic number theory and in the theory of special functions; see, for instance, [11] or [12]. When applied to a suitable rational kernel, the identities considered here generate a family of infinite series exhibiting several distinct structural regimes.

For a discrete set of parameter values an exact telescoping mechanism occurs, causing the series to collapse to finite sums that admit explicit evaluation. Outside this arithmetic locus the same series admit representations in terms of special functions and classical resolvent sums, while their asymptotic behaviour reflects deeper analytic structures related to Mellin-type expansions.

Thus the Diophantine identity studied here provides an example in which a purely arithmetic factorization gives rise to a broader analytic family of series combining telescoping phenomena, special-function representations, and asymptotic expansions involving classical constants.

The paper is organized as follows. Section 2 introduces the Diophantine equation and derives a reduced equation governing its solutions. Sections 3 and 4 describe two principal families of solutions, including generalized Sophie Germain identities and related Aurifeuillean factorizations. Section 5 applies these identities to construct the associated family of infinite series and analyzes its behaviour, including the telescoping subfamily and several analytic representations.

2. A Diophantine Equation for the Binomial $(a_1x^n)^2 + (a_2y^n)^2$

We investigate factorizations of the binomial

$$(a_1x^n)^2 + (a_2y^n)^2,$$

where $a_1, a_2, n, x, y \in \mathbb{Z}_{>0}$, and study the Diophantine equation naturally associated with such factorizations.

Definition 2.1 *Let*

$$a_1^2x^{2n} + a_2^2y^{2n} = \left(d_1x^n + d_2y^n + bx^{n-1}y^{n-1}\right)\left(d_1x^n + d_2y^n - bx^{n-1}y^{n-1}\right), \tag{2.1}$$

where $a_1, a_2, b, d_1, d_2, n, x, y \in \mathbb{Z}_{\geq 0}$.

When $a_2 = d_2 = n = 2$, $b = \pm 2$, and $a_1 = d_1 = 1$, equation (2.1) reduces to Sophie Germain's identity

$$x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy).$$

Besides its intrinsic algebraic interest, Sophie Germain's identity has found several applications in elementary number theory, for example in the evaluation of certain series or in proving the compositeness of integers of the form $k^4 + 4^k$; see, for instance, [7,8]. Related functional generalizations have also been considered (e.g. [9]).

Lemma 2.1 *Assume $a_1 = d_1$ and $a_2 = d_2$. For $x, y \neq 0$, equation (2.1) is equivalent to the reduced equation*

$$2a_1a_2 = b^2(xy)^{n-2}. \tag{2.2}$$

Proof: Expanding the right-hand side of (2.1) yields

$$a_1^2x^{2n} + a_2^2y^{2n} = (d_1x^n + d_2y^n)^2 - b^2x^{2n-2}y^{2n-2}.$$

Rearranging terms gives

$$x^{2n}(a_1^2 - d_1^2) + y^{2n}(a_2^2 - d_2^2) = x^ny^n\big(2d_1d_2 - b^2(xy)^{n-2}\big). \tag{2.3}$$

If $a_1 = d_1$ and $a_2 = d_2$, then (2.3) becomes $x^ny^n\big(2a_1a_2 - b^2(xy)^{n-2}\big) = 0$, and since $x, y \neq 0$, this is equivalent to (2.2). □

Proposition 2.1 *If y divides x and $z = x/y \in \mathbb{Z}$, then equation (2.3) is equivalent to*

$$z^n(a_1^2 - d_1^2) + \frac{1}{z^n}(a_2^2 - d_2^2) = 2d_1d_2 - b^2(zy^2)^{n-2}. \tag{2.4}$$

Proof: Dividing (2.3) by $(xy)^n$ gives

$$\left(\frac{x}{y}\right)^n(a_1^2 - d_1^2) + \left(\frac{y}{x}\right)^n(a_2^2 - d_2^2) = 2d_1d_2 - b^2(xy)^{n-2}.$$

Setting $z = x/y \in \mathbb{Z}$ yields (2.4). □

Corollary 2.1 *If $n > 2$ and $z > 1$, a necessary condition for the existence of integer solutions of (2.4) is that $a_2^2 - d_2^2$ be divisible by z^n.*

Proof: Let $\beta = n - 2 > 1$. Equation (2.4) can be written as

$$(a_1^2 - d_1^2)z^{\beta+2} + \frac{a_2^2 - d_2^2}{z^{\beta+2}} = 2d_1d_2 - b^2(zy^2)^{\beta}.$$

The right-hand side is an integer, while the left-hand side is an integer if and only if $z^{\beta+2}$ divides $a_2^2 - d_2^2$. □

A completely general study of the solutions of (2.1) is not attempted here. In what follows, we restrict attention to the subfamily defined by $a_1 = d_1 = 1$ and $a_2 = d_2 = a \in \mathbb{Z}$, for which (2.1) becomes

$$x^{2n} + a^2y^{2n} = (x^n + ay^n + bx^{n-1}y^{n-1})(x^n + ay^n - bx^{n-1}y^{n-1}), \tag{2.5}$$

and the reduced equation (2.2) takes the form

$$2a = b^2(xy)^{n-2}. \tag{2.6}$$

We now distinguish the cases $a = n$ and $a \neq n$.

3. Case I. $a = n$

Lemma 3.1 *For $n = 1$, there are no solutions of Eq. (2.5) with x and y odd primes.*

Proof: For $n = 1$, the reduced equation (2.6) becomes

$$2xy - b^2 = 0. \tag{3.1}$$

If x and y are odd primes, then $2xy$ is divisible by 2 but not by 4, hence it cannot be a perfect square. Therefore $2xy \neq b^2$ and no such solutions exist. □

Lemma 3.2 *For $n = 1$, there exist infinitely many nonnegative solutions with $\min\{b, x, y\} > 1$ such that $2xy$ is a perfect square.*

Proof: Let $a \in \mathbb{Z}^+$ be arbitrary. The triple

$$\{b, x, y\} = \{2a, a, 2a\}$$

satisfies $2xy = b^2$ identically. Since a is arbitrary, this yields infinitely many solutions. □

Some examples are $\{4, 2, 4\}$, $\{6, 3, 6\}$, $\{8, 4, 8\}$, $\{10, 5, 10\}$, and so on. Other solutions with different values also exist, for instance $\{6, 2, 9\}$, yielding

$$2^2 + 3^4 = \left(2 + 3^2 - 6\right)\left(2 + 3^2 + 6\right). \tag{3.2}$$

Lemma 3.3 *For $n > 2$, equation (2.5) has exactly two integer solutions with $\min\{x, y\} > 1$.*

Proof: Let $n > 2$ and write $\beta = n - 2 \geq 1$. Equation (2.6) can be rewritten as

$$2(\beta + 2) = b^2(xy)^\beta. \tag{3.3}$$

Assume first that $\beta \geq 2$ and $\min\{x, y\} > 1$. Then $xy \geq 4$, and therefore

$$b^2(xy)^\beta \geq 1 \cdot 4^\beta = 2^{2\beta}.$$

On the other hand, for $\beta \geq 2$ one has

$$2(\beta + 2) \leq 2\beta + 4 < 2^{2\beta}.$$

Hence (3.3) cannot hold for $\beta \geq 2$. Consequently the only possible case is $\beta = 1$, that is, $n = 3$.

For $n = 3$, equation (3.3) becomes

$$6 = b^2(xy).$$

Since $\min\{x, y\} > 1$, the product xy cannot be 1, 2, or 3, and the only possibility is $xy = 6$. It follows that $b^2 = 1$, hence $b = 1$, and therefore $(x, y) = (2, 3)$ or $(3, 2)$.

These give the two factorizations of Eq. (2.5):

$$2^6 + 3^8 = \left(2^3 + 3^4 + 6^2\right)\left(2^3 + 3^4 - 6^2\right),$$

$$3^4 + 2^6 = \left(3^2 + 2^3 + 3 \cdot 2^2\right)\left(3^2 + 2^3 - 3 \cdot 2^2\right).$$

In the second identity a common factor 3^2 has been extracted. □

4. Case II. $a \neq n$. Generalized Sophie Germain Identities

From the reduced equation

$$2a = b^2(xy)^{n-2}, \tag{4.1}$$

it follows that the case $n = 2$ plays a distinguished role, since it removes the dependence on x and y and leads to an infinite family of factorizations of Sophie Germain type.

Lemma 4.1 *For $n = 2$ there exist infinitely many nonnegative solutions of generalized Sophie Germain type.*

Proof: When $n = 2$, equation (4.1) reduces to

$$2a = b^2. \tag{4.2}$$

Thus b^2 must be an even perfect square, and every such choice determines a solution. Explicitly,

$$b^2 \in \{4, 16, 36, 64, 100, 144, \dots\}, \qquad a = \frac{b^2}{2}.$$

Since there are infinitely many even perfect squares, the claim follows. □

The classical Sophie Germain identity corresponds to the first case $b^2 = 4$, $a = 2$. Each subsequent choice generates a distinct identity of the same structural form.

Corollary 4.1 *Integers of the form $k^4 + (a^2)^k$, with $k > 1$ and $a = 2s^2$, $s \in \mathbb{Z}^+$, are composite.*

Proof: If k is even, the claim is immediate. Assume k is odd. Setting

$$\{b, x, y\} = \{\sqrt{2a},\, k,\, a^{\frac{k-1}{2}}\}$$

in the generalized Sophie Germain identity yields

$$k^4 + (a^2)^k = \left(k^2 + a^k + \sqrt{2a}\,k\,a^{\frac{k-1}{2}}\right)\left(k^2 + a^k - \sqrt{2a}\,k\,a^{\frac{k-1}{2}}\right). \tag{4.3}$$

Writing $a = 2s^2$ eliminates radicals and gives

$$k^4 + (4s^4)^k = \left(k^2 + (2s^2)^k + 2sk(2s^2)^{\frac{k-1}{2}}\right)\left(k^2 + (2s^2)^k - 2sk(2s^2)^{\frac{k-1}{2}}\right). \tag{4.4}$$

The first factor is strictly greater than 1 for all $k, s \in \mathbb{Z}^+$. For the second factor, define

$$f(k, s) = k^2 + (2s^2)^k - 2sk(2s^2)^{\frac{k-1}{2}}.$$

A direct inspection shows that $f(k, s) > 1$ for all $k > 1$, and also for $k = 1$ whenever $s > 1$. Hence both factors exceed 1, and the number is composite. □

For $s = 1$, this reduces to the classical statement that $k^4 + 4^k$ is composite for all $k > 1$ [7,8]. For instance, choosing $s = k - 1$ yields

$$k^4 + 4^k(k-1)^{4k} = \left(k^2 + 2^k(k-1)^{2k} + 2^{\frac{k+1}{2}}k(k-1)^k\right) \tag{4.5}$$

$$\times \left(k^2 + 2^k(k-1)^{2k} - 2^{\frac{k+1}{2}}k(k-1)^k\right). \tag{4.6}$$

Corollary 4.2 (Aurifeuillean factorization) *Let $\Phi_4(x) = x^2 + 1$ be the fourth cyclotomic polynomial. If $a = 2s^2$ and $k, s \in \mathbb{Z}^+$, then $\Phi_4(a^{2k+1})$ admits an Aurifeuillean factorization.*

Proof: Setting $x = 1$, $y = a^k$, $b = \sqrt{2a}$ and $n = 2$ in Eq. (2.5) gives

$$\Phi_4(a^{2k+1}) = \left(a^{2k+1} + 1 + \sqrt{2a}\,a^k\right)\left(a^{2k+1} + 1 - \sqrt{2a}\,a^k\right). \tag{4.7}$$

Writing $a = 2s^2$ removes the radicals and yields an explicit Aurifeuillean factorization. □

The special case $s = 1$ recovers the classical identity [10]

$$\Phi_4(2^{2k+1}) = \left(2^{2k+1} + 2^{k+1} + 1\right)\left(2^{2k+1} - 2^{k+1} + 1\right).$$

Lemma 4.2 *For $n > 2$, there exist infinitely many nonnegative solutions of the reduced equation.*

Proof: Let $\beta = n - 2 \geq 1$. The reduced equation becomes

$$2a = b^2(xy)^\beta.$$

Thus $b^2(xy)^\beta$ must be even. This occurs in infinitely many ways, for instance whenever at least one of b, x, y is even. Since this condition is independent of β, infinitely many solutions exist for all $n > 2$. □

Lemma 4.3 *For $n > 2$, there are no solutions with b, x, y all odd primes.*

Proof: If b, x, y are odd, then $b^2(xy)^\beta$ is odd for every $\beta \geq 1$. Hence the equation

$$2a = b^2(xy)^\beta$$

cannot hold, since the left-hand side is even. This contradiction proves the claim. □

5. A Family of Series with Telescoping and Zeta Structure

A classical application of Sophie Germain's identity is the finite-sum evaluation (5.1); see also [7,8].

$$\sum_{k=1}^{n} \frac{4k}{4k^4 + 1} = 1 - \frac{1}{1 + 2n + 2n^2}, \tag{5.1}$$

which follows from the factorization

$$4k^4 + 1 = (2k^2 - 2k + 1)(2k^2 + 2k + 1)$$

and a telescoping decomposition. Indeed,

$$\frac{4k}{4k^4 + 1} = \frac{4k}{(2k^2 - 2k + 1)(2k^2 + 2k + 1)} = \frac{1}{2k^2 - 2k + 1} - \frac{1}{2k^2 + 2k + 1},$$

so summing from $k = 1$ to n yields (5.1).

We show next that (5.1) is the first member of a wider family whose terms admit an Aurifeuillean factorization and hence a telescopic structure. We also describe what happens outside the telescopic locus: a natural representation in terms of special functions appears, explaining how transcendental constants arise.

5.1. A telescopic (rational) subfamily

Proposition 5.1 (Telescoping decomposition for a rational subfamily) *Let $b \in \mathbb{Z}_{>0}$ and set*

$$r = \frac{2}{b^2}, \qquad l = \sqrt{\frac{2}{r}} = b.$$

Define, for $k \in \mathbb{Z}_{>0}$,

$$a_k(r) = \frac{1}{1 - \sqrt{2r}\,k + rk^2}. \tag{5.2}$$

Then, for every $n \geq 1$,

$$\sum_{k=1}^{n} \frac{2\sqrt{2r}\,k}{1 + r^2k^4} = \sum_{k=1}^{n} \big(a_k(r) - a_{k+l}(r)\big) = \sum_{k=1}^{l} a_k(r) - \sum_{k=n+1}^{n+l} a_k(r). \tag{5.3}$$

In particular, the infinite sum

$$S(r) := \sum_{k=1}^{\infty} \frac{2\sqrt{2r}\,k}{1 + r^2k^4} \tag{5.4}$$

converges for every $r > 0$ and satisfies the finite evaluation

$$S(r) = \sum_{k=1}^{l} a_k(r) \qquad \text{for } r = \frac{2}{l^2},\ l \in \mathbb{Z}_{>0}. \tag{5.5}$$

Proof: The Aurifeuillean factorization

$$1 + r^2k^4 = (1 - \sqrt{2r}\,k + rk^2)(1 + \sqrt{2r}\,k + rk^2)$$

gives

$$\frac{2\sqrt{2r}\,k}{1 + r^2k^4} = \frac{1}{1 - \sqrt{2r}\,k + rk^2} - \frac{1}{1 + \sqrt{2r}\,k + rk^2}. \tag{5.6}$$

Now assume $r = 2/b^2$, so that $\sqrt{2r} = 2/b$ and $l = b \in \mathbb{Z}_{>0}$. A direct expansion yields the key shift identity

$$1 + \sqrt{2r}\,k + rk^2 = 1 - \sqrt{2r}\,(k + l) + r(k + l)^2. \tag{5.7}$$

Indeed,

$$1 - \sqrt{2r}\,(k + l) + r(k + l)^2 = 1 - \sqrt{2r}\,k - \sqrt{2r}\,l + rk^2 + 2rkl + rl^2,$$

and with $l = b$ and $r = 2/b^2$ one has $\sqrt{2r}\,l = 2$ and $rl^2 = 2$, while $2rl = \sqrt{2r} \cdot 2$ gives $-\sqrt{2r}\,l + 2rkl + rl^2 = -2 + 2\sqrt{2r}\,k + 2 = \sqrt{2r}\,k$, proving (5.7). Consequently,

$$\frac{1}{1 + \sqrt{2r}\,k + rk^2} = \frac{1}{1 - \sqrt{2r}\,(k + l) + r(k + l)^2} = a_{k+l}(r),$$

and substituting into (5.6) yields

$$\frac{2\sqrt{2r}\,k}{1 + r^2k^4} = a_k(r) - a_{k+l}(r).$$

Summing from $k = 1$ to n gives (5.3). Finally, since $a_k(r) = O(k^{-2})$ as $k \to \infty$, we have $a_k(r) \to 0$ and the limit $n \to \infty$ in (5.3) yields (5.5). □

Corollary 5.1 (Concrete example: the case converging to 3) *Taking $r = \frac{1}{2}$ (equivalently $b = l = 2$), one has*

$$\sum_{k=1}^{n} \frac{8k}{k^4 + 4} = a_1(1/2) + a_2(1/2) - a_{n+1}(1/2) - a_{n+2}(1/2), \tag{5.8}$$

and hence

$$\sum_{k=1}^{\infty} \frac{8k}{k^4 + 4} = a_1(1/2) + a_2(1/2) = 3.$$

Proof: Rewrite

$$\sum_{k=1}^{n} \frac{8k}{k^4+4} = \sum_{k=1}^{n} \frac{2k}{1+\frac{1}{4}k^4},$$

which is (5.3) with $r = \frac{1}{2}$ and $l = 2$. Then (5.8) follows, and taking $n \to \infty$ gives the limit 3 since $a_{n+1}(1/2), a_{n+2}(1/2) \to 0$. □

Remark 5.1 (Why the values are rational in the telescopic subfamily) *For $r = 2/b^2$ the closed form (5.5) is a* finite *sum of rational numbers, since $\sqrt{2r} = 2/b$ and $r = 2/b^2$ make each denominator in (5.2) rational. Hence $S(r) \in \mathbb{Q}$ for this subfamily. This explains the rational values structurally (not numerologically).*

First values and a table For $l \in \mathbb{Z}_{>0}$, define $r = 2/l^2$ and $S(l) := S(r)$. Then (5.5) gives

$$S(l) = \sum_{k=1}^{l} \frac{1}{1 - \frac{2}{l}k + \frac{2}{l^2}k^2}. \tag{5.9}$$

The first values are

$$S(1) = 1, \quad S(2) = 3, \quad S(3) = \frac{23}{5}, \quad S(4) = \frac{31}{5}, \quad S(5) = \frac{1721}{221}.$$

For convenience we record them in Table 2.

Table 1: First values in the telescopic family $S(l)$ (with $r = 2/l^2$).

l	$r = 2/l^2$	$S(l)$
1	2	1
2	1/2	3
3	2/9	23/5
4	1/8	31/5
5	2/25	1721/221

Asymptotic behaviour of the rational values $S(l)$

Proposition 5.2 (Asymptotic behaviour of the telescopic values) *Let*

$$S(l) = \sum_{k=1}^{l} \frac{1}{1 - \frac{2k}{l} + \frac{2k^2}{l^2}}.$$

Then, as $l \to \infty$,

$$S(l) = \frac{\pi}{2} l - \frac{1}{3l} + O\left(\frac{1}{l^3}\right). \tag{5.10}$$

In particular,

$$\frac{S(l)}{l} \to \frac{\pi}{2}.$$

Proof: Write the sum in the form

$$S(l) = \sum_{k=1}^{l} g\left(\frac{k}{l}\right), \qquad g(x) = \frac{1}{1 - 2x + 2x^2}.$$

Applying the Euler–Maclaurin summation formula (see, e.g., [15, Ch. 8] or [16, Ch. 3]) to the step size $1/l$ gives

$$\sum_{k=1}^{l} g\left(\frac{k}{l}\right) = l \int_0^1 g(x)\,dx + \frac{g(1) - g(0)}{2} + \frac{g'(1) - g'(0)}{12\,l} + O\left(\frac{1}{l^3}\right).$$

A direct computation shows that

$$\int_0^1 \frac{dx}{1-2x+2x^2} = \frac{\pi}{2},$$

while

$$g(0) = g(1) = 1, \qquad g'(0) = 2, \qquad g'(1) = -2.$$

Substituting these values into the Euler–Maclaurin expansion yields

$$S(l) = \frac{\pi}{2}l - \frac{1}{3l} + O(l^{-3}),$$

which proves the result. □

The asymptotic behaviour $S(l) \sim \frac{\pi}{2}l$ implies that the ratio $S(l)/l$ converges to $\pi/2$. The convergence is illustrated in Figure 2.Table 2 suggests that the normalized values $S(l)/l$ rapidly approach $\pi/2$. This behaviour is explained by the asymptotic expansion proved above.

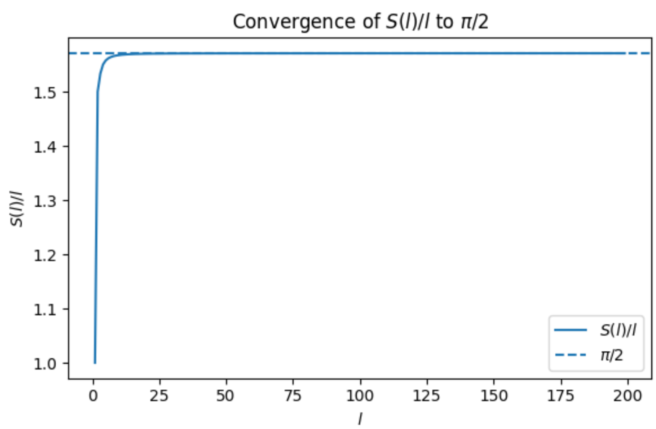

Figure 1: Numerical illustration of the convergence $\frac{S(l)}{l} \to \frac{\pi}{2}$. The dashed line indicates the limiting value $\pi/2$.

l	$S(l)$	$S(l)/l$	$S(l) - \frac{\pi}{2}l$
1	1	1.000000	-0.570796
2	3	1.500000	-0.141593
3	$\frac{23}{5}$	1.533333	-0.112389
4	$\frac{31}{5}$	1.550000	-0.083185
5	$\frac{1721}{221}$	1.557466	-0.066652
6	9.363158...	1.560526	-0.055865
7	10.946337...	1.563762	-0.046509

Table 2: Values of $S(l)$ and deviation from the linear asymptotic behaviour $\frac{\pi}{2}l$.

5.2. Beyond telescoping: special functions and transcendental constants

The telescoping mechanism above relies on the *integer* shift $l = \sqrt{2/r} \in \mathbb{Z}_{>0}$. When $\sqrt{2/r} \notin \mathbb{Z}$, the decomposition (5.6) still holds but no finite cancellation occurs. In this regime, the same series admits a natural closed form in terms of special functions.

Proposition 5.3 (A clean digamma form) *Let $r > 0$ and set*

$$u = \frac{1-i}{\sqrt{2r}}. \tag{5.11}$$

Then the series (5.4) converges absolutely and satisfies

$$S(r) = \frac{\sqrt{2}}{\sqrt{r}}\,\Im\bigl(\psi(1+u) + \psi(1-u)\bigr)\,, \tag{5.12}$$

where ψ is the digamma function.

Proof: Since

$$1 + r^2k^4 = r^2\Bigl(k^2 - \frac{i}{r}\Bigr)\Bigl(k^2 + \frac{i}{r}\Bigr) = r^2(k-u)(k+u)(k-\bar{u})(k+\bar{u}),$$

with

$$u = \frac{1-i}{\sqrt{2r}}, \qquad u^2 = -\frac{i}{r},$$

a direct partial-fraction computation gives

$$\frac{2\sqrt{2r}\,k}{1+r^2k^4} = -\frac{\sqrt{2}}{\sqrt{r}}\,\Im\Bigl(\frac{1}{k+u} + \frac{1}{k-u}\Bigr). \tag{5.13}$$

Summing (5.13) over $k \geq 1$ is justified by absolute convergence, since

$$\frac{2\sqrt{2r}\,k}{1+r^2k^4} = O(k^{-3}) \qquad (k \to \infty).$$

Using the classical identity

$$\sum_{k=1}^{\infty}\Bigl(\frac{1}{k+z} - \frac{1}{k}\Bigr) = -\psi(1+z) - \gamma, \qquad z \notin \{-1,-2,\dots\},$$

for $z = u$ and $z = -u$, and observing that the divergent harmonic terms are purely real and therefore disappear after taking imaginary parts, we obtain

$$\Im\sum_{k=1}^{\infty}\Bigl(\frac{1}{k+u} + \frac{1}{k-u}\Bigr) = -\Im\bigl(\psi(1+u) + \psi(1-u)\bigr)\,.$$

Substituting this into (5.13) yields (5.12). □

Remark 5.2 (Where π, logarithms, and other constants enter) *Formula (5.12) explains the "analytic" regime of $S(r)$: once the strict telescoping condition $r = 2/l^2$ is removed, values of $S(r)$ are governed by special-function values. For special algebraic r, the points $1 \pm u$ become algebraic (often quadratic), and reduction identities for ψ (reflection, multiplication; see [14, Sec. 5.5]) can introduce π and logarithms. More refined reductions may involve polygamma values at rational arguments and hence zeta values. This is the natural mechanism by which transcendental constants arise in this family.*

5.3. Monotonicity and asymptotics

Theorem 5.1 (Monotonicity for large r) *The function $S(r)$ is strictly decreasing on $[1/\sqrt{3}, \infty)$.*

Proof: Let

$$t_k(r) = \frac{2\sqrt{2r}\,k}{1+r^2k^4}.$$

A direct computation gives

$$t_k'(r) = \frac{\sqrt{2}}{\sqrt{r}}\,\frac{k\bigl(1 - 3r^2k^4\bigr)}{\bigl(1 + r^2k^4\bigr)^2}.$$

If $r \geq 1/\sqrt{3}$, then $3r^2k^4 \geq 3r^2 \geq 1$ for every $k \geq 1$, hence $t_k'(r) < 0$ for all k. Moreover, for $r \geq 1/\sqrt{3}$ we have the uniform estimate $|t_k'(r)| \leq C/k^3$ for some constant C, since for large k the derivative behaves like $O(k^{-3})$ uniformly on that interval. By the Weierstrass M-test, $\sum_{k\geq 1} t_k'(r)$ converges uniformly on $[1/\sqrt{3}, \infty)$, hence we may differentiate termwise:

$$S'(r) = \sum_{k=1}^{\infty} t_k'(r) < 0 \qquad (r \geq 1/\sqrt{3}).$$

□

Remark 5.3 (About global monotonicity) *Each individual term $t_k(r)$ increases for $0 < r < 1/(\sqrt{3}k^2)$ and decreases afterwards. Thus, a proof of monotonicity on $(0, 1/\sqrt{3})$ requires balancing finitely-many increasing terms against the decreasing tail. Numerically, $S(r)$ appears decreasing already on $(0, \infty)$, but we only claim the rigorous statement in Theorem 5.1.*

Theorem 5.2 (Asymptotic expansions) (i) *For $r > 1$ the following absolutely convergent expansion holds:*

$$S(r) = 2\sqrt{2}\sum_{m=0}^{\infty}(-1)^m \frac{\zeta(3+4m)}{r^{\frac{3}{2}+2m}} = \frac{2\sqrt{2}\,\zeta(3)}{r^{3/2}} - \frac{2\sqrt{2}\,\zeta(7)}{r^{7/2}} + \frac{2\sqrt{2}\,\zeta(11)}{r^{11/2}} - \cdots . \tag{5.14}$$

(ii) *As $r \to 0^+$,*

$$S(r) = \frac{\pi}{\sqrt{2r}} - \frac{\sqrt{2r}}{6} + O(r^{3/2}). \tag{5.15}$$

Proof: (i) For $r > 1$ we have $|1/(r^2k^4)| \leq 1/r^2 < 1$ for every $k \geq 1$, so

$$\frac{1}{1 + r^2k^4} = \frac{1}{r^2k^4}\cdot\frac{1}{1 + \frac{1}{r^2k^4}} = \frac{1}{r^2k^4}\sum_{m=0}^{\infty}(-1)^m \frac{1}{r^{2m}k^{4m}}$$

converges absolutely and uniformly in k. Multiplying by $2\sqrt{2r}\,k$ and summing gives

$$S(r) = 2\sqrt{2r}\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m \frac{k}{r^{2m+2}k^{4m+4}} = 2\sqrt{2}\sum_{m=0}^{\infty}(-1)^m \frac{1}{r^{\frac{3}{2}+2m}}\sum_{k=1}^{\infty}\frac{1}{k^{3+4m}},$$

where exchanging sums is justified by absolute convergence. This yields (5.14).

(ii) Write

$$S(r) = \frac{2\sqrt{2}}{\sqrt{r}}\sum_{k=1}^{\infty} F(k\sqrt{r}), \qquad F(x) = \frac{x}{1 + x^4}.$$

As $r \to 0^+$ this behaves as a Riemann sum for the integral

$$\int_0^{\infty} F(x)\,dx$$

with step size $\sqrt{r}$, giving the leading term $\pi/\sqrt{2r}$. A standard Euler–Maclaurin expansion for $\sum_{k\geq 1} F(k\sqrt{r})$ yields the next correction term $-\frac{1}{12}F'(0)\sqrt{r} = -\frac{1}{12}\sqrt{r}$; multiplying by $2\sqrt{2}/\sqrt{r}$ gives $-\sqrt{2r}/6$, proving (5.15). □

The asymptotic behaviour obtained in Theorem 5.2 is illustrated numerically in Figure 2.

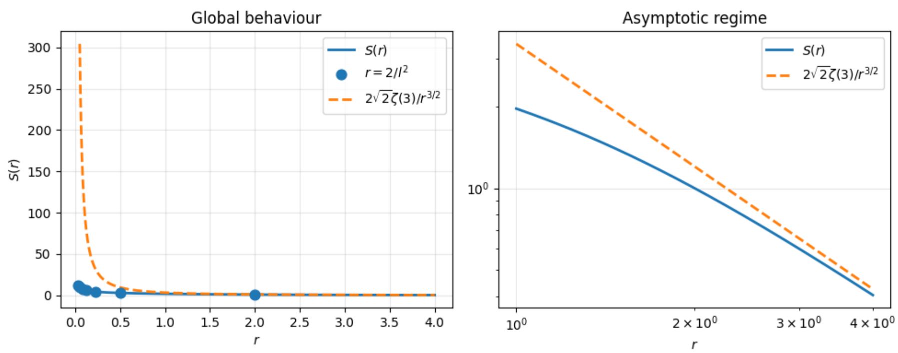

Figure 2: Numerical illustration of the function $S(r)$. The blue curve shows $S(r)$ for $r > 0$, while the dots mark the telescopic points $r = 2/\ell^2$, where the series collapses to finite rational sums. The dashed curve indicates the large-r asymptotic behaviour $S(r) \sim 2\sqrt{2}\zeta(3)/r^{3/2}$ from Theorem 5.2.

5.4. A resolvent representation

The series

$$S(r) = \sum_{k=1}^{\infty} \frac{2\sqrt{2r}\,k}{1+r^2k^4}, \qquad r > 0,$$

also admits a natural representation in terms of quadratic resolvent sums. Indeed, the elementary partial fraction identity

$$\frac{k}{1+r^2k^4} = \frac{1}{2ir}\left(\frac{1}{k^2-\frac{i}{r}} - \frac{1}{k^2+\frac{i}{r}}\right) \tag{5.16}$$

follows immediately from

$$\frac{1}{k^2-a} - \frac{1}{k^2+a} = \frac{2a}{k^4-a^2},$$

upon setting $a = i/r$.

Substituting (5.16) into the definition of $S(r)$ yields

$$S(r) = \frac{\sqrt{2}}{i\sqrt{r}} \sum_{k=1}^{\infty} \left(\frac{1}{k^2-\frac{i}{r}} - \frac{1}{k^2+\frac{i}{r}}\right), \tag{5.17}$$

showing that $S(r)$ may be expressed as the difference of two classical quadratic resolvent sums evaluated at the complex spectral parameters $z = \pm i/r$.

This observation admits a natural spectral interpretation. Let L denote the one–dimensional Dirichlet Laplacian on $(0,\pi)$, whose eigenvalues are $\lambda_k = k^2$, $k \geq 1$. Formally,

$$\sum_{k=1}^{\infty} \frac{1}{k^2+z} = \mathrm{Tr}\big((L+zI)^{-1}\big),$$

whenever z avoids the negative spectrum. Thus (5.17) shows that $S(r)$ can be viewed as a linear combination of resolvent traces of L.

A closed form follows from the classical identity (see, e.g., [11, p. 487] or [12, Sec. 1.421]):

$$\sum_{k=1}^{\infty} \frac{1}{k^2+a^2} = \frac{1}{2}\left(\frac{\pi}{a}\coth(\pi a) - \frac{1}{a^2}\right), \qquad a \notin i\mathbb{Z}. \tag{5.18}$$

Setting $a_\pm = e^{\mp i\pi/4} r^{-1/2}$ so that $a_\pm^2 = \mp i/r$, and using (5.18) in (5.17), one obtains

$$S(r) = \frac{\sqrt{2}\,\pi}{\sqrt{r}}\,\Im\left(\frac{1}{a_+}\coth(\pi a_+)\right). \tag{5.19}$$

This representation shows that the same family which exhibits an arithmetic telescoping locus for $r = 2/\ell^2$ also belongs naturally to the analytic framework of quadratic resolvent sums and spectral trace formulas.

5.5. Residues, contour methods, and an alternative closed form

The digamma representation explains the appearance of transcendental constants, but it is also useful to record a complementary derivation (often preferred in analytic number theory) based on residues. This is particularly natural here because the denominator $1 + r^2z^4$ has four simple complex roots.

Proposition 5.4 (Residue formulation) *Let $r > 0$ and let Ω be the set of roots of $1 + r^2z^4 = 0$, i.e.*

$$\Omega = \left\{ \frac{e^{i(\pi/4+m\pi/2)}}{\sqrt{r}} : m = 0, 1, 2, 3 \right\}.$$

Define the meromorphic function

$$H(z) = \frac{2\sqrt{2r}\, z}{1 + r^2z^4} \pi \cot(\pi z).$$

Then H has simple poles at $z \in \mathbb{Z}$ and at $z \in \Omega$, and one has the exact identity

$$S(r) = -\sum_{\omega\in\Omega} \operatorname{Res}\big(H(z), z = \omega\big). \tag{5.20}$$

Moreover, each residue admits the closed form

$$\operatorname{Res}\big(H(z), z = \omega\big) = \frac{2\sqrt{2r}\,\omega}{4r^2\omega^3} \pi \cot(\pi\omega) = \frac{\sqrt{2}}{2r^{3/2}} \cdot \frac{\pi}{\omega^2} \cot(\pi\omega). \tag{5.21}$$

Proof: The poles of $\pi \cot(\pi z)$ at $z \in \mathbb{Z}$ have residue 1, hence

$$\operatorname{Res}\big(H(z), z = k\big) = \frac{2\sqrt{2r}\, k}{1 + r^2k^4} \qquad (k \in \mathbb{Z}).$$

Choose a standard rectangular contour with vertices $\pm(N + \frac{1}{2}) \pm iT$ and let $T \to \infty$ then $N \to \infty$. Using the exponential decay of $\cot(\pi z)$ off the real axis and the fact that $2\sqrt{2r}\, z/(1 + r^2z^4) = O(|z|^{-3})$, the contour integral tends to 0. By the residue theorem, the sum of residues inside the contour vanishes; separating integer residues and the four nonreal poles Ω yields (5.20). Finally, each $\omega \in \Omega$ is a simple zero of $1 + r^2z^4$, so

$$\operatorname{Res}\left(\frac{2\sqrt{2r}\, z}{1 + r^2z^4}, z = \omega\right) = \frac{2\sqrt{2r}\,\omega}{(1 + r^2z^4)'|_{z=\omega}} = \frac{2\sqrt{2r}\,\omega}{4r^2\omega^3},$$

and multiplying by $\pi \cot(\pi\omega)$ gives (5.21). □

Remark 5.4 (Connection with special functions) *The residue representation (5.20) is equivalent to the digamma form (Proposition 5.3) after using the partial fraction expansion $\pi \cot(\pi z) = \frac{1}{z} + \sum_{k\neq 0} \left(\frac{1}{z-k} + \frac{1}{k}\right)$ and standard rearrangements, (see [11, p. 116]), which are precisely what lead to digamma differences.*

The family of series studied here therefore exhibits three distinct structural layers: an arithmetic telescoping locus, analytic representations in terms of special functions, and asymptotic expansions governed by odd values of the Riemann zeta function. This combination illustrates how a simple arithmetic factorization can generate a broader analytic structure involving telescoping series, special functions, and zeta-value asymptotics.

5.6. Perspectives and possible extensions

The family of series

$$S(r) = \sum_{k=1}^{\infty} \frac{2\sqrt{2r}\,k}{1 + r^2k^4}$$

exhibits a rather unusual combination of algebraic and analytic features. On the one hand, the Aurifeuillean factorization underlying Proposition 5.1 produces an exact telescoping mechanism for the discrete set of parameters $r = 2/l^2$, yielding finite rational evaluations. On the other hand, outside this arithmetic locus the same series admits natural representations in terms of special functions and analytic transforms. This section briefly outlines several perspectives that may deserve further study.

A Mellin-transform viewpoint. A useful way to interpret the structure of $S(r)$ is through Mellin analysis. Introducing the kernel

$$F(x) = \frac{x}{1 + x^4},$$

one may rewrite the series exactly in the form

$$S(r) = 2\sqrt{2}\sum_{k\geq 1} F(k\sqrt{r}).$$

In this representation $S(r)$ appears as a discrete Mellin-type sum of the function F evaluated along the geometric scale $k\sqrt{r}$. The Mellin transform of F is classical (see, e.g., [16, Ch. 3]) and is given by

$$\int_0^{\infty} \frac{x^s}{1 + x^4}\,dx = \frac{\pi}{4}\csc\left(\frac{\pi(s+1)}{4}\right), \qquad -1 < \Re(s) < 3.$$

From this viewpoint the asymptotic expansion of $S(r)$ for large r (Theorem 5.2) arises naturally from the poles of the Mellin kernel, which occur at

$$s = 3, 7, 11, \ldots.$$

These poles correspond precisely to the sequence of zeta values $\zeta(3 + 4m)$ appearing in the expansion (5.14).This pattern reflects the quartic symmetry of the kernel $x/(1 + x^4)$, whose Mellin transform has poles precisely at $s = 3 + 4m$. Thus the analytic behaviour of $S(r)$ reflects the Mellin structure of the function $x/(1 + x^4)$.

It is also instructive to consider the regime $r \to 0^+$. In that case the step size $\sqrt{r}$ becomes small and the discrete sum

$$\sum_{k\geq 1} F(k\sqrt{r})$$

approaches a Riemann sum for the integral of F. More precisely,

$$\sum_{k\geq 1} F(k\sqrt{r}) \sim \frac{1}{\sqrt{r}}\int_0^{\infty} F(x)\,dx \qquad (r \to 0^+),$$

which follows from the Euler–Maclaurin summation formula (or, equivalently, from the interpretation of the sum as a Riemann sum with step size $\sqrt{r}$), since the function $F(x) = x/(1 + x^4)$ is smooth and integrable on $(0, \infty)$. This leads to the asymptotic behaviour

$$S(r) \sim \frac{2\sqrt{2}}{\sqrt{r}}\int_0^{\infty} \frac{x}{1 + x^4}\,dx = \frac{\pi}{\sqrt{2r}}.$$

This clarifies the origin of the factor $r^{-1/2}$ in the small-r asymptotics of the series.

Relation with statistical-type integrals. The same series also admits an integral representation obtained from the identity

$$\frac{1}{1+a} = \int_0^{\infty} e^{-t} e^{-at}\, dt.$$

Applying this with $a = r^2 k^4$ gives

$$S(r) = 2\sqrt{2r} \int_0^{\infty} e^{-t} \sum_{k\geq 1} k\, e^{-r^2 k^4 t}\, dt.$$

Sums of the form

$$\sum_{k\geq 1} e^{-ak^p}$$

are closely related to partition functions and to integrals of Fermi–Dirac or Bose–Einstein type arising in statistical mechanics [17]. Although the exponent k^4 differs from the classical quadratic case, the analytic techniques used in the study of such integrals (Mellin transforms, Poisson summation, and saddle-point methods) may be adapted to the present setting. This viewpoint suggests that the analytic structure of $S(r)$ belongs to a broader class of series associated with generalized partition functions.

A spectral interpretation. The quartic growth in the denominator of $S(r)$ also admits a natural spectral interpretation. In one dimension, eigenvalues of fourth-order differential operators (such as the biharmonic operator Δ^2) typically grow like k^4. From this perspective, expressions of the form

$$\sum_{k\geq 1} \frac{1}{1+\tau k^4}$$

are reminiscent of resolvent-type traces associated with fourth-order spectral problems. The series $S(r)$ may therefore be viewed heuristically as a *weighted resolvent sum* corresponding to a fourth-order spectral model. Within this interpretation, the appearance of the zeta values $\zeta(3+4m)$ in the large-r expansion is consistent with the expected structure of spectral zeta functions associated with quartic eigenvalue growth.

Possible generalizations. These observations suggest several natural directions for further study. One possibility is to investigate analogous families of series of the form

$$S_m(r) = \sum_{k\geq 1} \frac{k^{\beta}}{1+r^2 k^{2m}}, \qquad m \geq 2,$$

which correspond heuristically to spectral models of order $2m$. It would be interesting to determine whether similar telescoping loci arise for special parameter values and how the associated analytic expansions depend on the exponent m. Such investigations would naturally involve Mellin-transform techniques, spectral zeta functions, and generalized Dirichlet series.

Taken together, these perspectives indicate that the family of series considered here lies at the intersection of several analytic frameworks, including Mellin analysis, special-function representations, and spectral-type summation formulas. The existence of a discrete telescoping locus inside this analytic family provides an intriguing example of an arithmetic structure emerging within a broader analytic context.

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